ORDER-PRESERVING FUNCTIONS; APPLICATIONS TO MAJORIZATION AND ORDER STATISTICS

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Let $\precsim$ be a partial ordering among the points of a set $D \subset R^n$. A real-valued function f defined on D is said to preserve $\precsim$ if $x, y \in D$, $x \precsim y$ implies $f(x) \leq f(y)$. The central theorem of this paper gives necessary and sufficient conditions for f to preserve $\precsim$ if $\precsim$ is a cone ordering, i.e. if there exists a convex cone C such that $x \precsim y$ if and only if $y - x \in C$. Corollaries to the theorem consider the case when f is differentiable and $\precsim$ is order isomorphic to a cone ordering under a differentiable mapping. It is seen that the ordering of majorization is a special case of a cone ordering and that a straightforward application of a corollary yields the results of Schur and Ostrowski on functions which preserve majorization. The corollaries are also applied to a partial ordering of positive semi-definite matrices and to certain partial orderings arising in the theory of order statistics.

In order to prove and extend the determinant inequality of Hadamard, I. Schur [22] defined a relation that is equivalent to the ordering of majorization [10, p. 49]. A real n-vector y is said to *majorize* the n-vector x (written $x \prec y$) if the components of x and y can be reordered so that

(i) $x_1 \geq x_2 \geq \cdots \geq x_n,\ y_1 \geq y_2 \geq \cdots \geq y_n$,

(ii) $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, $\quad k = 1, 2, \cdots, n-1$,

(iii) $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$.

Schur showed that a real-valued differentiable function f *preserves the ordering* $\prec$, i.e. $x \prec y$ implies $f(x) \leq f(y)$, if and only if f is symmetric and

(iv) $\dfrac{\partial f(x)}{\partial x_i} - \dfrac{\partial f(x)}{\partial x_{i+1}} \geq 0$, $\quad i = 1, 2, \cdots, n-1$,

for all x satisfying (i).[1] With the choice of particular functions f and the insight that the eigenvalues of a positive definite Hermitian matrix majorize its diagonal elements, he was able to obtain the determinant inequality of Hadamard as well as a number of related inequalities.

The importance of Schur's approach to the Hadamard inequality lies in the fact that his results and their refinements due to A. Ostrowski

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[1] Actually, Schur restricted his attention to the case that $x_i > 0$, $y_i > 0$; this restriction was removed by Ostrowski [19].

[19] can be applied to obtain a variety of inequalities wherever majorization arises. In matrix theory majorization also occurs in the comparison of eigenvalues with singular values [24], [11] and in the study of incidence matrices [21]. Majorization has also found applications in other contexts, e.g. in obtaining certain probabilistic inequalities [18], [20]. Other partial orderings occur in the study of condition numbers for matrices [16], order statistics [17], and positive definite matrices [14]. In each case the problem arises of determining necessary and sufficient conditions for a real-valued function f to be order preserving.

The purpose of this paper is to show how such conditions can be derived in general. Our approach is illustrated by the following simple interpretation of the ordering of majorization and Schur's condition. Vectors x and y satisfy (ii) and (iii) if and only if $y - x$ belongs to the convex cone C contained in the hyperplane $\sum_{i=1}^{n} z_i = 0$ and determined by the inequalities $\sum_{i=1}^{k} z_i \geq 0$, $k = 1, \cdots, n - 1$. Then (iv), as we shall see, is equivalent to the requirement that at each point x satisfying (i), the directional derivative of $f(x)$ is nonnegative in the direction of the edges of the convex cone C.

Section 2 is devoted to obtaining general conditions for a real-valued function to be order-preserving when the partial ordering is a cone ordering or is order isomorphic to a cone ordering. Section 3, by way of illustration, applies those results to the ordering of majorization and partial orderings found in [3], [14], and [17].

2. **Theory.** A relation $\precsim$ on a set D is a *partial ordering* if

$$x \precsim x \text{ for all } x \text{ in } D \quad \text{(reflexitivity)}$$
$$x \precsim y, \, y \precsim z \text{ implies } x \precsim z \quad \text{(transitivity)} .$$

Note that the condition

$$(1) \qquad x \precsim y, \, y \precsim x \text{ implies } x = y \quad \text{(antisymmetry)}$$

is *not* required.

