PACIFIC JOURNAL OF MATHEMATICS
Vol. 23, No. 3, 1967

ON A THEOREM OF NIKODYM WITH APPLICATIONS TO WEAK CONVERGENCE AND VON NEUMANN ALGEBRAS

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The theorem of the title is a "striking improvement of the principle of uniform boundedness" in the space of countably additive measures on a sigma algebra. It says that if a set T of countably additive measures μ on a sigma algebra S is pointwise bounded: $\sup_{\mu \in T} |\mu(E)| < \infty, E \in S$, then it is uniformly bounded: $\sup_{\mu \in T}(\sup_{E \in S} |\mu(E)|) < \infty$.

Notice that the content of Nikodym's theorem is not changed by assuming that T is a countable set and recall that a countably additive complex valued measure on a sigma algebra is bounded and finitely additive.

An elementary example is given which illustrates that the theorem can not be extended to the case of bounded and countably additive measures on an algebra of sets. Next the theorem is extended, via a "sliding hump" argument, to the case where the measures are bounded and finitely additive on a sigma algebra. Then, after some remarks concerning weak convergence, the extended theorem is applied to extend recent results of Aarnes for normal functionals on a von Neumann algebra to the general case.

In order to set the notation, let us begin by stating our extension of Nikodym's theorem [4, Th. 8, p. 309-311].

THEOREM. *If $\{\mu_n\}$ is a sequence of bounded and finitely additive measures on a sigma algebra S of subsets of a set X such that for each element E of S $\sup_n |\mu_n(E)| < \infty$, then $\sup_n(\sup_{E \in S} |\mu_n(E)|) < \infty$.*

As mentioned above, before establishing the theorem let us consider the following example. Suppose X is the set of nonnegative integers and a subset E of X is in S if either E or X-E is a finite subset of the positive integers. Let $\{\mu_n\}$ be defined on S by $\mu_n(E) = n$ if E is a finite set containing n, $u_n(E) = 0$ if E is a finite set not containing n, and $\mu_n(E) = -\mu_n(X - E)$. Then $\{\mu_n\}$ is a sequence of bounded and countably additive measures on S satisfying (i) for each $E \in S$ $\lim_n |\mu_n(E)| = 0$, but also (ii)

$$\lim_n (\sup_{E \in S} \mu_n(E)) = \infty .$$

Proof of theorem. Suppose on the contrary that

$$\sup_n (\sup_{E \in S} |\mu_n(E)|) = \infty .$$

Observe that this supposition implies that if $p > 0$, then there exists a positive integer n and a partition (E, F) of X in S(i.e., (E, F) is a pair of pairwise disjoint elements of S such that $E \cup F = X$) such that $\min(|\mu_n(E)|, |\mu_n(F)|) > p$ because if $|\mu_n(E)| > \sup_k |\mu_k(X)| + p$, then $|\mu_n(F)| \geqq |\mu_n(E)| - |\mu_n(X)| > p$. Let n_1 be the least positive integer such that there exists a partition (E_1, F_1) of X in S satisfying $\min(|\mu_{n_1}(E_1)|, |\mu_{n_1}(F_1)| > 2$. At least one of $\sup_n(\sup_{E \in S} \mu_n(E \cap E_1))$ and $\sup_n(\sup_{E \in S} \mu_n(E \cap F_1))$ is infinite; suppose that the former sup is infinite. Now, letting E_1 play the role of X, let n_2 be the least integer greater than n_1 such that there exists a partition (E_2, F_2) of E_1 in S for which $\sup_n(\sup_{E \in S} \mu_n(E \cap E_2)) = \infty$ and

$$|\mu_{n_2}(F_2)| > \sup_n |\mu_n(F_1)| + 3 .$$

Iterating this process and relabeling if necessary $(n_k \rightarrow k)$, we obtain a sequence $\{F_k\}$ of pairwise disjoint elements of S such that

