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DIOPHANTINE SYSTEMS

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We concern ourselves in this paper with integral solutions of three Diophantine systems, generalizations of

$$x + y + z = u + v + w, xyz = uvw$$

and of $xy + xz + yz = uv + uw + vw, xyz = uvw$. **The solutions are given in terms of parameters and are integral for an integral choice of the parameters. Throughout the paper the integer** n **will be greater than 1.**

Heron [3] in the first century B. C. considered the problem of finding two rectangles such that the area of the first is three times the area of the second and the perimeter of the second is three times of perimeter of the first. He also considered a second problem which results in the Diophantine system $x + y = u + v, xy = 4uv$. Planude [3] discussed the system $x + y = u + v, xy = buv$, and Cantor [3] gave general solutions to this problem. Tannery [3] generalized the two problems of Heron. Moessner [7] and [8] gave particular solutions, while Dickson [4] and Gloden [6] gave parametric solutions of the system

$$\begin{aligned} x + y + z &= u + v + w \,, \\ xyz &= uvw \,, \end{aligned} \tag{1}$$

Bini [1] considered a system equivalent to (1) and Buquet [2] extended this system to $2n$ unknowns.

All of the above systems are special cases of the system

$$\begin{aligned} A(x, y) &= 0 \,, \\ cP(x) &= dP(y) \,, \end{aligned} \tag{2}$$

where $A(\alpha, \beta) = \sum_{i=1}^{n} (a_i\alpha_i - b_i\beta_i)$, $P(\alpha) = \prod_{i=1}^{n} \alpha_i$, a_i, b_i are integers, and c and d are nonzero integers. We make the following definitions:

$$A_p(\alpha, \beta) = A(\alpha, \beta) - (a_p\alpha_p - b_p\beta_p),$$

$P_p(\alpha) = P(\alpha)/\alpha_p$, $\pi_1(\alpha, \beta) = cb_pP_p(\alpha) - da_pP_p(\beta)$, p is a fixed integer, $1 \leqq p \leqq n$, and the α's and β's are arbitrary integers.

We agree that solutions in which some unknown vanishes, or those for which $a_ix_i = b_iy_i$, $(i = 1, \cdots, n)$ are trivial solutions.

THEOREM. 1. *Any nontrivial integral solution of* (2) *is proportional to a solution given by*

$$
(3) \qquad \begin{aligned}
x_i &= \pi_1(\alpha, \beta)\alpha_i , && i \neq p , \\
y_i &= \pi_1(\alpha, \beta)\beta_i , && i \neq p , \\
x_p &= dP_p(\beta)A_p(\alpha, \beta) , \\
y_p &= cP_p(\alpha)A_p(\alpha, \beta) .
\end{aligned}
$$

Proof. Since the solution is nontrivial there is an integer p, $1 \leqq p \leqq n$, such that $a_px_p \neq b_py_p$. If for $i \neq p$ we set

$$
(4) \qquad \begin{aligned}
x_i &= \pi_1(\alpha, \beta)\alpha_i , \\
y_i &= \pi_1(\alpha, \beta)\beta_i ,
\end{aligned}
$$

then (2) becomes

$$
\begin{aligned}
a_px_p - b_py_p &= - \pi_1(\alpha, \beta)A_p(\alpha, \beta) , \\
cP_p(\alpha)\pi_1^{n-1}(\alpha, \beta)x_p - dP_p(\beta)\pi_1^{n-1}(\alpha, \beta)y_p &= 0 .
\end{aligned}
$$

The solution of this system is

$$
(5) \qquad \begin{aligned}
x_p &= dP_p(\beta)A_p(\alpha, \beta) , \\
y_p &= cP_p(\alpha)A_p(\alpha, \beta) .
\end{aligned}
$$

From (4) and (5) it follows that (3) is a solution of (2).

