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UNSTABLE POINTS IN THE HYPERSPACE OF CONNECTED SUBSETS

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A topological property which has proved useful is that of possessing an unstable point. It is thus interesting to see which topological spaces consist entirely of unstable points. The purpose of this paper is to describe a class of such spaces. This is done in the

THEOREM. If X is a finite simplicial complex then the hyperspace $C(X)$ consists entirely of unstable points if and only if X has no free 1-simplex.

The proof given here is for the case where X is connected —the more general theorem follows obviously from this case.

2. Definitions and remarks. A point p in a space Z is called *unstable* if for each open neighborhood U of p there is a homotopy $h_t: Z \to Z$ such that $h_0 = 1$, $p \notin h_1(Z)$, and for all t, $h_t|_{Z \setminus U} = 1$ and $h_t(U) \subset U$. (Here 1 denotes the identity mapping on Z.) A point which is not unstable is called *stable*. (Synonyms for "unstable" are "labile" and "homotopically labil"; see, for example, [2].)

If (X, d) is a compact metric space we denote by 2^X the collection of all nonempty, closed subsets of X. When furnished with the Hausdorff metric [3, p. 167] 2^X is called the *hyperspace* of closed subsets of X. The Hausdorff metric can be defined as follows: for $x \in X$ and $E \subset X$ we define $\text{dist}(x, E) = \inf\{d(x, y) \mid y \in E\}$. For $\varepsilon > 0$ we define $V_\varepsilon(E) = \{x \in X \mid \text{dist}(x, E) < \varepsilon\}$. Then the Hausdorff metric ρ on 2^X is given by $\rho(E, E') = \inf\{\varepsilon > 0 \mid E \subset V_\varepsilon(E') \text{ and } E' \subset V_\varepsilon(E)\}$. Note that in order to show $\rho(E, E') < \varepsilon$ it suffices to show the following: $\text{dist}(x, E') < \varepsilon$ for each $x \in E$ and $\text{dist}(x', E) < \varepsilon$ for each $x' \in E'$. The subspace of $(2^X, \rho)$ consisting of the nonempty, closed, connected subsets of X is denoted by $C(X)$. (We still use the metric ρ on $C(X)$.) It is well known that $C(X)$ and 2^X are compact.

A function $f: (X, d) \to (X, d)$ is called an ε-mapping of X if $d(x, f(x)) < \varepsilon$ for each x in X.

Finite simplicial complex denotes a geometric realization of an abstract finite simplicial complex, and has the Euclidean topology.

A 1-simplex is called *free* if it is not a proper face of some other simplex in the complex.

If (X, d) is a metric space then d is said to be a *convex metric* [1, p. 1101] if for each pair of points a, b in X there is a point z in X such that $d(a, z) = d(z, b) = 1/2\, d(a, b)$. In case (X, d) is complete this is equivalent to: for each pair of points a, b in X there is a set

$J \subset X$ which is isometric with the closed interval $[0; d(a, b)]$ under an isometry h such that $h(0) = a$ and $h(d(a, b)) = b$. (There may be more than one such set.) This last is, in turn, equivalent to: for each pair of points a, b in X and positive numbers α, β such that $\alpha + \beta = d(a, b)$ there is a point z in X such that $d(a, z) = \alpha$, $d(z, b) = \beta$.

The following two theorems will be used in the sequel:

THEOREM A. *If Y is a compact* ANR *and $p \in Y$ then p is unstable if for every $\varepsilon > 0$ there is a continuous ε-mapping $f: Y \to Y$ such that $p \notin f(Y)$. The proof is found in* [2].

THEOREM B. *If Y is a Peano continuum then there exists a convex metric for Y which is compatible with the topology on Y. The proof is found in* [1].

It is known [4] that when X is a Peano continuum, $C(X)$ is an AR. Thus, if X is a connected finite simplicial complex (and hence a Peano continuum), $C(X)$ is a compact ANR, and Theorem A applies—to show a point in $C(X)$ is unstable we need only construct the appropriate ε-mapping.