A *convex cone* is a nonempty set $C \subset R^n$ such that $p, q \in C$ implies $\lambda_1 p + \lambda_2 q \in C$ for all $\lambda_1, \lambda_2 \geq 0$. A convex cone C is said to be *pointed* if $x \in C$, $-x \in C$ implies $x = 0$. A *cone ordering* on $D \subset R^n$ (*induced by* a convex cone C) is a relation $\precsim$ on D defined by:

$$(2) \qquad x \precsim y \text{ if and only if } y - x \in C .$$

Any cone ordering $\precsim$ on R^n is a partial ordering and satisfies

$$(3) \qquad x \precsim y \text{ implies } x + z \precsim y + z, \text{ for all } x, y, z \text{ in } R^n$$

$$(4) \qquad x \precsim y \text{ implies } \lambda x \precsim \lambda y, \text{ for all } x, y \text{ in } R^n \text{ and } \lambda \geq 0 .$$

Conversely, if $\precsim$ is a partial ordering on R^n satisfying (3) and (4), then $\precsim$ is the cone ordering induced by the convex cone $C = \{x : x \succsim 0\}$. Moreover, C is pointed if and only if $\precsim$ satisfies (1).

A set $S \subset C$ is said to *span* the convex cone C *positively* if every point of C can be expressed as a positive linear combination of a finite number of points in S. A set $T \subset C$ is called a *frame* of C, if T, but no proper subset of T, spans C positively. A convex cone which possesses a finite frame is called a *convex polyhedral cone*.

It can be shown (though we shall not specifically require this result) that every closed convex cone C in R^n possesses a frame, but not necessarily a finite one. In particular, any frame for C can be obtained by combining a (necessarily finite) frame for the maximal linear subspace L contained in C (see McKinney [15]) and a frame for the pointed cone $C \bmod L$. The existence of a frame for a pointed closed convex cone in R^n follows from an application of the Krein-Milman theorem to a bounded set obtained by intersecting the cone with an appropriate linear variety. Results along this line have been obtained by Klee [13].

A particularly simple but important cone ordering is the componentwise ordering $\leqq$ on R^n given by

$$x \leqq y \text{ if and only if } x_i \leqq y_i \text{ for } i = 1, \cdots, n ,$$

where R^n is interpreted as the space of column n-tuples and x_i denotes the ith component of x. It is readily checked that $\leqq$ is the cone ordering induced by the positive orthant $R_+^n = \{x : x \geqq 0\}$, and that the unit coordinate vectors e^i, $i = 1, \cdots, n$, constitute a frame for R_+^n.

In applications the cone C is often given in terms of m simultaneous linear inequalities, i.e. $C = \{x : Ax \geqq 0\}$ where A is an $m \times n$ matrix. According to a theorem of Weyl [8], [9], C is a convex polyhedral cone. In general, the task of computing a frame for C given the $m \times n$ matrix A is not easy; it is equivalent to determining the extreme points of a convex polytope given the equations of bounding hyperplanes, or vice versa. See for example [2] and [23]. However, the matter is simple when A is nonsingular.

PROPOSITION 1. *If the matrix A is nonsingular, the columns of A^{-1} constitute a frame for the convex cone $C = \{x : Ax \geqq 0\}$.*

Proof. The cone $C = \{x : Ax = y, \, y \geqq 0\}$ can be rewritten as

$$C = \{x : x = A^{-1}y, \, y \geqq 0\} .$$

Thus the columns of A^{-1} span C positively. But clearly no proper subset spans C.

Our fundamental result on cone orderings is contained in the following

THEOREM 2. *Suppose f is a real-valued function defined on a convex set $D \subset R^n$ with interior, f is continuous at boundary points of D and $\precsim$ is a cone ordering on D induced by the convex cone $C \subset R^n$ with frame T. Then f preserves $\precsim$ if and only if*

$$(5) \qquad \begin{aligned} &f(x + \lambda t) - f(x) \geqq 0, \quad \text{for all } x \in D \\ &\text{and all } t \in T \text{ and } \lambda > 0 \text{ such that } x + \lambda t \in D. \end{aligned}$$

Proof. "Only if" is obvious. Suppose therefore that (5) holds. Fix $x, y \in D$ such that $x \precsim y$. We must show $f(y) - f(x) \geqq 0$. Since $v = y - x \in C$, there are $t_1, \cdots, t_k$ in T and positive scalars $\mu_1, \cdots, \mu_k$ such that

$$v = \sum_{i=1}^{k} \mu_i t_i.$$

If $x, y \in \operatorname{int} D$ (the interior of D), then for suitable $\alpha > 0$ the cylindrical neighborhood (See Figure 1.)

$$E = \{z \in R^n : \|z - w\| < \alpha \text{ for some } w \text{ on the line segment } [x, y]\}$$

is contained in D, where $\|\cdot\|$ is the Euclidean norm on R^n. See Figure 1. It is now easy to see that there exists an integer N and a polyhedral path from x to y lying entirely in E and having $Nk + 1$ nodes, each of which is displaced from the preceding node by one of the vectors $(1/N)\mu_i t_i$, $i = 1, \cdots, k$.