Suppose now that $x_i = \lambda_i$, $y_i = \mu_i$ is a non trivial integral solution of (2). Then $A(\lambda, \mu) = 0$, $cP(\lambda) = dP(\mu)$, and $a_p\lambda_p \neq b_p\mu_p$. If in (3) we choose $\alpha_i = \lambda_i$, $\beta_i = \mu_i$ we get

$$
\begin{aligned}
x_i &= \pi_1(\lambda, \mu)\lambda_i , && (i = 1, \cdots, n) , \\
y_i &= \pi_1(\lambda, \mu)\mu_i , && (i = 1, \cdots, n) ,
\end{aligned}
$$

which is proportional to the solution $x_i = \lambda_i$, $y_i = \mu_i$, since $\pi_1(\lambda, \mu)$ is integral and $\pi_1(\lambda, \mu) = cP(\lambda)/\lambda_p\mu_p\,(b_p\mu_p - a_p\lambda_p) \neq 0$.

Dickson [5] has given solutions of the system

$$
(6) \qquad \begin{aligned}
xy + xz + yz &= uv + uw + vw , \\
xyz &= uvw .
\end{aligned}
$$

He [4] also indicates that this system may be solved by the same method he used to solve (1). Our second theorem generalizes (6).

We wish to solve the system

$$
(7) \qquad \begin{aligned}
\sum_{i=1}^{n} \left[\frac{a_iP(x)}{x_i} - \frac{b_iP(y)}{y_i} \right] &= 0 , \\
cP(x) &= dP(y) ,
\end{aligned}
$$

where a_i, b_i, c, d, $P(\alpha)$, $P_p(\alpha)$ are the same as in Theorem 1. We set

$$
P_{pi}(\alpha) = \frac{P(\alpha)}{\alpha_p\alpha_i}, \; B(x) = \sum_{i=1}^{n}{}' a_iP_{pi}(x), \; C(y) = \sum_{i=1}^{n}{}' b_iP_{pi}(y) ,
$$

where $\sum'$ indicates that the p^{th} term is omitted from the summations, $\pi_2(\alpha, \beta) = cP_p(\alpha)C(\beta) - dP_p(\beta)B(\alpha)$, p is a fixed integer, $1 \leqq p \leqq n$, and the α's and β's are arbitrary integers.

We agree that solutions in which some unknown vanishes or any solution for which $a_iP(x)/x_i = b_iP(y)/y_i$, $(i = 1, \cdots, n)$, are trivial solutions.

THEOREM[1] 2. *Any nontrivial integral solution of* (7) *is proportional to a solution given by*

$$\begin{aligned} x_i &= \pi_2(\alpha, \beta)\alpha_i , \qquad i \neq p , \\ y_i &= \pi_2(\alpha, \beta)\beta_i , \qquad i \neq p , \\ x_p &= dP_p(\beta)\,(a_pP_p(\alpha) - b_pP_p(\beta)) , \\ y_p &= cP_p(\alpha)\,(a_pP_p(\alpha) - b_pP_p(\beta)) . \end{aligned} \tag{8}$$

Proof. Since the solution is nontrivial there is an integer p, $1 \leqq p \leqq n$, such that $a_pP(x)/x_p \neq b_pP(y)/y_p$. If for $i \neq p$ we set

$$\begin{aligned} x_i &= \pi_2(\alpha, \beta)\alpha_i , \\ y_i &= \pi_2(\alpha, \beta)\beta_i , \end{aligned} \tag{9}$$

then (7) becomes

$$\pi_2^{n-2}(\alpha, \beta)B(\alpha)x_p - \pi_2^{n-2}(\alpha, \beta)C(\beta)y_p = \pi_2^{n-1}(\alpha, \beta)\,(b_pP_p(\beta) - a_pP_p(\alpha)) ,$$
$$c\pi_2^{n-1}(\alpha, \beta)P_p(\alpha)x_p - d\pi_2^{n-1}(\alpha, \beta)P_p(\beta)y_p = 0 .$$

The solution of this system is

$$\begin{aligned} x_p &= dP_p(\beta)\,(a_pP_p(\alpha) - b_pP_p(\beta)) , \\ y_p &= cP_p(\alpha)\,(a_pP_p(\alpha) - b_pP_p(\beta)) . \end{aligned} \tag{10}$$

It follows from (9) and (10) that (8) is a solution of (7).

