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A WILD CANTOR SET IN THE HILBERT CUBE

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Let E^n be the Euclidean n-space. A Cantor set C is a set homeomorphic with the Cantor middle-third set. Antoine and Blankinship have shown that there exists a "wild" Cantor set in any E^n for $n \geqq 3$, where "wild" means that $E^n - C$ is not simply connected. However it is also known that no "wild" Cantor set (in fact, compact set) can exist in many infinite dimensional spaces, such as s (the countably infinite product of lines) or the Hilbert space l_2. A result of this paper provides a positive answer for a generalization of Blankinship's result in the Hilbert cube.

If X is a space, we denote by X^n the space $\prod_{i=1}^{n} X_i$ and X^∞ the space $\prod_{i=1}^{\infty} X_i$ with $X_i = X$. Let τ_n denote the projecting function of X^∞ onto X^n and π_n the projecting function of X^∞ onto X_n. Let $J, \dot{J}$ denote intervals $[-1, 1], (-1, 1)$ respectively. The Hilbert cube is the space J^∞ under the metric $\rho(x, y) = \sum_{i \geq 1}(|x_i - y_i|)/2^i$. Hilbert space, l_2, is the space of all square summable sequences of real numbers with metric $d((x_i), (y_i)) = \sqrt{[\sum_{i=1}^{\infty}(x_i - y_i)^2]}$. The space $\dot{J}^\infty$ is also denoted by s. Let $E^n = \prod_{i=1}^{n} E_i$ be the Euclidean n-space.

A Cantor set is a set homeomorphic with the Cantor middle-third set. The existence of a Cantor set C in E^n ($n \geqq 3$) such that $E^n - C$ is not simply connected was first demonstrated by Antoine [4] in 1921 and constructed by W. A. Blankinship [5] in 1951. It is known that every Cantor set is s (or in l_2) must be tame, in the sense that its complement in s (or in l_2) is topologically as nics as the space itself. In fact it has been proved (by V. Klee in the case of l_2 [9] and by R. D. Anderson [1] in the case of s, using Klee's method) that if K is a compact set in X (for $X = s$ or l_2), then $X - K \approx X$. The question as to whether a finite dimensional closed set can leave the Hilbert cube multiply connected (in particular, whether a Cantor set can have this property) was then raised in [5] by Blankinship and was also later mentioned in [7] by Klee. In this paper we shall give such a question a positive answer by constructing a Cantor set C in the Hilbert cube J^∞ such that $J^\infty - C$ is not homotopically trivial. In fact, we shall apply the result of Blankinship [5] to show that $J^\infty - C$ has nontrivial 1st-Homotopy group. We remark that such a set C cannot be constructed as a subset of $\dot{J}^\infty$. Note that Anderson [1] (by using Klee's method) proved that any Cantor set C (in fact, any compact set) in $\dot{J}^\infty$ can be carried into an end-face, say $K_1 = \{x \in J^\infty \mid \pi_1(x) = 1\}$, by a homeomorphism on J^∞. It is quite clear that the complement of any Cantor subset (in fact, any compact subset) of K_1 in J^∞ is homotopically trivial, therefore, if the complement of C in J^∞ is to be homotopically nontrivial, C must, in a sense, join various end-faces of J^∞.

2. Some notation and lemma.

All homeomorphisms concerned are assumed to be geometric homeomorphisms, and when a homeomorphism has domain in E^n, it is assumed to be linear. Two subsets of E^n are similar if they are homeomorphic under some homeomorphism. Let Δ denote the boundary of the unit square in E^2. A *-circle is a set homeomorphic to Δ. An n-tube, $n \geqq 3$, is a set homeomorphic to the product of a circular 2-cell with $(n-2)$ *-circles.

We shall choose a fixed set of positive real numbers $r_1, r_2, \cdots$ with the properties that (1) $r_1 > 1$ and (2) $r_{n+1} > 2(\sum_{i=1}^{n} r_i)$. Let $L_i = [r_i = 1, r_i + 1] \subset E_i$ and $L^n = \prod_{i=1}^{n} L_i \times (r_{n+1}, r_{n+2}, \cdots)$. We shall regard E^n as a subset of E^{n+1} by considering E^n as Ex^n0.

LEMMA 1. *If X is a Hausdorff space and $A_1, A_2, \cdots$ is a decreasing sequence of compact subsets of X such that each A_i is dense in itself, then $\bigcup_{i=1}^{\infty} A_i$ is dense in itself.*

Proof. If x is an isolated point of $\bigcap_{i=1}^{\infty} A_i$, then for some i, x is an isolated point of A_i, contrary to the hypothesis.

