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AN EXTREMAL LENGTH CRITERION FOR THE PARABOLICITY OF RIEMANNIAN SPACES

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It is the purpose of this paper to show that a given Riemannian space satisfying a regularity condition is parabolic if and only if the extremal distance of a fixed ball in the space from the ideal boundary of the space is infinite.

We will also show that the harmonic modulus of a space bounded by two sets of boundary components coincides with the extremal distance between the two sets.

STATEMENTS OF MAIN RESULTS

1. Regularity condition. Throughout this paper we denote by R a noncompact C^∞ Riemannian space with the ideal boundary β. We always assume that R is orientable and connected. Let A be the complement of a regular subregion of R with the relative boundary α. We also assume that A-α is connected. We consider the following regularity condition for R (more precisely, for A):

For any nonconstant harmonic function u defined on a region $\Omega \subset A$, the set $\{x \in \Omega \,|\, |\nabla u(x)| = 0\}$ has zero capacity.

This condition is always satisfied if the dimension of R is two. This is also true, for example, when the metric tensor g_{ij} is real analytic on A-α. A typical case is furnished by a locally flat A-α.

In this paper we only consider those spaces R for which the above regularity condition is met.

2. Extremal length. Let ρ be a density, i.e. a nonnegative Borel function on A, and let Γ be a family of curves γ which issue from a point in α and lie in A-α. We define the *harmonic extremal length*, or simply the extremal length of Γ, by

$$\lambda(\Gamma) = \sup_{\rho \not\equiv 0} \frac{L(\Gamma, \rho)^2}{V(A, \rho)}, \tag{1}$$

where $V(A, \rho) = \int_A \rho^2 dV$ and $L(\Gamma, \rho) = \inf_\Gamma \int_\gamma \rho\, ds$. Here dV and ds are the volume and the line element

We are particularly interested in the family $\Gamma_\beta \subset \Gamma$ of all curves $\gamma \in \Gamma$ terminating at β.

3. Parabolicity. We call R parabolic, $R \in O_G$, if R carries no nonconstant positive superharmonic function. The main object of this paper is to prove:

THEOREM 1. *The space R is parabolic if and only if* $\lambda(\Gamma_\beta) = \infty$.

4. Moduli. Let Ω be a regular subregion of R with relative boundary $\beta_\Omega \subset A - \alpha$, and let u_Ω be the continuous function on $\bar{\Omega} \cap A$ which is harmonic in the interior of $\bar{\Omega} \cap A$ with $u_\Omega | = 0$ and $u_\Omega | \beta_\Omega = 1$. The constant μ_Ω given by

$$\log \mu_\Omega = 1/\int_{\bar{\Omega} \cap A} du_\Omega \wedge * du_\Omega \tag{2}$$

is called the *harmonic modulus*, or simply the modulus of $\bar{\Omega} \cap A$ with respect to α. It is easy to see that

$$\mu_\Omega \leqq \mu_{\Omega'} \tag{3}$$

for $\Omega \subset \Omega'$. Therefore, we can define μ_R, the *harmonic modulus* of A with respect to α, as the directed limit

$$\mu_R = \lim_{\Omega \to R} \mu_\Omega . \tag{4}$$

It is again easy to see that $u_R = \lim_{\Omega \to R} u_\Omega$ exists and is continuous on A, harmonic on $A - \alpha$ with $u_R | \alpha = 0$. Moreover,

$$\log \mu_R = 1/\int_A du_R \wedge * du_R . \tag{5}$$

It can be seen that $R \in 0_G$ if and only if $\mu_R = \infty$ (Glasner [3]). Thus Theorem 1 may be considered as a special case of

THEOREM 2. *The following identity is valid*:

$$\lambda(\Gamma_\beta) = \log \mu_R . \tag{6}$$

The proof will be given in 5–9.

