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THE REGULAR OPEN CONTINUOUS IMAGES OF COMPLETE METRIC SPACES

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This article characterizes the regular T_0 open continuous images of complete metric spaces. These images are shown to be the regular T_0-spaces having monotonically complete bases of countable order. This follows from a theorem of Worrell and Wicke and a theorem below which shows that every regular T_0-space having a monotonically complete base of countable order is an open continuous image of a complete metric space.

The class of regular T_0-spaces having monotonically complete bases of countable order is equivalent to a class of spaces Aronszajn introduced axiomatically in [4]. This class includes the complete metric spaces and spaces satisfying R. L. Moore's Axiom 1 [9]. Theorem 2 provides contrast to the theorem of Ponomarev [10]: every T_0 first countable space is an open continuous image of a metric space. A result related to Theorem 3 is Arhangel'skiĭ's characterization of the T_1 open compact continuous images of metrizable spaces as the metacompact developable T_1-spaces [2]. In connection with this result, it may be noted that an open compact continuous T_1 image of a regular T_0-space having a base of countable order also has a base of countable order [11], and a T_1 metacompact space having a base of countable order is developable [12].

For notation and terminology the reader is referred to [7], [9], and [12]. *Space* is used here to mean *topological space*. The null set convention is not used. A base for the topology of a space S will be referred to as a *base for* S. Recall that a collection of sets is said to be *perfectly decreasing* [12], if and only if each of its elements properly includes an element of the collection; and that a *base of countable order* for a space [3], can be defined as a base B for the space such that if P is a point common to the elements of a perfectly decreasing subcollection K of B, any open set containing P includes an element of K; i.e., the elements of K form a base at P. By a *monotonically complete base* [11], is meant a base B such that the closures of the elements of any monotonic subcollection of B have a point in common. Recall also that regular T_0-spaces are T_1, as Koutský remarked [5, p. 826].

2. Regular spaces having monotonically complete bases of countable order.

THEOREM 1. *A regular T_0-space S has a monotonically complete base of countable order if and only if there exists a sequence $G_1, G_2, \cdots$ of bases for the topology of S such that if $g_1, g_2, \cdots$ is a sequence such that, for each n, g_n belongs to G_n and $\bar{g}_{n+1}$ is a subset of g_n, then there exists a point P in each g_n such that the collection of terms of $g_1, g_2, \cdots$ is a base at P.*

Proof. Suppose V is a monotonically complete base of countable order for S. There exists a sequence $H_1, H_2, \cdots$ of well-ordered subcollections of V covering S such that these conditions are satisfied:

(1) For each n and h in H_n there exists a point $P_{n,h}$ belonging to h such that no element of H_n precedes h and contains $P_{n,h}$.

(2) If $n < k$, the closure of the first element h of H_k containing the point P is a subset of the first element h' of H_n containing P; and if P is in a proper subset of h', $\bar{h}$ is a proper subset of h'. By an argument similar to that used in the proof of Theorem 1 of [12], it follows that the collections $G_n = H_n + H_{n+1} + \cdots$ are bases for S. If $g_1, g_2, \cdots$ is a sequence as in the statement of Theorem 1, there exists a first h_n in H_n that includes a term of $g_1, g_2, \cdots$. For each n, there exists $j > n + 1$ such that g_j is a subset of h_n and h_{n+1}. For some $k \geqq j$, g_j belongs to H_k. Let P denote the point P_{k,g_j}. If h is the first element of H_n to contain P then h includes g_j. Thus h does not precede h_n. Since h_n contains P it follows that $h = h_n$. Similarly, h_{n+1} is the first element of H_{n+1} to contain P and thus $\bar{h}_{n+1}$ is a subset of h_n. If $h_n = h_{n+1}$ for some n, then $h_n = \{P\}$ for some point P, and thus $g_k = \{P\}$ for some k, and $\{g_k\}$ is a base at P. If $h_n \neq h_{n+1}$, for any n, the terms of $h_1, h_2, \cdots$ form a monotonic subcollection of V and thus there exists a point P common to each $\bar{h}_n$. Since $\bar{h}_{n+1}$ is a subset of h_n, P is in each h_n. If D is open and contains P, there exists some h_n which is a subset of D and thus some g_k is included in D. Hence P is in $\bar{g}_k$ for all k, and since $\bar{g}_k$ is a subset of g_{k-1} for all $k > 1$, it follows that P is in each g_k. Since the h_n's form a base at P so do the g_n's.

If $G_1, G_2, \cdots$ is a sequence as in the statement of Theorem 1 there exists a sequence $H_1, H_2, \cdots$ of well-ordered collections covering S such that for each n: (1) H_n is a subcollection of G_n. (2) Each element h of H_n contains a point belonging to no predecessor of h in H_n. (3) If $n < k$ and P is a point, the closure of the first element of H_k containing P is a subset of the first element of H_n doing so. $V = H_1 + H_2 + \cdots$ is a base for S and can be shown to be a base of countable order by an argument used in Theorem 2 of [12]. A technique similar to one employed there and álso in the preceding paragraph, shows that V is monotonically complete.

