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OSCILLATION OF SOLUTIONS TO SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

JACK W. MACKI AND JAMES S. W. WONG

A solution $y(t)$ of

$$y'' + f(t, y) = 0 \tag{1}$$

is said to be oscillatory if for every $T > 0$ there exists $t_0 > T$ such that $y(t_0) = 0$. Let $\mathscr{F}$ be the class of solutions of (1) which are indefinitely continuable to the right, i.e. $y \in \mathscr{F}$ implies $y(t)$ exists as a solution to (1) on some interval of the form $[T_y, \infty)$. Equation (1) is said to be oscillatory if each solution from $\mathscr{F}$ is oscillatory. If no solution in $\mathscr{F}$ is oscillatory, equation (1) is said to be nonoscillatory.

THEOREM 1. **Let $f(t, x)$ be continuous and satisfy $b(t)\Psi(x) \geqq f(t, x) \geqq a(t)\Phi(x)$ for $0 \leqq t < \infty$, $-\infty < x < \infty$, where**
(i) $a(t) \geqq 0$, $b(t) \geqq 0$ are both locally integrable,
(ii) $\Phi(x)$, $\Psi(x)$ are nondecreasing and satisfy $x\Phi(x) > 0$, $x\Psi(x) > 0$ for $x \neq 0$ and, for some $\alpha \geqq 0$, $\int_\alpha^\infty [\Phi(u)]^{-1} du < \infty$, $\int_{-\alpha}^{-\infty} [\Psi(u)]^{-1} du < \infty$. Then equation (1) is oscillatory if and only if $\int^\infty ta(t)dt = \int^\infty tb(t)dt = \infty$.

Conditions on $f(t, x)$ are also given (Theorem 2) which are sufficient for equation (1) to be nonoscillatory.

We are interested principally in the nonlinear equation, and in particular with extending the results of Atkinson [1], who treated the case $f(t, x) = a(t)x^{2n-1}$, with $a(t)$ continuous and positive, and $n > 1$. For this particular equation, Atkinson proved that (1) is oscillatory if and only if

$$\int^\infty ta(t)dt = \infty \; . \tag{2}$$

He also proved that the conditions

$$a(t) \in C^1[0, \infty) \; , \qquad a'(t) \leqq 0 \; , \qquad \int^\infty t^{2n-1}a(t)dt < \infty \tag{3}$$

are sufficient for (1) to be nonoscillatory. We shall state analogous results for a considerable larger class of functions $f(t, x)$. Results of this type are numerous, the principle ones being due to Jones [4], Walman [7], Das [3], Bhatia [2], Nehari [5], and Utz [6]. Our results can be specialized to yield certain of the theorems presented by the first two authors, and have some points of contact with the others.

THEOREM 1. *Let* $f(t, x)$ *be continuous on* $S = [0, \infty) \times (-\infty, \infty)$, *with* $b(t)\Psi(x) \geqq f(t, x) \geqq a(t)\Phi(x)$ *for* $(t, x) \in S$, *where*

(a) $a(t)$ *and* $b(t)$ *are nonnegative locally integrable functions,*

(b) $\Phi(x)$ *and* $\Psi(x)$ *are nondecreasing, with* $x\Phi(x) > 0$ *and* $x\Psi(x) > 0$ *for* $x \neq 0$, *on* $(-\infty, \infty)$, *and for some* $\alpha \geqq 0$,

$$\int_\alpha^\infty [\Phi(u)]^{-1} du < \infty , \qquad \int_{-\alpha}^{-\infty} [\Psi(u)]^{-1} du < \infty .$$

Then equation (1) *is oscillatory if and only if*

$$\int_0^\infty tb(t)dt = \int_0^\infty ta(t)dt = \infty .$$

Proof. The proof is modelled on that of Atkinson. Suppose (1) has a nonoscillatory solution, $y(t)$, from the class $\mathscr{F}$, say $y(t) > 0$ for $t > T$. Then, for $T \leqq s \leqq t$,

$$y'(t) - y'(s) = -\int_s^t f(u, y(u))du \leqq 0 , \tag{4}$$

and so $y'(t)$ is nonincreasing. This, and the fact that $y(t) > 0$, imply that $\lim_{t\to\infty} y'(t) = L$ exists, where $0 \leqq L < \infty$. Let $t \to \infty$ in (4) to get, for $s \geqq T$,

$$y'(s) = L + \int_s^\infty f(u, y(u))du \geqq \int_s^\infty f(u, y(u))du \geqq 0 .$$