Note that stability is a local property: if U is a neighborhood of y in a space Y and V is a neighborhood of z in a space Z and f is a homeomorphism of U onto V such that $f(y) = z$ then y is unstable in Y if and only if z is unstable in Z.

3. Some lemmas.

LEMMA 1. *If X is a Peano continuum and if $g: X \to 2^X$ is continuous, then the function $f: 2^X \to 2^X$ defined by $fE = \bigcup\{g(x) \mid x \in E\}$ is continuous. Furthermore, if $\alpha > 0$ and $\rho(\{x\}, g(x)) < \alpha$ for each $x \in X$ then f is a α-mapping.*

Proof. (i) Each fE is closed: suppose $y \in \overline{fE}$. Pick

$$y_n \in fE \cap S_{1/n}(y)\ , \qquad n = 1, 2, 3, \cdots$$

Then a sequence $\{x_n\}$ can be found in E so that $y_n \in g(x_n)$, $n = 1, 2, 3, \cdots$. Because E is compact we may assume the sequence $\{x_n\}$ has a limit x in E. Then $g(x_n)$ has limit $g(x) \subset fE$. We show $y \in g(x)$. Suppose the contrary. Then for some $\delta > 0$ we have $\operatorname{dist}(y, g(x)) \geqq 2\delta$. But since $g(x_n) \to g(x)$ we have $g(x_n) \subset V_\delta(g(x))$ for sufficiently large n. Thus, for sufficiently large n, $\operatorname{dist}(y, g(x_n)) \geqq \operatorname{dist}(y, V_\delta(g(x))) \geqq \delta$. But $\operatorname{dist}(y, g(x_n)) \leqq d(y, y_n)$ for all n. Since $d(y, y_n) \to 0$ we have $\operatorname{dist}(y, g(x_n)) \to 0$, a contradiction.

(ii) f is continuous: Let $\varepsilon > 0$. Choose $\delta > 0$ so that $\rho(g(x), g(y)) < \varepsilon$ when $d(x, y) < \delta$. Suppose $\rho(E, E') < \delta$. If $y \in fE$ then $y \in g(x)$ for some $x \in E$. Because $\rho(E, E') < \delta$ there is a point $x' \in E'$ with $d(x, x') < \delta$. Then $\rho(g(x), g(x')) < \varepsilon$, so $\operatorname{dist}(y, g(x')) < \varepsilon$. Thus there is a point $y' \in g(x')$ with $d(y, y') < \varepsilon$. But $y' \in fE'$, so we have $\operatorname{dist}(y, fE') < \varepsilon$. An entirely similar argument shows that $y' \in fE'$ implies $\operatorname{dist}(y', fE) < \varepsilon$. By a remark above we then have $\rho(fE, fE') < \varepsilon$. Thus f is continuous.

(iii) f is an α-mapping: we must show $\rho(E, fE) < \alpha$ for each E in 2^X. If $x \in E$ then, since $\rho(\{x\}, g(x)) < \alpha$, we have $\operatorname{dist}(x, g(x)) < \alpha$. Then $d(x, y) < \alpha$ for some $y \in g(x)$. Since $g(x) \subset fE$, $\operatorname{dist}(x, fE) < \alpha$. Now suppose $y \in fE$. Then $y \in g(x)$ for some $x \in E$. Because $\rho(\{x\}, g(x)) < \alpha$ we have $d(x, y) < \alpha$ so $\operatorname{dist}(y, E) < \alpha$. Thus $\rho(E, fE) < \alpha$.

LEMMA 2. *If X is a Peano continuum and if $g: X \to C(X)$ is continuous, then the function $f: C(X) \to C(X)$ defined by $fE = \bigcup\{g(x) \mid x \in E\}$ is continuous. Furthermore, if $\rho(\{x\}, g(x)) < \alpha$ for each $x \in X$ then f is an α-mapping.*

Proof. We need only show that fE is connected for each $E \in C(X)$. Let $E \in C(X)$ and suppose fE is not connected. Then we can write $fE = A_1 \cup A_2$ where A_1 and A_2 are nonempty, $A_1 \cap A_2 = \varnothing$ and A_1 and A_2 are closed in fE (and hence in X). Let $E_1 = \{x \in E \mid g(x) \subset A_1\}$, $E_2 = \{x \in E \mid g(x) \subset A_2\}$. Because $g(x)$ is connected for each x in E we must have either $g(x) \subset A_1$ or $g(x) \subset A_2$. Thus $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \varnothing$. Also, $E_1 \neq \varnothing$: if $E_1 = \varnothing$ then $E_2 = E$ and $fE \subset A_2$, a contradiction. Similarly, $E_2 \neq \varnothing$. Finally, continuity of g insures that E_1 and E_2 are closed. Thus E is not connected, contrary to the choice of E. Therefore f is continuous.