3. Brief outline of the construction.

The construction is an inductive modification of the construction by Antoine [4] and by Blankinship [5]. The Cantor set C will be the intersection of a decreasing sequence of compact subsets $K_1, K_2, \cdots$ of the Hilbert cube $L^\infty = \prod_{i=1}^{\infty} L_i$. For each $n \geqq 3$, K_n will be the product of a compact subset K_n' of L^n with $\prod_{i=n+1}^{\infty} L_i$. K_3' is the intersection of a simple chain of linking 3-tubes of E^3 with L^3. K_4' will be contained in $K_3' \times L_4$ and is the intersection of a simple chain of linking 4-tubes of E^4 with L^4 and so on.

4. Construction of K_3.

DEFINITION. Let r, s be positive integers and d_r an arbitrary real number. Let S be a compact subset of $E^\infty (= \prod_{i=1}^{\infty} E_i)$ such that $\pi_r(S) = d_r$. We say $\tilde{S}$ is the set generated by rotating S about the hyperplane $x_r = d_r$ and $x_s = 0$ if

$$\tilde{S} = \left\{ \begin{array}{l} x \in E^\infty : \exists y \in S \ni (x_r, x_s) \in \mathrm{Bd}([d_r - y_s, d_r + y_s] \times [-y_s, y_s]) \\ \qquad\qquad \text{and } x_i = y_i \text{ for } i \neq r, s \end{array} \right\}$$

where $[d_r - y_s, d_r + y_s] \subset E_r$, $[-y_s, y_s] \subset E_s$.

The following Lemma is evident:

LEMMA 2. *Suppose S is the set defined above and $\pi_s(S) > 0$, then $\tilde{S}$ is homeomorphic to the product of S with a *-circle.*

DEFINITION. Let

$$T^2 = \{x \in E^\infty : (x_1 - r_1)^2 + (x_2 - r_2)^2 \leqq \left(\frac{1}{4}\right)^2 \text{ and } x_i = r_i \text{ for } i \geqq 3\}$$

$$\Delta_0 = \{x \in E^\infty : (x_1 - r_1)^2 + (x_2 - r_2)^2 = \left(\frac{1}{2}\right)^2 \text{ and } x_i = r_i \text{ for } i \geqq 3\} .$$

For $n \geqq 3$, define T^n inductively to be the set generated by rotating T^{n-1} about the hyperplane $x_{n-1} = 0$, $x_n = r_n$.

LEMMA 3. *For $n \geqq 2$, $\min \pi_n(T^n) \geqq 1$.*

Proof. It is clear for $n = 2$. For $n \geqq 3$, it follows from the fact $\min \pi_n(T^n) = r_n - (1/4 + r_2 + \cdots + r_{n-1})$ and from the hypothesis of r_i.

LEMMA 4. *For $n \geqq 3$, T^n is an n-tube in E^n.*

Proof. $\pi_2(T^2) > 0$ by Lemma 3. Then by Lemma 2, T^3 is a 3-tube. Inductively, T^n is an n-tube.

LEMMA 5. *For $n \geqq 3$, $T^n \cap L^n = \tau_2(T^2) \times \prod_{i=3}^n L_i \times (r_{n+1}, r_{n+2}, \cdots)$.*

Proof. This is a consequence of Lemma 3.

Let $\{t_i^3\}_{i=1}^l$ be a chain of cylically linked disjoint 3-tubes contained in the interior of T^3 and looping once around the axis of T^3. We assume (1) they are all similar to T^3, (2) $l \equiv 0 \pmod 4$ and l is large enough so that each t_i^3 can be regarded as the set generated by rotating a small circular 2-cell t_i^2 along a small *-circle Δ_i, (3) $\operatorname{diam}(t_i^3) <$ 1/3(diam T^3) for all i, and (4) Only two members of $\{t_i^3\}_{i=1}^l$ intersect $\operatorname{Bd}(L^3)$(one in each side) and the intersection of each such t_i^3 with Bd (K^3) is exactly two disjoint 2-cells. Let $A_3 = \bigcup_{i=1}^k t_i^3$, $K_3' = A_3 \cap L^3$ and $K_3 = K_3' \times \prod_{i=4}^\infty L_i$.

5. Construction of $K_4, K_5, \cdots$.

For the purpose of simplicity, we shall give only the construction of K_4 and assert that for $n \geqq 5$, K_n can be inductively constructed.

Step 1. For each i, let h_i be a (linear) homeomorphism of T^3

onto t_i^3. Hence $\{t_{ij}^3 = h_i(t_j^3)\}_{j=1}^l$ is a similar chain of cyclically linked disjoint 3-tubes in t_i^3. We require that each h_i is so chosen that (1) if t_i^3 is a member that intersects $\mathrm{Bd}(L^3)$, then only two members of $\{t_{ij}^3\}_{j=1}^l$ intersect $\mathrm{Bd}(L^3)$ and the intersection of each such member with $\mathrm{Bd}(L^3)$ is exactly two disjoint 2-cells and (2) $\mathrm{diam}(t_{ij}^3) < (1/3^2)\mathrm{diam}(T^3)$ for all ij.