By (5), f is nondecreasing along each arc of the path, hence $f(y) - f(x) \geqq 0$. Next suppose x and y are points of D on the boundary. Let $z \in \operatorname{int} D$, and for each $\delta \in (0, 1)$ define

$$x_\delta = (1 - \delta)x + \delta z, \quad y_\delta = (1 - \delta)y + \delta z.$$

Since f is continuous at x and y, for each $\varepsilon > 0$ there exists some δ such that $f(x_\delta) \geqq f(x) - \varepsilon$ and $f(y_\delta) \leqq f(y) + \varepsilon$. Moreover, $x_\delta \precsim y_\delta$ with $x_\delta, y_\delta \in \operatorname{int} D$. Hence by the argument above, $f(y_\delta) - f(x_\delta) \geqq 0$. It follows that $f(y) - f(x) \geqq -2\varepsilon$ for all $\varepsilon > 0$, thus $f(y) - f(x) \geqq 0$. If exactly one of the points x, y lies in $\operatorname{int} D$, essentially the same proof applies if z is taken to be the one in the interior.

The following theorem shows that the hypotheses on D in Theorem 2 are essentially the weakest that can be imposed without involving C.

THEOREM 3. *Suppose D is a nonempty closed subset of R^n such that, for every convex cone C with frame T and every continuous*
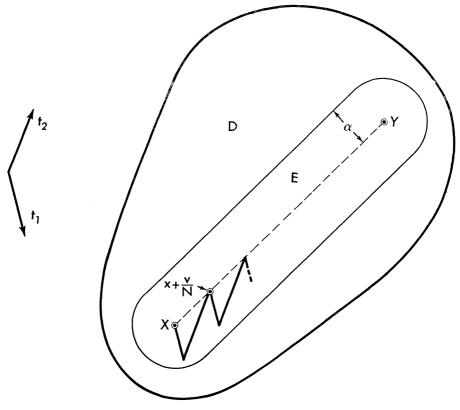
FIGURE 1.

function f on D, condition (5) is sufficient to insure that f preserves the cone ordering on D induced by C. Then D is a point or D is convex with nonempty interior.

Outline of proof. It suffices to show that a "counterexample" to Theorem 2 can be constructed whenever D has at least two points and is not convex with interior. We illustrate the construction for $n = 2$ when D is not convex; the generalization to dimension n involves no essential difficulties. Consider the closed subset D of R^2 shown shaded in Figure 2. Since D is not convex there exist points p and q on the boundary of D such that the open interval (p, q) is contained in the complement of D (which is open). Select p', q' sufficiently close to p and q respectively on the extension of the line segment $[p, q]$, and vectors t_1, t_2 sufficiently close to parallel to $q - p$. Let C be the convex cone spanned by $\{t_1, t_2\}$ and let $\precsim$ be the cone ordering on R^2 induced by C. If p', q', t_1, and t_2 are properly chosen, there exist distinct points p'', q'' in $[p, q]$ such that $p' \precsim p'' \precsim q'' \precsim q'$ and such that $x \in D$ and $p' \precsim x \precsim q'$ implies $p' \precsim x \precsim p''$ or $q'' \precsim x \precsim q'$, i.e. the open set E in Figure 2 is disjoint from D. A continuous function f

can be defined on the complement of E as follows: f is constant on the unstriped regions in Figure 2, with value 0 or 1 as indicated, and linearly interpolated on the striped regions. It can be verified that the restriction of f to D satisfies (5), but f is not order preserving since $p \precsim q$ and $f(p) > f(q)$.
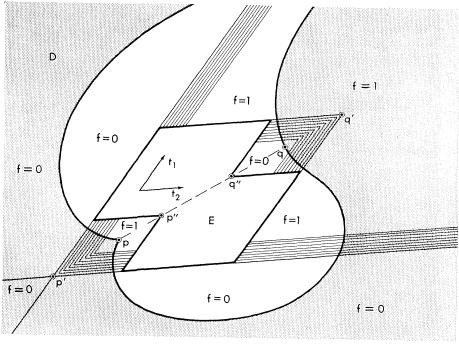

FIGURE 2.

It remains to be observed that if D is a convex set without interior, then *any* function f will satisfy (5) trivially if the convex cone C and its frame T are chosen so that no member of T is parallel to a line segment in D.

In application, it is generally more convenient to use a differential form of Theorem 2. For any function f defined on $D \subset R^n$, we adopt the convention that the gradient

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \cdots, \frac{\partial f(x)}{\partial x_n} \right)$$

at $x \in \operatorname{int} D$ is a row vector, since it is a point of the dual of the column vector space R^n. This convention will make transposition of vectors unnecessary when writing inner products, as in equation (6) below, where $\nabla f(x) \cdot t$ denotes the inner product of $\nabla f(x)$ and t.