$$|\mu_k(F_k)| > \Sigma_{j<k} |\mu_k(F_j)| + k + 1 .$$

Decomposing, if you wish, $\{F_k\}_{k \geqq 2}$ into a sequence of subsequences, it becomes clear that since μ_1 is bounded there is a subsequence $\{F_{k_i}\}_{i \geqq 1}$ of $\{F_k\}_{k \leqq 2}$ verifying $\sup_{E \in S} |\mu_1(E \cap [\bigcup_i F_{k_i}])| < 1$. Repeating this process we obtain a subsequence $n_1 = 1, n_2 = k_1, \cdots$ satisfying

$$\sup_{E \in S} |\mu_{n_j}(E \cap [\bigcup_{i>j} F_{n_i}])| < 1 .$$

Finally, invoking now the hypothesis that S is a sigma algebra, let $G = \bigcup_i F_{n_i}$ and notice that the contradiction

$$|\mu_{n_i}(G)| \geqq |\mu_{n_i}(F_{n_i})| - \Sigma_{j<i} |\mu_{n_i}(F_{n_j})| - |\mu_{n_i}(\bigcup_{j>i} F_{n_j})| > n_i$$

obtains.

Toward a promised application, suppose that $\lim_n \mu_n(E)$ exists for $E \in S$. While existence of these limits is a necessary condition for the sequence $\{\mu_n\}$ to converge weakly, our example shows that if S is merely an algebra of sets, then the sequence $\{\mu_n\}$ need not be bounded and, hence, certainly not weakly convergent. Nevertheless, suppose S is a sigma algebra and $\mu(E) = \lim_n \mu_n(E)$. Then by the Theorem μ is a bounded and finitely additive measure on S and if $\nu_n = \mu_n - \mu$, then $\{\nu_n\}$ is a sequence of bounded and finitely additive measures on S satisfying $\lim_n \nu_n(E) = 0, E \in S$. We shall show that the sequence $\{\nu_n\}$ converges weakly to zero. To this end it is sufficient in view of [6, Th. 3.1] (see also [3]) to show that if $\{F_k\}$ is a sequence of pairwise disjoint elements of S, then $\lim_n(\Sigma_k \nu_n(F_k)) = 0$ which is an easy consequence of [5, Lemma 3.3] (identity F_k with k). The following corollary is thus established.

COROLLARY. *A sequence* $\{\mu_n\}$ *of bounded and finitely additive measures on a sigma algebra* S *is weakly convergent if, and only if,* $\lim_n \mu_n(E)$ *exists,* $E \in S$.

REMARK. Alternately, a very short proof of our theorem can be given using weak convergence theory and Phillips' Lemma as in our proof of the corollary: Suppose $\sup_n |\mu_n(E)| < \infty$, $E \in S$, and

$$\sup_n (\sup_{E \in S} |\mu_n(E)|) = \infty .$$

Then there exists a subsequence $\{\mu_{n_k}\}$ such that if $\lambda_k = 1/k\, \mu_{n_k}$, then $\{\lambda_k\}$ satisfies $\lim_k |\lambda_k(E)| = 0$, $E \in S$, (which implies weak convergence and, hence, boundedness) and the contradictory condition

$$\lim_k (\sup_{E \in S} |\lambda_k(E)|) = \infty .$$

It should also be noted that Andô [2] recognized the role of Phillips' Lemma and pointwise convergence in the general case.

Turning now to Aarnes results, we refer the reader to [1] for a discussion of the setting and restrict our attention to a sketch of the technical modifications which are necessary to establish the following theorems.

THEOREM 1′. *If* $\mathfrak{F}$ *is a family of functionals on a von Neumann algebra* $\mathfrak{A}$, *which is pointwise bounded on the projections in* $\mathfrak{A}$, *then* $\mathfrak{F}$ *is uniformly bounded on bounded sets of* $\mathfrak{A}$.

Proof. Let A be a self-adjoint operator in a von Neumann algebra $\mathfrak{A}$, and let $\mathfrak{B}$ be the commutative von Neumann sub-algebra of $\mathfrak{A}$ it generates. Suppose now that $\mathfrak{F}$ is a family of linear functionals on $\mathfrak{A}$ which is pointwise bounded on the projections in $\mathfrak{A}$. A fortiori $\mathfrak{F}$ is then pointwise bounded on the projections in $\mathfrak{B}$.

