

IDEAL NEIGHBOURHOODS IN A RING

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A group topology on a ring is said to have *ideal closure property* in case the closure of an ideal is the intersection of all maximal ideals containing it. Hinrichs considered such group topologies on rings $C(X)$ of continuous real-valued functions defined over completely regular Hausdorff spaces. He gave a characterization of such topologies with ideal neighbourhoods at zero in $C(X)$, and showed that there exists in $C(X)$ a group topology with ideal closure property with the largest collection of open ideals. His results are indeed true in a wider class of rings—viz. semisimple commutative rings with unit element whose structure spaces of maximal ideals (with hull-kernel topology) are Hausdorff. This generalization is achieved by making use of a characterization by Gillman of such rings.

In this paper, a ring will always mean a semisimple commutative ring with unit. With every maximal ideal M of a ring R , let us associate an ideal $\pi(M)$ consisting of all elements x in R such that x belongs to all maximal ideals forming a certain neighbourhood of M in the structure space $\mathfrak{M}(R)$ of maximal ideals of R [1]. The ideal $\pi(M)$ can be algebraically characterized as the set $\{x \in R \mid \text{there exists } y \in M \text{ such that } xy = 0\}$. It is known [1] that the structure space $\mathfrak{M}(R)$ of maximal ideals of R is Hausdorff if and only if for every $M \in \mathfrak{M}(R)$, M is the unique maximal ideal containing $\pi(M)$. In what follows, a ring R shall also be such that $\mathfrak{M}(R)$ is Hausdorff. We may mention that this class of rings includes all (von Neumann) regular rings, rings of all (bounded) uniformly continuous real-(complex-) valued functions defined over a uniform space, and $L^1(G)$'s (with unit element adjoined whenever needed) for locally compact Abelian groups G .

By a group topology on a ring we mean the topology which makes the additive group structure of the ring a topological group (not necessarily Hausdorff). The ideal closure property of a group topology on a ring is written shortly as I.C.P. Obviously, the closed ideals in a ring with a group topology with I.C.P. are precisely the intersections of maximal ideals. We will make use of the well-known fact that any open subgroup of a topological group is closed.

THEOREM 1. *Let R be a ring with some group topology τ . If τ has I.C.P., then $\text{Cl } \pi(M) = M$ for every maximal ideal M of R . The converse is true if further τ has an ideal basis of neighbourhoods at zero.*

Proof. We need prove only the converse since the first part is easy as $\mathfrak{M}(R)$ is assumed to be Hausdorff. Consider any ideal I , all of whose maximal divisors are $\{M_\alpha\}$. Let V be any open ideal. Taking $x \in \cap M_\alpha$, we see that for each α , there exists an element $a_\alpha \in V$ such that $x + a_\alpha \in \pi(M_\alpha)$. Therefore there exist elements $b_\alpha \in M_\alpha$ such that $(x + a_\alpha)b_\alpha = 0$. That is, $xb_\alpha \in V$. Since the ideal $(I, \dots, b_\alpha, \dots)$ cannot be contained in any maximal ideal of R , we will have some $i \in I$ such that $i +$ a finite (ring) linear combination of b_α 's equal to 1. So, $x +$ an element of V belongs to I . This means that, under the hypothesis of ideal basis of neighbourhoods at zero (an ideal neighbourhood of zero is necessarily an open ideal an conversely), x belongs to $\text{Cl } I$. As every maximal ideal is evidently closed, $\text{Cl } I$ is contained in $\cap M_\alpha$; whence the desired result follows.

LEMMA. Suppose that I_1 and I_2 are two ideals of a ring R such that for some maximal ideal M of R and some $x \in M$, $\{x + I_j\} \cap \pi(M)$ is nonnull, $j = 1, 2$. Then $\{x + I_1 \cap I_2\} \cap \pi(M)$ is nonnull.