Suppose now that $x_i = \lambda_i$, $y_i = \mu_i$ is a nontrivial integral solution of (7). Then $\lambda_p\beta(\lambda) + a_pP_p(\lambda) = \mu_pC(\mu) + b_pP_p(\mu)$, $cP(\lambda) = dP(\mu)$, and $a_pP(\lambda)/\lambda_p \neq b_pP(\mu)/\mu_p$. If in (8) we choose $\alpha_i = \lambda_i$, $\beta_i = \mu_i$ we obtain

$$\begin{aligned} x_i &= \pi_2(\lambda, \mu)\lambda_i , \qquad (i = 1, \cdots, n) , \\ y_i &= \pi_2(\lambda, \mu)\mu_i , \qquad (i = 1, \cdots, n) , \end{aligned}$$

which is proportional to the solution $x_i = \lambda_i$, $y_i = \mu_i$ since $\pi_2(\lambda,\mu)$ is integral and

$$\pi_2(\lambda, \mu) = \frac{cP(\lambda)}{\lambda_p\mu_p}\left[\frac{a_pP(\lambda)}{\lambda_p} - \frac{b_pP(\mu)}{\mu_p}\right] \neq 0 .$$

[1] This theorem also solves the problems of Heron and Planude.

The method of the two preceding theorems may be used to obtain solutions of the system

$$
\begin{aligned}
(11)\qquad & \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\frac{a_{ij}x_i}{x_j}-\frac{b_{ij}y_i}{y_j}\right]=0\ ,\\
& cP(x)=dP(y)\ ,
\end{aligned}
$$

where $c, d, P(\alpha)$ are defined above and the a_{ij}, b_{ij} are integers. We define

$$
\begin{aligned}
P_{ij}(\alpha) &= \frac{P(\alpha)}{\alpha_i\alpha_j}\ ,\\
D(\alpha, \beta) &= \sum_{j=2}^{n} a_{1j}P_{1j}(\alpha)P_1(\beta)\ ,\\
E(\alpha, \beta) &= \sum_{j=2}^{n} b_{1j}P_1(\alpha)P_{1j}(\beta)\ ,\\
F(\alpha, \beta) &= \sum_{i=2}^{n-1}\sum_{j=i+1}^{n}(a_{ij}\alpha_iP_{1j}(\alpha)P_1(\beta)-b_{ij}\beta_iP_1(\alpha)P_{1j}(\beta))\ ,\\
\pi_3(\alpha, \beta) &= cP_1(\alpha)E(\alpha, \beta)-dP_1(\beta)D(\alpha, \beta)\ ,\\
G(\alpha, \beta) &= \sum_{i=1}^{n-1} a_{in}\alpha_iP_n(\alpha)P_n(\beta)\ ,\\
H(\alpha, \beta) &= \sum_{i=1}^{n-1} b_{in}\beta_iP_n(\alpha)P_n(\beta)\ ,\\
I(\alpha, \beta) &= \sum_{j=2}^{n-1}\sum_{i=1}^{j-1}(a_{ij}\alpha_iP_{nj}(\alpha)P_n(\beta)-b_{ij}\beta_iP_n(\alpha)P_{nj}(\beta))\ ,\\
\pi_4(\alpha, \beta) &= cdP_n(\alpha)P_n(\beta)I(\alpha, \beta)\ .
\end{aligned}
$$