Integrating from $s = T$ to $s = t$, we obtain

$$y(t) \geqq y(t) - y(T) \geqq \int_T^t \int_s^\infty f(u, y(u))duds \geqq \int_T^t (u - T)f(u, y(u))du ,$$

which implies that

$$y(t) \geqq \int_T^t (u - T)a(u)\Phi(y(u))du . \tag{5}$$

From the monotonicity of Φ, we have

$$\Phi(y(t))\left[\Phi\left(\int_T^t (u - T)a(u)\Phi(y(u))du\right)\right]^{-1} \geqq 1 ,$$

and if we multiply by $(t - T)a(t)$ and integrate from r to s, we get, after a change of variable on the left,

$$\int_L^U [\Phi(u)]^{-1} du \geqq \int_r^s (t - T)a(t)dt , \tag{6}$$

where

$$L = \int_T^r (u - T)a(u)\Phi(y(u))du , \quad U = \int_T^s (u - T)a(u)\Phi(y(u))du .$$

If, by an appropriate choice of r, we can make $L \geqq \alpha$, than the left-hand side of (6) is bounded above for all $s > r$, hence $\int_0^\infty ta(t)dt < \infty$. If this is not possible, then for all $r \geqq T$,

$$\alpha > \int_T^r (u - T)a(u)\Phi(y(u))du \geqq \Phi(y(T))\int_T^r (u - T)a(u)du ,$$

and the result again follows.

For the case when $y(t) < 0$ for $t \geqq T$, the procedure is the same, the changes in detail being that $y'(t)$ now goes to a finite nonpositive limit, the inequalities (4) through (5) are reversed while the inequality (6) is in the same direction, and the functions $a(t)$ and $\Phi(x)$ are replaced in (5) and (6) by the functions $b(t)$ and $\Psi(x)$.

To prove the other half of the theorem, we must show that if either $\int_0^\infty ta(t)dt < \infty$ or $\int_0^\infty tb(t)dt < \infty$, then (1) has a nonoscillatory solution. Suppose $\int_0^\infty tb(t)dt < \infty$, and consider the integral equation

$$y(t) = 1 - \int_t^\infty (s - t)f(s, y(s))ds . \tag{7}$$

If (7) has a nonnegative continuous solution on some interval $[T, \infty)$, it is clear that $y'' + f(t, y(t)) = 0$ on this interval. Also, since the improper integral in (7) would converge, we would have $\lim_{t\to\infty} y(t) = 1$, that is, $y(t)$ would be nonoscillatory.

Let a positive integer T be chosen such that $\Psi(1)\int_T^\infty sb(s)ds \leqq 1/2$. we define, for N a positive integer, $N \geqq T$,

$$\begin{aligned} y_N(t) &= 1 && \text{for } t \geqq N , \\ y_N(t) &= 1 - \int_{t+(1/N)}^\infty \left(s - t - \frac{1}{N}\right) f(s, y_N(s))ds && \text{for } T \leqq t \leqq N . \end{aligned}$$

This formula defines $y_N(t)$ successively on the intervals $[N - (k/N), N - (k - 1)/N]$ for $k = 1, 2, \cdots, N(N - T)$; hence $y_N(t)$ is defined on $[T, \infty)$. For $N - 1/N \leqq t < \infty$, we have

$$0 \leqq \int_{t+(1/N)}^\infty \left(s - t - \frac{1}{N}\right) f(s, y_N(s))ds \leqq \Psi(1)\int_T^\infty sb(s)ds \leqq \frac{1}{2} ,$$

hence $1/2 \leqq y_N(t) \leqq 1$ on this interval. Any easy induction then shows that $1/2 \leqq y_N(t) \leqq 1$ on the entire interval $[T, \infty)$. Consequently, for $t \geqq T$,

$$| y'_N(t) | = \left| \int_{t+(1/N)}^{\infty} f(s, y_N(s))ds \right| \leqq \Psi(1) \int_{t+(1/N)}^{\infty} b(s)ds \leqq \frac{1}{2}$$