Proof. We have $x + a_j \in \pi(M)$, $a_j \in I_j$, $j = 1, 2$. So there exist $b_j \in M$, $j = 1, 2$ such that $(x + a_j)b_j = 0$, which gives that $xb_1b_2 \in I_1 \cap I_2$. Since $(b_1b_2, \pi(M)) = R$, there exists an element $h \in R$ such that $hb_1b_2 - 1 \in \pi(M)$. Obviously then, $\{x + I_1 \cap I_2\} \cap \pi(M)$ is nonnull.

THEOREM 2. The family of all maximal ideals of a ring R defines a ring topology in R having I.C.P. This is a Hausdorff topology whose open ideals are precisely the finite intersections of maximal ideals.

Proof. Any nonempty family of ideals in R will define a ring topology on it, if they are considered as a subbase for the neighbourhood system at zero. The ring topology given by all maximal ideals is Hausdorff because the ring is semisimple.

In view of the preceding theorem and lemma, to see that this topology has I.C.P., it is sufficient to prove that, if M_1 and M_2 are any two maximal ideals and $x \in M_1$ then $\{x + M_2\} \cap \pi(M_1)$ is nonnull. If $M_1 = M_2$, this is obvious; we take $-x \in M_2$. If M_1 and M_2 are distinct we have $\pi(M_1) + M_2 = R$; hence, $a + b = 1$, $a \in \pi(M_1)$, $b \in M_2$. That is, $x - xb = xa$, as required.

Now, finite intersections of maximal ideals are evidently open. Conversely, if I is any open ideal then I contains a finite intersection of maximal ideals; consequently, any maximal divisor of I is one of them, because a maximal ideal is prime. Therefore, $\text{Cl } I$ is the intersection of a finite number of maximal divisors of I , using I.C.P. But,

since I is an open subgroup it is closed. Thus the result follows.

Let σ denote the family of all ideals I of a ring R satisfying the condition: For every maximal ideal M of R and for every $x \in M$, $(x + I) \cap \pi(M)$ is nonnull. From the previous lemma, σ is closed under finite intersection. Thus as a basis of neighbourhoods at zero, σ defines a ring topology on R .

THEOREM 3. *Among the group topologies with I.C.P. on a ring R , there is one possessing the largest collection of open ideals—viz. σ .*

Proof. By direct appeal to the definition of σ , $\text{Cl } \pi(M) = M$ for every maximal ideal M . So, I.C.P. of σ follows invoking Theorem 1. Obviously, any open ideal in whatever group topology with I.C.P. on R belongs to σ .

REMARK 1. The topology on R given by σ is Hausdorff. In fact, the σ -topology is finer than any group topology with I.C.P. and having an ideal basis of neighbourhoods at zero.

REMARK 2. The following statements are equivalent in a ring R .

- (i) σ consists of all ideals.
- (ii) σ -topology is discrete.
- (iii) R is a (von Neumann) regular ring.
- (iv) $\pi(M) = M$ for every maximal ideal M .

The proof is evident by the use of [1, Th. 3.16].

REMARK 3. It does not seem possible to algebraise the pseudo-compactness of a topological space so that the results of [2] in this regard may also be generalized.

The following information is in a similar vein to that of Theorem 3. Now of course, the ring need not be semisimple, nor its structure space need be Hausdorff; and the topologies in the ring may well disregard the compatibility criteria.

THEOREM 4. *Among the topologies with I.C.P. in a ring R , there is one which is the coarsest.*

Proof. Define a topology τ in the ring R by taking all maximal ideals to be a subbase for the closed sets. It suffices to show that τ has I.C.P. We observe first that an ideal I is contained in the set union of a finite number of prime ideals P_i only if it is contained in

some one P_r . For, assume without losing generality that any pair of prime ideals in this finite collection are incomparable. Then we may choose elements $x_r \in (\bigcap_{s \neq r} P_s) - P_r$. If $i_r \in I - P_r$ would also exist, it will amount to a contradiction viz., $\sum i_r x_r$ belongs to I but not to any P_r . Now, the closure of an ideal is the intersection of basic closed sets in τ —which are finite unions of maximal ideals. Therefore, $\text{Cl } I$ must be an intersection of maximal ideals. The proof is complete.

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