COROLLARY 4. *Suppose f is a continuous real-valued function defined on a convex set $D \subset R^n$ with gradient $\nabla f(x)$ at each point in*

the nonempty interior of D, and $\precsim$ is a cone ordering on D induced by the convex cone $C \subset R^n$ with frame T. Then f preserves $\precsim$ if and only if

$$(6) \qquad\qquad \nabla f(x) \cdot t \geq 0$$

for all t in T and all x in the interior of D.[2]

Proof. It is clear from the proof that Theorem 2 remains true when condition (5) is modified to read "$\cdots$for all $x \in \text{int } D \cdots$". To say that $\nabla f(x)$ exists at $x \in \text{int } D$ is to say that the directional derivative $\lim_{\lambda \downarrow 0} [f(x + \lambda t) - f(x)]/\lambda$ can be expressed as the inner product $\nabla f(x) \cdot t$ (in fact, the convergence is uniform in t for t in a bounded neighborhood of the origin). The corollary now follows from the relationship between the directional derivative and monotonicity of f on line segments $[x, x + \lambda t]$, $t \in T$, $x + \lambda t \in D$.

We remark that Corollary 4 remains true if, in the last line, T is replaced by a dense subset of T or a dense subset of a frame T' obtained from T by normalization. A similar remark is valid for Theorem 2 provided f is required to be continuous.

It may happen that a partial ordering $\precsim$ on a set $D \subset R^n$ is not a cone ordering, but can be related to one. Specifically, suppose G is a one-to-one mapping of D onto $G(D) \subset R^n$ and that $\precsim_C$ is a cone ordering on $G(D)$ such that $x \precsim y$ if and only if $G(x) \precsim_C G(y)$. Such a G can be viewed as an *order isomorphism* between $(D, \precsim)$ and $(G(D), \precsim_C)$. It is clear that a function f defined on D preserves $\precsim$ if and only if the composite function $\phi = fG^{-1}$ preserves $\precsim_C$. Since G maps into R^n, it can be described in terms of its n real-valued *coordinate functions* g_i, where $g_i(x)$ is the ith coordinate of $G(x)$. We give some conditions on these g_i sufficient to insure that Corollary 4 and the chain rule of differentiation may be applied to fG^{-1}.

DEFINITION 5. Let $\mathscr{G}_n$ be the class of all mappings G such that
(i)   the domain $D \subset R^n$ of G is contained in the closure of its interior,
(ii) G is a homeomorphism of D onto $G(D) \subset R^n$,
(iii)   on $\text{int } D$, G is of class C^1 and the derivative matrix

$$J_G(x) = \left(\frac{\partial g_i(x)}{\partial x_j} \right)$$

is nonsingular.

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[2] Although, in the interest of simplicity, we avoid introducing the concept of the polar (dual) C^* of a convex cone C, we cannot help remarking that equation (6) is equivalent to the statement that $\nabla f(x)$ belongs to $C^* = \{z : z \cdot y \geq 0, \text{ for all } y \in C\}$.

PROPOSITION 6. *If* $G \in \mathscr{G}_n$, *then* $G(\text{int } D) = \text{int } G(D)$, $G^{-1} \in \mathscr{G}_n$, *and* $J_{G^{-1}}(G(x)) = [J_G(x)]^{-1}$ *for all* $x \in \text{int } D$.

Proof. From the Invariance of Domain Theorem [12, p. 95; or 1, p. 156] it follows that $G(\text{int } D) = \text{int } G(D)$. The rest of the proposition follows from the simpler results for the restrictions of G and G^{-1} to the open sets $\text{int } D$ and $\text{int } G(D)$ respectively. See for example [6, Sections 5.5, 5.7].

It is now obvious from Corollary 4 and Proposition 6 that:

COROLLARY 7. *Suppose* $G \in \mathscr{G}_n$ *has domain* D *and convex range* $G(D)$, $\precsim_C$ *is a cone ordering on* $G(D)$ *induced by the convex cone* C *with frame* T, *and* $\precsim$ *is a partial ordering on* D *such that* $x \precsim y$ *if and only if* $G(x) \precsim_C G(y)$. *If* f *is a continuous real-valued function defined on* D *with gradient* ∇f *on the interior of* D, *then* f *preserves* $\precsim$ *if and only if*

$$(7) \qquad\qquad \nabla f(x) \cdot [J_G(x)]^{-1} \cdot t \geqq 0$$

for all t *in* T *and all* x *in the interior of* D.