THEOREM 3. *Any nonzero integral solution of* (11) *which does not satisfy*

$$
(12)\qquad \sum_{j=2}^{n}\left[\frac{b_{1j}\mu_1}{\mu_j}-\frac{a_{1j}\lambda_1}{\lambda_j}\right]=0\ ,
$$

is proportional to a solution given by

$$
(13)\qquad
\begin{aligned}
x_i &= \pi_3(\alpha, \beta)\alpha_i\ , \qquad i\neq 1\ ,\\
y_i &= \pi_3(\alpha, \beta)\beta_i\ , \qquad i\neq 1\ ,\\
x_1 &= dP_1(\beta)F(\alpha, \beta)\ ,\\
y_1 &= cP_1(\alpha)F(\alpha, \beta)\ ,
\end{aligned}
$$

and any non-zero integral solution which does not satisfy

$$
(14)\qquad \sum_{i=1}^{n-1}\left[\frac{b_{in}\mu_i}{\mu_n}-\frac{a_{in}\lambda_i}{\lambda_n}\right]=0\ ,
$$

is proportional to a solution given by

$$(15)\qquad \begin{aligned} x_i &= \pi_4(\alpha, \beta)\alpha_i , \qquad i \neq n , \\ y_i &= \pi_4(\alpha, \beta)\beta_i , \qquad i \neq n , \\ x_n &= dP_n(\beta)\,(dP_n(\beta)H(\alpha, \beta) - cP_n(\alpha)G(\alpha, \beta)) , \\ y_n &= cP_n(\alpha)\,(dP_n(\beta)H(\alpha, \beta) - cP_n(\alpha)G(\alpha, \beta)) . \end{aligned}$$

Proof. If we multiply the first of the equations in (11) by $P_1(x)P_1(y)$ and for $i \neq 1$ set

$$(16)\qquad \begin{aligned} x_i &= \pi_3(\alpha, \beta)\alpha_i , \\ y_i &= \pi_3(\alpha, \beta)\beta_i , \end{aligned}$$

the system becomes

$$\begin{aligned} \pi_3^{2n-3}(\alpha, \beta)D(\alpha, \beta)x_1 - \pi_3^{2n-3}(\alpha, \beta)E(\alpha, \beta)y_1 &= -\pi_3^{2n-2}(\alpha, \beta)F(\alpha, \beta) , \\ c\pi_3^{n-1}(\alpha, \beta)P_1(\alpha)x_1 - d\pi_3^{n-1}(\alpha, \beta)P_1(\beta)y_1 &= 0 , \end{aligned}$$

which has as solution

$$(17)\qquad \begin{aligned} x_1 &= dP_1(\beta)F(\alpha, \beta) , \\ y_1 &= cP_1(\alpha)F(\alpha, \beta) . \end{aligned}$$

From (16) and (17) it follows that (13) is a solution of (11).

Suppose now that $x_i = \lambda_i, y_i = \mu_i$ is any nonzero integral solution of (11) which does not satisfy (12). If we choose $\alpha_i = \lambda_i, \beta_i = \mu_i$ then (13) becomes

$$\begin{aligned} x_i &= \pi_3(\lambda, \mu)\lambda_i , \qquad (i = 1, \cdots, n) , \\ y_i &= \pi_3(\lambda, \mu)\mu_i , \qquad (i = 1, \cdots, n) , \end{aligned}$$

which is proportional to the solution $x_i = \lambda_i, y_i = \mu_i$ since $\pi_3(\lambda, \mu)$ is integral and

$$\pi_3(\lambda, \mu) = \frac{cP(\lambda)P_1(\lambda)P_1(\mu)}{\lambda_1\mu_1} \sum_{j=2}^{n} \left[\frac{b_{1j}\mu_1}{\mu_j} - \frac{a_{1j}\lambda_1}{\lambda_j} \right] \neq 0 .$$