PARABOLIC CASE

5. A general inequality. We start with proving

$$\lambda(\Gamma_\beta) \geqq \log \mu_R . \tag{7}$$

Let Γ_{B_Ω} be the family of curves $\gamma \in \Gamma$ which lie in $\bar{\Omega} \cap A$ and terminate at a point of β_Ω. Define ρ as $(\log \mu_\Omega) | \nabla u_\Omega |$ in the interior of $\bar{\Omega} \cap A$ and as zero elsewhere in R. For $\gamma \in \Gamma_{\beta_\Omega}$,

$$\int_\gamma \rho \, ds = \int_\gamma (\log \mu_\Omega) \, | \nabla u_\Omega | \, ds \geqq (\log \mu_\Omega) \int_\gamma \frac{dh}{ds} \, ds = \log \mu_\Omega .$$

Therefore

$$L(\Gamma_{\beta_\Omega}, \rho) = \inf_{\Gamma} \int_\gamma \rho ds \geqq \log \mu_\Omega .$$

By (2) we also obtain

$$V(A, \rho) = \int_{\bar\Omega \cap A} (\log \mu_\Omega)^2 |\nabla u_\Omega|^2 dV = (\log \mu_\Omega)^2 \int_{\bar\Omega \cap A} du_\Omega \wedge * du_\Omega = \log \mu_\Omega ,$$

and infer by (1) that

(8) $$\lambda(\Gamma_{\beta_\Omega}) \geqq \log \mu_\Omega .$$

Since every $\gamma \in \Gamma_\beta$ contains a $\gamma' \in \Gamma_{\beta_\Omega}$, we can easily see that $\lambda(\Gamma_\beta) \geqq \gamma(\Gamma_{\beta_\Omega})$ (cf. Ahlfors-Sario [1, p. 222]). Thus (8) implies that $\lambda(\Gamma_\beta) \geqq \log \mu_\Omega$ for every Ω. On letting $\Omega \to R$ we obtain (7).

6. Now suppose that $R \in 0_G$. Then since $\mu_R = \infty$, (7) implies that

(9) $$\lambda(\Gamma_\beta) = \log \mu_R = \infty .$$

In order to complete the proofs of Theorems 1 and 2, we have only to show the validity of (6) under the assumption $R \notin 0_G$. Note that in our discussion thus far we have not made any use of the regularity condition.

Hyperbolic case

7. u-lines. Hereafter we assume that $R \notin 0_G$. Then u_R, to be denoted simply by u, is not constant on A. Since $u | \alpha = 0$ and $u | A - \alpha > 0$, we infer that $|\nabla u|$ can be extended continuously to all of A and that $|\nabla u| | \alpha \neq 0$.

For each $x \in \alpha$ we consider the unique curve l_x issuing from x and such that $l_x - x \subset A - \alpha$, $* du = 0$ on l_x, $|\nabla u| \neq 0$ on l_x. Moreover we require that l_x either terminates at β or at a point of A at which $|\nabla u| = 0$. Such an l_x will be called a u-line. As y traces l_x, $u(y)$ increases. Thus we can classify points of α as follows:

$$\alpha_0 = \{x \in \alpha \mid \lim_{y \to \beta, y \in l_x} u(y) < 1\} ,$$

$$\alpha_1 = \{x \in \alpha \mid \lim_{y \to \beta, y \in l_x} u(y) = 1\} ,$$

with

(10) $$\alpha = \alpha_0 \cup \alpha_1 .$$

8. Vanishing surface area. We denote by dS the surface element of α. We wish to show that

$$S(\alpha_0) = \int_{\alpha_0} dS = 0 . \tag{11}$$

Let F_{-1} be the set of points $x \in \alpha$ such that l_x terminates at some point of R. Clearly $F_{-1} \subset \alpha_0$, and we set $F_0 = \alpha_0 - F_{-1}$. By the regularity condition in §1, we see that $S(F_{-1}) = 0$ (cf. Brelot-Choquet [2]). Therefore we only have to show that $S(F_0) = 0$. Let

$$F_n = \left\{x \in F_0 \,\middle|\, \lim_{y\to\beta, y\in l_x} (1 - u(y)) \geqq \frac{1}{n}\right\} \qquad (n = 1,2,\cdots) .$$

Since $F_0 = \bigcup_1^\infty F_n$, it is sufficient to show that $S(F_n) = 0$.

We can find a positive harmonic function ω in the interior of A with the following properties (cf. Nakai [4]): (a) ω has the boundary values 0 on α, (b) $\lim_{y\to\beta, y\in l_x} \omega(y) = \infty$ for $x \in F_0$, (c) $\int_A |\nabla\omega_c|^2 dV \leqq c$, with $\omega_c = \min(\omega, c)$ for every positive number c.

Fix a $c > 0$ arbitrarily and a point $y_x \in l_x$ with $w_c(y_x) = c$ for each $x \in F_n$.