In Corollary 7, G may be interpreted as a reparametrization of D which converts $\precsim$ into a cone ordering. A somewhat different interpretation is possible when the coordinate functions g_i are used to define the partial ordering $\precsim$ on D as in

COROLLARY 8. *Suppose* $\precsim$ *is a partial ordering on a set* $D \subset R^n$ *given by*

$$x \precsim y \text{ if and only if:} \quad \begin{array}{l} g_i(x) \leqq g_i(y), 1 \leqq i \leqq k \\ g_i(x) = g_i(y), k < i \leqq n - l \, . \end{array}$$

Suppose further that the mapping G *whose coordinate functions are* $g_1, \cdots, g_n$ *belongs to* $\mathscr{G}_n$ *and has convex range* $G(D) \subset R^n$. *If* f *is a continuous real-valued function defined on* D *with gradient* ∇f *on the interior of* D, *then* f *preserves* $\precsim$ *if and only if the first* k *components of the vector* $\nabla f \cdot [J_G(x)]^{-1}$ *are nonnegative and the last* l *components are zero for all* x *in the interior of* D.

Proof. Corollary 7 applies with

$$C = \{z : z_i \geqq 0, 1 \leqq i \leqq k, \text{ and } z_i = 0, k < i \leqq n - l\} \, .$$

Thus it suffices to observe that the vectors

$$e^i, \, 1 \leqq i \leqq k \,,$$
$$e^i, \, n - l < i \leqq n \,,$$
$$-e^i, \, n - l < i \leqq n \,,$$

constitute a frame for the cone C.

3. Applications. We apply the results of §2 to four examples of vector partial orderings, together with certain minor variations. The first example is the ordering of majorization already investigated by Schur [22], Ostrowski [19], and others. The second two examples arise in the theory of order statistics and here some of the results are new. The fourth example is the matrix ordering induced by the cone of positive semi-definite matrices, which has been studied by Loewner [14].

In all cases we confine ourselves to the characterization of order preserving functions which are continuous on a set D and have a gradient on the interior of D. This must be understood, although to avoid repetitions we do not continue to mention it.

EXAMPLE 1. In the notation of §2, let

$$D = \{x \in R^n : x_1 \geqq x_2 \geqq \cdots \geqq x_n\} \,.$$

On D, define

Order 1a. *$x \precsim_{1a} y$ if and only if*

$$\sum_{i=1}^{k} x_i \leqq \sum_{i=1}^{k} y_i, \, k = 1, 2, \cdots, n, \, x, y \in D \,.$$

This is a cone ordering since $x \precsim_{1a} y$ if and only if $y - x \in C = \{z : Az \geqq 0\}$ where $A = [a_{ij}]$ is an $n \times n$ lower triangular matrix with $a_{ij} = 1$ for $i \geqq j$ and $a_{ij} = 0$ for $i < j$. By Proposition 1 and Corollary 4, f preserves $\precsim_{1a}$ if and only if $\nabla f(x) A^{-1} \geqq 0$ for $x \in \text{int } D$. Since $A^{-1} = [\alpha_{ij}]$, where $\alpha_{ij} = 1$ if $i = j$, $\alpha_{ij} = -1$ if $i = j + 1$ and $\alpha_{ij} = 0$ otherwise, we have

Condition 1a. *f preserves $\precsim_{1a}$ if and only if*

$$\frac{\partial f(x)}{\partial x_1} \geqq \frac{\partial f(x)}{\partial x_2} \geqq \cdots \geqq \frac{\partial f(x)}{\partial x_n} \geqq 0$$

for all x in the interior of D.

In the literature more attention has been devoted to a modification of $\precsim_{1a}$, defined on the same set D, namely:

Order 1b. $x \precsim_{1b} y$ *if and only if*

$$\sum_{i=1}^{k} x_i \leqq \sum_{i=1}^{k} y_i, \ k = 1, 2, \cdots, n - 1, \ and \ \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i, \ x, y \in D \ .$$

We apply Corollary 8 with $k = n - 1$, $l = 0$, and $G(x) = Ax$ where A is the same as for order 1a. Since $[J_G(x)]^{-1} = A^{-1}$ it follows that

Condition 1b. *f preserves $\precsim_{1b}$ if and only if*

$$(8) \qquad \frac{\partial f(x)}{\partial x_1} \geqq \frac{\partial f(x)}{\partial x_2} \geqq \cdots \geqq \frac{\partial f(x)}{\partial x_n}$$

for all x in the interior of D.

Any partial ordering $\precsim$ on $D = \{x : x_1 \geqq x_2 \geqq \cdots \geqq x_n\}$ can be extended to a partial ordering $\precsim^\dagger$ on R^n by defining $x \prec^\dagger y$ if $x^\dagger \prec y^\dagger$, where $x^\dagger$ is the vector obtained from x by rearranging the components in decreasing order. We note the obvious result:

PROPOSITION 9. If $\precsim$ is a partial ordering on

$$D = \{x : x_1 \geqq x_2 \geqq \cdots \geqq x_n\}$$

then a function f preserves $\precsim^\dagger$ if and only if f is symmetric and preserves $\precsim$.