LEMMA 3. *If (X, d) is a compact metric space with convex metric d and if $\alpha > 0$ then the function f defined on $C(X)$ by $fE = \{x \in X \mid \operatorname{dist}(x, E) \leqq \alpha/2\}$ is an α-mapping of $C(X)$ into itself.*

(i) f is well-defined: it is clear that fE is closed. To see that fE is connected we merely observe that each point of fE is joined to E by a line segment lying in fE, and that E itself is connected.

(ii) f is continuous: we show that $\rho(fE, fE') < \beta$ whenever $\rho(E, E') < \beta$. Suppose $\rho(E, E') < \beta$. Let $y \in fE$. Because E is compact we can choose a point x in E such that $d(y, x) = \operatorname{dist}(y, E) \leqq \alpha/2$. Since $\rho(E, E') < \beta$ we can find a point $x' \in E'$ such that $d(x, x') < \beta$. Then $d(x', y) \leqq d(x', x) + d(x, y) < \beta + \alpha/2$. If $d(x', y) \leqq \alpha/2$ then $y \in fE'$, so $\operatorname{dist}(y, fE') = 0$. If $d(x', y) = \alpha/2 + \delta > \alpha/2$ then there is a point $y' \in X$ such that $d(x', y') = \alpha/2$ and $d(y', y) = \delta$. Then $y' \in fE'$

and (since $\delta < \beta$), $\operatorname{dist}(y, fE') \leqq d(y, y') < \beta$. Thus $y \in fE$ implies $\operatorname{dist}(y, fE') < \beta$. By interchanging E and E' in this argument we have that $y' \in fE'$ implies $\operatorname{dist}(y', fE) < \beta$. Thus $\rho(fE, fE') < \beta$.

(iii) f is an α-mapping: this follows immediately from the obvious inclusions valid for all $E \in C(X)$: $E \subset fE$ and $V_\alpha(E) \supset fE$.

LEMMA 4. *If X is the closed unit interval $[0; 1]$ then in $C(X)$ the stable points are precisely those intervals $[a; b]$ such that $0 < a < b < 1$.*

Proof. Let T denote the 2-simplex in the coordinate plane whose vertices are $(0, 0)$, $(1, 0)$, $(1/2, 1/2)$. Consider the map $g: C(X) \to T$ defined by $g([a; b]) = ((a + b)/2, (b - a)/2)$. Then g is a homeomorphism onto T.

Since the unstable points of T are precisely the boundary points of T, we see that the unstable points of $C(X)$ are just those that are mapped by g into these boundary points. Since for $(x, y) \in T$ we have $(x, y) \in$ boundary T if and only if (1) $y = 0$ or (2) $x = y$ or (3) $y = 1 - x$ we see that $[a; b]$ is unstable in $C(X)$ if and only if

(1) $\dfrac{b-a}{2} = 0$ (that is, $[a; b]$ is a point) or

(2) $\dfrac{a+b}{2} = \dfrac{b-a}{2}$ (that is, $a = 0$) or

(3) $\dfrac{b-a}{2} = 1 - \dfrac{a+b}{2}$ (that is, $b = 1$).

Thus $[a; b]$ is unstable if and only if $a = b$ or $a = 0$ or $b = 1$.

4. Proof of the theorem. As mentioned above, we assume X is connected. First suppose that X does not contain a free 1-simplex. Let $A \in C(X)$. The method of proof depends on whether or not A has empty interior, A^0.