Set $v = 1 - u$ on A. In a neighborhood of a point in α with respect to A we may incorporate v into a coordinate system, say $v = x^1$, while $x^2, \cdots, x^m$ are $m - 1$ linearly independent parameters for α. Then

$$|\Delta v|^2 = g^{11}\left(\frac{\partial v}{\partial x^1}\right)^2 = g^{11} .$$

Since $*dv = |\nabla v|\, dS = \sqrt{g^{11}}\, dS$ on α, $S(F_n) = 0$ is equivalent to $\int_{F_n} *dv = 0$. Observe that

$$\begin{aligned} c\int_{F_n} *dv &\leqq \int_{F_n}\left(\int_x^{y_x} \frac{\partial\omega_c}{\partial v}\, dv\right) * dv \\ &= \int_{F_n}\int_x^{y_x} \left|\frac{\partial\omega_c}{\partial v}\right| dv \wedge *dv \\ &= \int_{F_n}\int_x^{y_x} \left|\frac{\partial\omega_c}{\partial x^1}\right| g^{11} dV. \end{aligned}$$

By the Schwarz inequality we have

$$\int_{F_n}\int_x^{y_x} \left|\frac{\partial\omega_c}{\partial x^1}\right| g^{11} dV \leqq \left(\int_{F_n}\int_x^{y_x} \left|\frac{\partial\omega_c}{\partial x^1}\right|^2 g^{11} dV\right)^{1/2} \left(\int_{F_n}\int_x^{y_x} g^{11} dV\right)^{1/2}$$

$$\leqq \left(\int_A |\nabla \omega_c|^2 dV\right)^{1/2} \left(\int_A |\nabla v|^2 dV\right)^{1/2}$$
$$\leqq \sqrt{c}\left(\int_A du \wedge *du\right)^{1/2} .$$

From this we infer that

$$\left|\int_{F_n} *dv\right| \leqq 1/\sqrt{\mu_R c} .$$

Since the number c can be arbitrarily large, we have $\int_{F_n} *dv = 0$, and (11) follows.

9. Let ρ be a density with $\rho \not\equiv 0$ on A. Since

$$du \wedge *du = |\nabla u| dV ,$$

we can compute

$$V(A, \rho) = \int_A \rho^2 dV = \int_A \frac{\rho^2}{|\nabla u|^2} du \wedge *du$$
$$\geqq \int_{\alpha_1}\left(\int_{l_x} \frac{\rho^2}{|\nabla u|^2} du\right) * du$$
$$= \int_{\alpha_1}\left(\int_{l_x} \frac{\rho^2}{|\nabla u|^2} du \cdot \int_{l_x} 1^2 du\right) * du$$
$$\geqq \int_{\alpha_1}\left(\int_{l_x} \frac{\rho}{|\nabla u|} du\right)^2 * du .$$

On $l_x (x \in \alpha_1)$ we have $du = |\nabla u| ds$, and thus

$$V(A, \rho) \geqq \int_{\alpha_1}\left(\int_{l_x} \rho ds\right)^2 * du .$$

From $l_x \in \Gamma_\beta$ for $x \in \alpha_1$ we obtain $\int_{l_x} \rho ds \geqq L(\Gamma_\beta, \rho)$, and therefore

$$V(A, \rho) \geqq L(\Gamma_\beta, \rho)^2 \int_{\alpha_1} * du . \tag{12}$$

On the other hand, by (11), we have $\int_{\alpha_1} * du = \int_\alpha * du$. Take an arbitrary regular region Ω with $\beta_\Omega \subset A - \alpha$. Then

$$\int_\alpha * du = \lim_{\Omega \to R} \int_\alpha * du_\Omega .$$

Here we see that

$$\int_\alpha * du_\Omega = \int_{\beta_\Omega} * du_\Omega = \int_{\beta_\Omega - \alpha} u_\Omega * du_\Omega = \int_{\bar{\Omega} \cap A} du_\Omega \wedge * du_\Omega ,$$

and infer that

$$\int_\alpha * du = \lim_{\Omega \to R} \int_{\bar{\Omega} \cap A} du_\Omega \wedge * du_\Omega = \int_A du \wedge * du \ .$$

This together with (5) and (12) implies the inequality

$$\log \mu_R \geqq \frac{L(\Gamma_\beta, \rho)^2}{V(A, \rho)} \ .$$

Since ρ was arbitrary, we now conclude that

$$\log \mu_R \geqq \lambda(\Gamma_\beta) \ .$$

We combine this with (7) and obtain (6).

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