Thus, f preserves $\precsim_{1a}^\dagger$ (or $\precsim_{1b}^\dagger$) if f is symmetric and satisfies condition 1a (or condition 1b). This result was obtained by Ostrowski [19]. The partial ordering $\precsim_{1a}^\dagger$ has also been considered by K. Fan [7] who showed that $\phi(x) \leqq \phi(y)$ for all symmetric gauge functions ϕ if and only if $x \precsim_{1a}^\dagger y$. The ordering $\precsim_{1b}^\dagger$ (i.e., majorization) has been discussed by Hardy, Littlewood and Pólya [10, pp. 44-49, p. 89], and by Beckenbach and Bellman [4, pp. 30-32]. The reader of [4] should note that §29, p. 31 applies to the partial ordering $\precsim_{1b}^\dagger$ and §31, p. 32 applies to $\precsim_{1b}$.

EXAMPLE 2. Let $D_+ = \{x \in R^n : x_1 \geqq x_2 \geqq \cdots \geqq x_n > 0\}$. We define

Order 2a. $x \precsim_{2a} y$ *if and only if*

$$\frac{x_i}{x_{i+1}} \leqq \frac{y_i}{y_{i+1}}, \ i = 1, \cdots, n - 1, \ x, y \in D_+ \ .$$

A geometric interpretation of $\precsim_{2a}$ can be given. If $x, y \in \text{int} \, D_+$, then $x \precsim_{2a} y$ if and only if there exists a starshaped function ϕ such that $y_i = \phi(x_i)$. A function ϕ defined on $(0, \infty)$ is *starshaped* if the line segment joining a point on the graph of ϕ to the origin lies entirely

on or above the graph.

The partial ordering $\precsim_{2a}$ has arisen in the theory of reliability [17]. In [5] a justification is given for saying that a component "wears out" if the distribution F of its time to failure is starshaped with respect to the exponential distribution H, i.e. if $H^{-1}F$ is starshaped. Order statistics $X_1 > X_2 > \cdots > X_n > 0$ and $Y_1 > Y_2 > \cdots > Y_n > 0$ from such distributions F and H, respectively, have the property that X_i/X_{i+1} is stochastically less then Y_i/Y_{i+1}. The functions preserving $\precsim_{2a}$ are of particular interest, since they are exactly those yielding tests of the hypothesis of no wear-out which are unbiased at all levels of significance. Moreover, it has been shown in [3] that $(EX_1, EX_2, \cdots, EX_n) \precsim_{2a} (EY_1, EY_2, \cdots, EY_n)$.

In order to apply Corollary 8 to $\precsim_{2a}$, we take $g_i(x) = x_i/x_{i+1}$ for $i = 1, \cdots, n-1$. There is some latitude in choosing g_n. A convenient choice is $g_n(x) = x_n$. The map G whose coordinate functions are $g_1, \cdots, g_n$ takes D_+ homeomorphically onto the convex set

$$\{z : z_i \geqq 1, \, i = 1, \cdots, n-1, \, z_n > 0\} \, ,$$

which has interior. The inverse of G is given by $x_k = \prod_{i=k}^{n} g_i(x)$. It is readily computed that $[J_G(x)]^{-1} = J_{G^{-1}}(G(x)) = [a_{ij}]$, where $a_{ij} = x_i x_{j+1}/x_j$ for $1 \leqq i \leqq j < n$, $a_{ij} = 0$ for $1 \leqq j < i \leqq n$, and $a_{in} = x_i/x_n$ for $1 \leqq i \leqq n$. From Corollary 8 with $k = n-1$, and $l = 1$ we have

Condition 2a. f preserves $\precsim_{2a}$ if and only if

$$(9) \qquad \sum_{i=1}^{j} x_i \frac{\partial f(x)}{\partial x_i} \geqq 0 \, , \qquad j = 1, \cdots, n-1$$

$$(10) \qquad \sum_{i=1}^{n} x_i \frac{\partial f(x)}{\partial x_i} = 0$$

for all $x \in \operatorname{int} D_+$.

Note that since $g_1, \cdots, g_{n-1}$ are homogeneous, any function f which preserves $\precsim_{2a}$ must satisfy $f(x) = f(\lambda x)$ for all $\lambda > 0$, or equivalently $[\partial f(\lambda x)/\partial \lambda]_{\lambda=1} = 0$, which is just (10).