Step 2. For each i, j, let t_{ij}^4 be the 4-tube in T^4 generated by rotating t_{ij}^3 about planes $x_3 = 0, x_4 = r_4$. We now regard each t_{ij}^3 as the set generated by rotating a small 2-cell t_{ij}^2 along a small *-circle. We assume further that t_{ij}^2 is contained in L^3 whenever t_{ij}^3 intersects L^3. Let $\tilde{t}_{ij}^2$ be the set generated by rotating t_{ij}^2 about planes $x_2 = 0$, $x_4 = r_4$. Then t_{ij}^4 can be regarded as the geometric product of $\tilde{t}_{ij}^2$ with Δ_{ij}. $\tilde{t}_{ij}^2$ is a 3-tube. Let h_{ij} be a linear homeomorphism of T^3 onto $\tilde{t}_{ij}^2$. Let $t_{ijk}^3 = h_{ij}(t_k)$, $k = 1, 2, \cdots, l$. We require each h_{ij} is so chosen that (1) if $t_{ij}^2 \subset L^3$, then only two members of $\{t_{ijk}^3\}_{k=1}^l$ intersect $L^3 \times \mathrm{Bd}(L_4)$ (one in each side) and the intersection of each such member with $L^3 \times \mathrm{Bd}(L_4)$ is exactly two disjoint 2-cells and (2) diam $(t_{ijk}) < (1/3)(\mathrm{diam}\ T^3)$. Let t_{ijk}^4 denote the geometric product of t_{ijk}^3 with Δ_{ij}. Let $A_4 = \bigcup_{i,j,k=1}^l t_{ijk}^4$, $K_4' = A_4 \cap L^4$ and $K_4 = K_4' \times \prod_{i=5}^\infty L_i$.

6. THEOREM 1. *Let $C = \bigcap_{i=3}^\infty K_i$. Then C is a Cantor set in L^∞.*

Proof. It follows from the construction that $K_3, K_4, \cdots$ is a decreasing sequence of compact subset of L^∞ and each K_i is dense in itself. Hence C is dense in itself by Lemma 1. Furthermore, each K_i is a finite union of disjoint compact subsets whose diameters are uniformly small and tend to zero as $i \to \infty$. We conclude then that C is a compact zero-dimensional space which is dense in itself, hence is a Cantor set.

THEOREM 2. *If F is a mapping of $\Delta_0 \times I$ into L^n $(n \geqq 3)$ such that $F|_{\Delta_0 \times 0}$ = identity on Δ_0 and $F(\Delta_0 \times 1)$ is a point, then $F(\Delta_0 \times I) \cap K_n' \neq \phi$.*

Proof. The proof is due to [5]. Basically Blankinship had constructed a Cantor set C' in A_n such that C' links Δ_0 in E^n, hence A_n also links Δ_0 in E^n. As a consequence, $K_n' = A_n \cap L^n$ links Δ_0 in L^n.

THEOREM 3. *$L^\infty - C$ has nontrivial lst-Homotopy group.*

Proof. Let F be a mapping of $\Delta_0 \times I$ into L^∞ such that $F|_{\Delta_0 \times 0} =$

identity on Δ_0 and $F(\Delta_0 \times 1)$ is a point. For each $n \geqq 3$, $\tau_n(F)$ is a mapping of $\Delta_0 \times I$ into L^n satisfying $(\tau_n F)_{\Delta_0 \times 0} =$ identity on Δ_0 and $(\tau_n F)(\Delta_0 \times 1)$ is a point. Hence by Theorem 2, $(\tau_n F)(\Delta_0 \times I) \cap K'_n \neq \phi$. This implies $F(\Delta_0 \times I) \cap K_n \neq \phi$, hence $F(\Delta_0 \times I) \cap C \neq \phi$.

THEOREM 4. *There exist two Cantor sets in the Hilbert cube such that no homeomorphism of one onto the other can be extended to a homeomorphism on the whole Hilbert cube.*

Let $\dot{L}_i = \mathrm{Int}(L_i)$ and let $(\dot{L})^\infty = \prod_{i=1}^\infty \dot{L}_i$. Let $V'_n = K'_n \cap \mathrm{Int}(L^n)$ and $V_n = V'_n \times \prod_{i=n+1}^\infty \dot{L}_i$. Then each V_n is a closed subset of $(\dot{L})^\infty$ and hence $C_0 = \bigcap_{n=3}^\infty V_n$ is both zero-dimensional and closed in $(\dot{L})^\infty$. By similar reasoning C_0 links Δ_0 in $(\dot{L})^\infty$. Finally, using the fact $s \simeq (\dot{L})^\infty$ and $l_2 \cong s$ [2], we conclude:

THEOREM 5. *s and l_2 contain zero-dimensional closed sets whose complements are not simply-connected.*

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