By the representation of $\mathfrak{B}$ as $L_C^\infty(S, \mu)$ for some S and μ, this transfers to the statement that for each measurable set $E \subseteqq S$ there is a constant $K(E) < \infty$ such that

$$|\widetilde{\varphi}(E)| < K(E)$$

for all $\widetilde{\varphi} \in ba(S, \Sigma_1, \mu_1)$ corresponding to members of $\mathfrak{F}$[4, Ch IV, 8.16, p. 296]. Then it follows, by our Theorem that we can find a constant $K < \infty$ such that

$$|\widetilde{\varphi}(E)| < K$$

for all measurable sets E in S and the same class of functions $\{\widetilde{\varphi}\}$ It immediately follows that the norms of the elements of $\{\widetilde{\varphi}\}$ must be uniformly bounded. Hence, by the isometric character of the map $\varphi \rightarrow \tilde{\varphi}$ we obtain in particular that the set $\{\varphi(A); \varphi \in \mathfrak{F}\}$ is bounded. But then, by the Banach-Steinhaus theorem and the fact that every operator in $\mathfrak{A}$ can be written as the linear sum of two self-adjoint operators, it follows that $\mathfrak{F}$ is uniformly bounded on bounded sets in $\mathfrak{A}$.

In order to make our concluding remarks comprehensible, let's first state Aarnes' Theorem 2.

THEOREM 2. *Let $\{\varphi_n\}_{n \in N}$ be a sequence of normal linear functionals on $\mathfrak{A}$, and suppose that for every projection $P \in \mathfrak{A}$, $\lim_{n\to\infty} \varphi_n(P)$ exists as a finite complex number, which we denote by $\varphi(P)$. Then:*

(i) *φ has a unique extension to all of $\mathfrak{A}$ as an element of $\mathfrak{A}^*$, and* lim *$\varphi_n(A)$ exists and is equal to $\varphi(A)$ for every $A \in \mathfrak{A}$.*

(ii) *φ is completely additive, and consequently normal.*

(iii) *The restrictions $\{\varphi_n \mid \mathfrak{P} \cap \mathfrak{B}\}_{n \in N}$ is equicontinuous in* 0 *with respect to the relativized weak operator topology on any commutative von Neumann sub-algebra $\mathfrak{B} \subseteq \mathfrak{A}$.*

(iv) *The family $\{\varphi_n\}_{n \in N}$ is uniformly completely additive.*

Because the measures $\tilde{\varphi}_n$ need not be countably additive unless the functionals φ_n are normal (see discussion in [1]), (ii) and (iv) can't be expected to carry over. However, the proof of (i) as given in [1] carries over if Theorem 1′ is used instead of Aarnes' Theorem 1 and (iii) follows from the weak convergence theory for finitely additive measures on a sigma algebra. Thus the following obtains.

THEOREM 2′. *Let $\{\varphi_n\}_{n \in N}$ be a sequence of linear functionals on $\mathfrak{A}$, and suppose that for every projection $P \in \mathfrak{A}$, $\lim_{n\to\infty} \varphi_n(P)$ exists as a finite complex number, which we denote by $\varphi(P)$. Then:*

(i) *φ has a unique extension to all of $\mathfrak{A}$ as an element of $\mathfrak{A}^*$, and* lim *$\varphi_n(A)$ exists and is equal to $\varphi(A)$ for every* A *$\in \mathfrak{A}$.*

(ii) *The restrictions $\{\varphi_n \mid \mathfrak{P} \cap \mathfrak{B}\}_{n \in N}$ is equicontinuous in* 0 *with respect to the relativized weak operator topology on any commutative von Neumann sub-algebra $\mathfrak{B} \subseteq \mathfrak{A}$.*

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Received March 22, 1967. The author is partially supported by a National Science Foundation grant.

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