If we define

Order 2b. $x \precsim_{2b} y$ *if and only if*

$$x \precsim_{2a} y \text{ and } x_n \leqq y_n, x, y \in D_+ \, ,$$

then by a minor variation of the application of Corollary 8 to $\precsim_{2a}$, we have

Condition 2b. f preserves $\precsim_{2b}$ *if and only if*

$$\sum_{i=1}^{j} x_i \frac{\partial f(x)}{\partial x_i} \geq 0 , \qquad j = 1, \cdots, n$$

for all $x \in \operatorname{int} D_+$.

This result can be obtained by an alternate method. Write $u_i = \ln x_i$ and $v_i = \ln y_i$ so that u and v range over

$$D = \{u : u_1 \geq u_2 \geq \cdots \geq u_n\} .$$

Then $x \precsim_{2b} y$ if and only if $u \precsim_{1c} v$, where

$u \precsim_{1c} v$ *if and only if*
$u_i - u_{i+1} \leq v_i - v_{i+1}, i = 1, \cdots, n - 1,$ and $u_n \leq v_n, u, v \in D .$

This is the cone ordering induced by the cone $C^* = \{x : (A^{-1})^T x \geq 0\}$, where A is the matrix used in defining the cone C which induces $\precsim_{1a}$. Hence by Proposition 1 and Corollary 4, f preserves $\precsim_{1c}$ if and only if $\nabla f(u) \cdot A^T \geq 0$, i.e.

$$\sum_{i=1}^{j} \frac{\partial f(u)}{\partial u_i} \geq 0 , \qquad j = 1, \cdots, n ,$$

for all $u \in \operatorname{int} D$. The change of variables yields Condition 2b. Observe that the members of the frame of C (the columns of A^{-1}) determine the bounding inequalities for C^*, and conversely, the members of the frame of C^* (the rows of A) determine the bounding inequalities for C. These facts reflect a duality between $\precsim_{1a}$ and $\precsim_{1c}$. A similar, though not so obvious, duality exists between $\precsim_{1b}$ (majorization) and $\precsim_{2a}$ (the ordering used in [17]) expressed as a cone ordering on the $u's$.

In [17] the following variant of $\precsim_{2a}$ is also considered.

Order 2c. $x \precsim_{2c} y$ *if and only if*

$$\frac{x_i}{x_{i+1}} \leq \frac{y_i}{y_{i+1}}, i = 1, \cdots, n - 1, \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i, x, y \in D_+ .$$

The map G whose coordinate functions are

$$g_i(x) = x_i/x_{i+1}, i = 1, \cdots, n - 1, g_n(x) = \sum_{1}^{n} x_i$$

is again a homeomorphism of D_+ onto the same convex set as in Example 2a. With somewhat more labor than before we find

$$[J_G(x)]^{-1} = \left[a_{ij} \Big/ \sum_{1}^{n} x_k \right] ,$$

where $a_{ij} = x_i x_{j+1} (\sum_{j+1}^{n} x_k)/x_j$ for $1 \leq i \leq j < n$, $a_{ij} = -x_i x_{j+1} (\sum_{1}^{j} x_k)/x_j$

for $1 \leq j < i \leq n$, and $a_{in} = x_i$ for $1 \leq i \leq n$.

From Corollary 8 with $k = n - 1$ and $l = 0$ we have

Condition 2c. *f preserves $\precsim_{2c}$ if and only if*

$$(11) \quad \left(\sum_{i=1}^{j} x_i \frac{\partial f(x)}{\partial x_i}\right)\left(\sum_{k=j+1}^{n} x_k\right) \geq \left(\sum_{i=j+1}^{n} x_i \frac{\partial f(x)}{\partial x_i}\right)\left(\sum_{k=1}^{j} x_k\right) \quad j = 1, \cdots, n - 1$$

for all $x \in \operatorname{int} D_+$.

We remark that the use of $g_n(x) = \sum_1^n x_i$ in computing Condition 2a would have led to the more involved set of conditions (11) and (10). However, it is possible to reduce (11) to (9) with the use of (10).

Suppose two partial orderings $\precsim_a$ and $\precsim_b$ of the type considered in this paper are defined on a set D, and $x \precsim_a y$ implies $x \precsim_b y$. Suppose further that comparatively simple conditions are known for a function to preserve $\precsim_b$. Then any function which satisfies these conditions preserves $\precsim_a$. This observation, coupled with the result given in Proposition 10 below, was used in [17, Th. 2.4] to show that a function satisfying (8) preserves $\precsim_{2c}$. We give here an alternate proof of Proposition 10 which illustrates an interesting application of Corollary 8.

PROPOSITION 10. *If* $x, y \in D_+$ *and* $x \precsim_{2c} y$, *then* y *majorizes* x, *i.e.* $x \precsim_{1b}^{\dagger} y$.