If $A^0 = \varnothing$ we assume the metric on X is convex. Let $\varepsilon > 0$. By Lemma 3 the map $f: C(X) \to C(X)$ defined by

$$fE = \{x \in X \mid \operatorname{dist}(x, E) \leqq \varepsilon/2\}$$

is a continuous ε-mapping of $C(X)$. Also, $A \neq fE$ for each $E \in C(X)$ because each fE clearly has nonempty interior. Then, by Theorem A, A is an unstable point of $C(X)$.

If $A^0 \neq \varnothing$ then some point q in A has a neighborhood in X which is an open Euclidean n-ball for some $n \geqq 1$. Since X contains no free 1-simplex we must have $n \geqq 2$. Let B_1 be an open Euclidean n-ball neighborhood of q, centered at q, with radius $r_1 < \varepsilon/2$, and such that $B_1 \subset X$. Let B_2 be a second open n-ball, also centered at

q, and with radius $r_2 < r_1$. Let S_1 and S_2 be the $(n-1)$-spheres that are the boundaries of B_1 and B_2, respectively. Then $S_1 \cup \bar{B}_2$ is a closed subset of $\bar{B}_1$.

Because $n \geqq 2$, $\bar{B}_1 \backslash B_2$ is a Peano continuum, so $C(\bar{B}_1 \backslash B_2)$ is an AR. Define $g_1: S_1 \cup \bar{B}_2 \to C(\bar{B}_1 \backslash B_2)$ by

$$g_1(x) = \begin{cases} \{x\}, \text{ if } x \in S_1 . \\ S_2, \text{ if } x \in \bar{B}_2 . \end{cases}$$

Clearly, g_1 is continuous. Since $S_1 \cup \bar{B}_2$ is closed in $\bar{B}_1$ and $C(\bar{B}_1 \backslash B_2)$ is an AR, we have a continuous extension g_2 of g_1, $g_2: \bar{B}_1 \to C(\bar{B}_1 \backslash B_2)$. Now define $g: X \to C(X)$ by

$$g(x) = \begin{cases} \{x\}, \text{ if } x \in X \backslash B_1 \\ g_2(x), \text{ if } x \in \bar{B}_1 . \end{cases}$$

Then g is well-defined because $\bar{B}_1 \cap (X \backslash B_1) = S_1$, and for each $x \in S_1$, $g_2(x) = g_1(x) = \{x\}$. Clearly, g is continuous. Then by Lemma 2, the function $f: C(X) \to C(X)$ defined by $fE = \bigcup \{g(x) \mid x \in E\}$ is continuous. Also, $fE \neq A$ for each E in $C(X)$ because, while $q \in A$, the construction of g shows that $q \notin g(x)$ for each x in X, and hence $q \notin fE$ for each E in $C(X)$. Finally, f is an ε-mapping because g satisfies the second hypothesis of Lemma 2: for each $x \in X$ we have either $\rho(\{x\}, g(x)) = 0$ or $\rho(\{x\}, g(x)) = \rho(\{x\}, g_2(x))$. Since $g_2: \bar{B}_1 \to C(\bar{B}_1 \backslash B_2)$ and diameter $\bar{B}_1 < \varepsilon$, $\rho(\{x\}, g_2(x)) < \varepsilon$.

To complete the proof we suppose that X has a free 1-simplex S, and we exhibit a stable point of $C(X)$. Since S is free there is a point q which has an open neighborhood which is a Euclidean 1-ball B of radius α for some $\alpha > 0$. Furthermore we may assume that B is identical with the open metric ball about q of radius α; $B(q, \alpha)$, and that $0 < \alpha < 1$. Then $\overline{B(q, \alpha/2)}$ is a closed interval in S, which we denote by A. Then $B(A, \alpha/4)$ is a neighborhood (in $C(X)$) of A, and this neighborhood is homeomorphic to the open neighborhood in $C([0:1])$ of radius $\alpha/4$ about the closed interval $[1/4; 3/4]$ in $C([0:1])$ under a homeomorphism which carries $[1/4; 3/4]$ to A. Now by Lemma 4, $[1/4; 3/4]$ is a stable point of $C([0:1])$, so A is a stable point of $C(X)$.

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