THEOREM 2. *A regular T_0-space having a monotonically complete base of countable order is an open continuous image of a complete metric space.*

Proof. Let S denote a regular T_0-space having a monotonically complete base of countable order. By Theorem 1 there exists a sequence $G_1, G_2, \cdots$ of bases for S with the property stated in that theorem. Form the Baire space M [6] over the collections $G_1, G_2, \cdots$. The elements of M are sequences $\xi = (g_1, g_2, \cdots)$ where g_n belongs to G_n. If $\xi = (g_1, g_2, \cdots)$ and $\xi' = (g'_1, g'_2, \cdots)$ the distance $\rho(\xi, \xi')$ is defined to be $1/k$ if there exists a first positive integer k such that $g_k \neq g'_k$. Otherwise $\rho(\xi, \xi') = 0$. Designate by $O_{a_1 \cdots a_k}$ the collection of all sequences $(a'_1, a'_2, \cdots)$ such that $a_i = a'_i$, $i = 1, \cdots, k$. Let W denote the collection of all elements in M of the form $(g_1, g_2, \cdots)$ where for each n, $\bar{g}_{n+1}$ is a subset of g_n. Then, by the condition on $G_1, G_2, \cdots$, there exists a unique point P common to the terms of $g_1, g_2, \cdots$. If $\xi = (g_1, g_2, \cdots)$ is in W, define $f\xi$ to be the unique point P common to the g_n's. If P is a point of S, by regularity there exists an element ξ of W such that P is common to the terms of ξ. Hence f is a mapping of W onto S. Suppose W intersects the set $O_{g_1 \cdots g_k}$. Then $\bar{g}_{i+1}$ is a subset of g_i for all $i \leqq k-1$. Clearly, $f(W \cdot O_{g_1 \cdots g_k})$ is a subset of g_k. If P is an element of g_k, there exists $g_{k+1}, g_{k+2}, \cdots$ such that $\bar{g}_{k+n}$ is a subset of g_{k+n-1} for all $n \geqq 1$. Hence $f(W \cdot O_{g_1 \cdots g_k}) = g_k$. Since the collection of all sets $W \cdot O_{g_1 \cdots g_k}$ is a base for W and by the property of $G_1, G_2, \cdots$, f is open and continuous on W. (This argument is related to one used by Ponomarev [10].)

Suppose $P_1, P_2, \cdots$ is a sequence of points of W satisfying the Cauchy convergence criterion. For each n, there exists a positive integer m_n such that $\rho(P_k, P_j) < 1/n$, provided $k, j \geqq m_n$. It may be assumed that $m_{n+1} > m_n$ for every n. Let $a_1^n, a_2^n, \cdots, a_n^n$ denote the first n coordinates of P_{m_n}. Let a_n denote a_n^n for each n. Then if $k \geqq m_n$, the first n coordinates of P_k are $a_1, \cdots, a_n$. For if $n = 1$, $a_1 = a_1^1$ is the first coordinate of P_{m_1}. If $k > m_1$, then $\rho(P_k, P_{m_1}) < 1$, and thus a_1 is the first coordinate of P_k. Suppose the statement is true for n. If $k \geqq m_{n+1}$, then $\rho(P_k, P_{m_{n+1}}) < 1/(n+1)$. Since $m_{n+1} > m_n$, the first n coordinates of $P_{m_{n+1}}$ are $a_1, \cdots, a_n$, by the assumption, and the $(n+1)^{\text{st}}$ coordinate is a_{n+1}. Let P denote $(a_1, a_2, \cdots)$. It follows that P is the sequential limit point of $P_1, P_2, \cdots$. Moreover, since P_{m_n} is in W, the coordinates $a_1, a_2, \cdots, a_n$ satisfy the condition that $\bar{a}_{k+1}$ is a subset of a_k for all $k \leqq n-1$. Since this is true for all n, it follows that P is in W, and thus W is complete with respect to ρ.

REMARK. From the proof of the above theorem it may be seen that the complete metric space of the theorem may be taken to be of zero dimension and of the same weight as the image space. (The *weight* of a topological space is the minimum cardinal number m such that the space has a base of power m [1].)

3. The characterization theorem. In [11] Worrell and Wicke define a λ-base for a topological space as a base B of countable order for the space such that if K is a perfectly decreasing monotonic subcollection of B, there exists a point P such that any open set containing P includes an element of K. A regular T_0-space has a λ-base if and only if it has a monotonically complete base of countable order [11]. A principal theorem of [11] is that an open continuous (essentially) T_1 image of a space having a λ-base also has a λ-base.

THEOREM 3. *A regular T_0-space is an open continuous image of a complete metric space if and only if it has a monotonically complete base of countable order.*

Proof. The sufficiency follows from Theorem 2. The necessity is a consequence of the theorems cited in the paragraph preceding the statement of Theorem 3, and the facts that a regular T_0-space is T_1 and that a complete metric space has a λ-base.

THEOREM 4. *The following conditions on a regular T_0-space are equivalent.*

(a) *The space has a monotonically complete base of countable order.*

(b) *The space satisfies Aronszajn's axiom* [4, p. 231].

(c) *The space has a λ-base.*

(d) *The space is an open continuous image of a complete metric space.*

Proof. The equivalence of (a), (b), and (c) is stated in [11], and may be established by methods used in the proof of Theorem 1 above. Theorem 3 above shows the equivalence of (a) and (d).

By using techniques similar to those used above, the following theorem may be proved. (The sufficiency is a joint result of Worrell and Wicke given in [11].)

THEOREM 5. *A T_1-space S has a base of countable order if and only if there exists a metric space (M, d) and an open continuous mapping f of M onto S such that for each x in S, $f^{-1}(x)$ is complete with respect to the metric d.*

This result and a theorem of Arhangel'skiǐ [3] imply the following theorem of Michael [8]:

If f is an open continuous mapping of a metric space E onto a T_2 paracompact space F such that $f^{-1}(y)$ is complete for every y in F, then F is metrizable.

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