Proof. Since D_+ is contained in the domain of definition of $\precsim_{1b}$, it suffices to show that $\precsim_{2c}$ is preserved by each of the functions

$$f_k(x) = \sum_{i=1}^{k} x_i \qquad k = 1, \cdots, n - 1$$

$$f_n(x) = \sum_{i=1}^{n} x_i$$

$$f_{n+1}(x) = -\sum_{i=1}^{n} x_i ,$$

used in defining $\precsim_{1b}$. It is easily verified that these functions satisfy (11). In fact, f_n and f_{n+1} satisfy (11) with equality. This is a direct consequence of the fact $f_n(x) = f_n(y)$ is required for both $x \precsim_{2c} y$ and $x \precsim_{1b} y$.

EXAMPLE 3. Let $D_+^\circ = \{x : x_1 > x_2 > \cdots > x_n > 0\}$ and define

Order 3. *$x \precsim_3 y$ if and only if $x, y \in D_+^\circ$ and*

$$\frac{x_i - x_{i+1}}{x_{i+1} - x_{i+2}} \leqq \frac{y_i - y_{i+1}}{y_{i+1} - y_{i+2}}, \ i = 1, \cdots, n - 2 \ ,$$

and

$$\frac{x_{n-1} - x_n}{x_n} \leqq \frac{y_{n-1} - y_n}{y_n} \ .$$

Just as $\precsim_{2a}$ arises from the study of distribution functions F which are starshaped with respect to a distribution H, $\precsim_3$ arises from distributions F which are convex with respect to a distribution H, i.e. for which $H^{-1}F$ is convex. Observe that $x \precsim_3 y$ if and only if $y_i = \phi(x_i)$ for some nondecreasing convex function ϕ such that $\phi(0) = 0$. Thus $x \precsim_3 y$ implies $x \precsim_{2a} y$. We define the map G by

$$g_i(x) = \frac{x_i - x_{i+1}}{x_{i+1} - x_{i+2}}, \ i = 1, 2, \cdots, n \ ,$$

where we adopt the convention that $x_{n+1} = 0$ and $x_{n+2} = -1$. G maps D°_+ homeomorphically onto $\{x : x_i > 0, \ i = 1, \cdots, n\}$. We have $x_i = \sum_{k=i}^{n} \prod_{j=k}^{n} g_j(x)$ and $[J_G(x)]^{-1} = [a_{ij}]$, where

$$a_{ij} = (x_i - x_{j+1})(x_{j+1} - x_{j+2})/(x_j - x_{j+1})$$

for $j \geqq i$ and $a_{ij} = 0$ for $j < i$. By Corollary 8 we have

Condition 3. *f preserves* $\precsim_3$ *if and only if*

$$\sum_{i=1}^{j} (x_i - x_{j+1}) \frac{\partial f(x)}{\partial x_i} \geqq 0, \ j = 1, \cdots, n - 1 \ ,$$

$$\sum_{i=1}^{n} x_i \frac{\partial f(x)}{\partial x_i} = 0$$

for all x *in* D°_+.

EXAMPLE 4. We identify $D = R^{n(n+1)/2}$ with the set of all $n \times n$ symmetric matrices, and on D define

Order 4. $A \precsim_4 B$ *if and only if* $B - A$ *is positive semi-definite.*

This ordering, which was introduced by C. Loewner [14], is the cone ordering induced by the convex cone of positive semi-definite matrices. Though this cone does not have a finite frame, the set of all positive semi-definite matrices $M = [m_{ij}]$ of rank one, normalized by the condition trace $M = 1$, does constitute a frame. Such matrices can be written in the form $mm^T = [m_i m_j]$ where m is a unit column vector, and any positive semi-definite matrix A has the representation $A = \sum_{i=1}^{r} \alpha_i M_i$ where r is the rank of A and $\alpha_i > 0$.

We apply Corollary 4. Note that (6) can be written as

$$\sum_{i,j} \frac{\partial f(A)}{\partial a_{ij}} m_i m_j = m^T \left[\frac{\partial f(A)}{\partial a_{ij}} \right] m \geqq 0 \ .$$

Thus we obtain that

Condition 4. *f preserves $\precsim_4$ if and only if the gradient matrix* $[\partial f(A)/\partial a_{ij}]$ *is positive semi-definite.*

A related result has been obtained by Loewner [14], who considered the case that f is orthogonally invariant, i.e., $f(A) = f(\Gamma A \Gamma^T)$ for all orthogonal matrices Γ. He showed that such functions preserve $\precsim_4$ if and only if

$$\frac{\partial f(A)}{\partial \lambda_i} \geqq 0, \ i = 1, 2, \cdots, n$$

where λ_i are the eigenvalues of A.

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