We may now write (11) as

$$(18)\qquad \begin{aligned} \sum_{j=2}^{n} \sum_{j=1}^{j-1} \left[\frac{a_{ij}x_i}{x_j} - \frac{b_{ij}y_i}{y_j} \right] &= 0 , \\ cP(x) &= dP(y) . \end{aligned}$$

If we multiply the first equation in (18) by $P_n(x)P_n(y)$ and for $i \neq n$ set

$$(19)\qquad \begin{aligned} x_i &= \pi_4(\alpha, \beta)\alpha_i , \\ y_i &= \pi_4(\alpha, \beta)\beta_i , \end{aligned}$$

the system becomes

$$\frac{1}{x_n}\pi_4^{2n-1}(\alpha,\beta)G(\alpha,\beta)-\frac{1}{y_n}\pi_4^{2n-1}(\alpha,\beta)H(\alpha,\beta)=-\pi_4^{2n-2}(\alpha,\beta)I(\alpha,\beta)\,,$$
$$\frac{1}{x_n}\pi_4^{n-1}(\alpha,\beta)dP_n(\beta)-\frac{1}{y_n}\pi_4^{n-1}(\alpha,\beta)cP_n(\alpha)=0$$

which has solution

$$\begin{aligned} x_n &= dP_n(\beta)\,(dP_n(\beta)H(\alpha,\beta)-cP_n(\alpha)G(\alpha,\beta))\,,\\ y_n &= cP_n(\alpha)\,(dP_n(\beta)H(\alpha,\beta)-cP_n(\alpha)G(\alpha,\beta))\,. \end{aligned} \tag{20}$$

It follows from (19) and (20) that (15) is a solution of (18) and hence of (11).

Suppose now that $x_i=\lambda_i$, $y_i=\mu_i$ is any nonzero integral solution of (18) which does not satisfy (14). If in (15) we choose $\alpha_i=\lambda_i$, $\beta_i=\mu_i$ we get

$$\begin{aligned} x_i &= \pi_4(\lambda,\mu)\lambda_i\,, \qquad (i=1,\cdots,n)\,,\\ y_i &= \pi_4(\lambda,\mu)\mu_i\,, \qquad (i=1,\cdots,n)\,, \end{aligned}$$

which is proportional to the solution $x_i=\lambda_i$, $y_i=\mu_i$ since $\pi_4(\lambda,\mu)$ is integral and

$$\pi_4(\lambda,\mu)=cdP_n^2(\lambda)P_n^2(\mu)\sum_{i=1}^{n-1}\left[\frac{b_{in}\mu_i}{\mu_n}-\frac{a_{in}\lambda_i}{\lambda_n}\right]\neq 0\,.$$

The following example shows that not all systems of type (11) can be solved by the method of this paper. The system

$$\sum_{i=1}^{3}\sum_{j=i+1}^{4}\left[\frac{a_{ij}x_i}{x_j}-\frac{b_{ij}y_i}{y_j}\right]=0\,,$$
$$c\prod_{i=1}^{4}x_i=d\prod_{i=1}^{4}y_i\,,$$

where $a_{12}=3$, $a_{13}=-2$, $a_{14}=3$, $a_{23}=4$, $a_{24}=4$, $a_{34}=3$, $b_{12}=6$, $b_{13}=-2$, $b_{14}=3$, $b_{23}=8$, $b_{24}=2$, $b_{34}=3$, $c=2$, $d=1$, has the solution

$$x_1=2,\ x_2=3,\ x_3=-4,\ x_4=-2,\ y_1=4,\ y_2=-3,\ y_3=-4,\ y_4=2\,,$$

which also satisfies

$$\sum_{j=2}^{4}\left[\frac{a_{1j}x_1}{x_j}-\frac{b_{1j}y_1}{y_j}\right]=0$$

and

$$\sum_{i=1}^{3}\left[\frac{a_{i4}x_i}{x_4}-\frac{b_{i4}y_i}{y_4}\right]=0\,.$$

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