Since the family $\{y_N(t)\}$ is equicontinuous and uniformly bounded on $[T, \infty)$, we extract a uniformly convergent subsequence $\{y_k(t)\}$, $\lim_{k\to\infty} y_k(t) = y(t)$.

We now choose any large real number B, and write

$$y_k(t) = 1 - \int_{t+\{1/N(k)\}}^{B} \left(s - t - \frac{1}{N(k)}\right) f(s, y_k(s))ds + \in(k, B) ,$$

where

$$| \in(k, B) | \leqq \Psi(1) \int_B^{\infty} sb(s)ds .$$

If we let $k \to \infty$, we have

$$\liminf_{k\to\infty} \in(k, B) \leqq y(t) - 1 + \int_t^B (s - t) f(s, y(s))ds \leqq \limsup_{k\to\infty} \in(k, B) .$$

If we now let $B \to \infty$, it is clear from the above bound on $\in(k, b)$ that the lim inf and lim sup terms go to zero, and so $y(t)$ satisfies equation (7).

For the case

$$\int_0^{\infty} ta(t)dt < \infty ,$$

we consider the integral equation

$$y(t) = -1 - \int_t^{\infty} (s - t) f(s, y(s))ds ,$$

and the procedure is the same, except that $-1 \leqq y_k(t) \leqq 0$, and $a(t)\Phi(x)$ replaces $b(t)\Psi(x)$.

COROLLARY 1. *Let $f(t, x)$ be continuous on S, with $xf(t, x) > 0$ for $x \neq 0$. If $| f(t, x) | \geqq a(t) | \Phi(x) |$ for $(t, x) \in S$, where $a(t)$ is locally integrable and continuous on $[0, \infty)$, while $\Phi(x)$ is nondecreasing, $x\Phi(x) > 0$ for $x \neq 0$, on $(-\infty, \infty)$, and*

$$\int_{\alpha}^{\infty} [\Phi(u)]^{-1} du < \infty , \qquad \int_{-\alpha}^{-\infty} [\Phi(u)]^{-1} du < \infty , \qquad \textit{for some} \quad \alpha \geqq 0 ,$$

then

$$\int_0^{\infty} ta(t)dt = \infty \Longrightarrow (1) \textit{ is oscillatory} .$$

COROLLARY 2. *Let $f(t, x)$ be continuous on S, with $xf(t, x) > 0$ for $x \neq 0$. If $| f(t, x) | \leqq a(t) | \Phi(x) |$ for $(t, x) \in S$, where $a(t)$ is locally* *integrable and continuous on* $[0, \infty)$, *while* $\Phi(x)$ *is nondecreasing and* $x\Phi(x) > 0$ *for* $x \neq 0$, *on* $(-\infty, \infty)$, *then*

$$(1) \text{ is oscillatory} \Rightarrow \int_0^\infty ta(t)dt = \infty .$$

These corollaries are obtained by closely examing which conditions on $f(t, x)$ are used in the two halves of Theorem 1. The importance of the theorems and the corollaries lies in the fact that they show it is the global behaviour of $f(t, x)$, rather than its local behaviour, which determines the oscillation properties of (1).

Jones [4] considers the equation

$$y'' + \sum_{k=2}^{n} a_k(t) y^{2k-1} = 0 ,$$

with suitable conditions on the $a_k(t)$. His result follows from Theorem 1 by setting

$$\Psi(y) = \begin{cases} y^3 , & y \leqq -1 \\ y^{2n-1} , & -1 \leqq y \leqq 0 \\ y^3 , & 0 \leqq y \leqq 1 \\ y^{2n-1} , & 1 \leqq y , \end{cases} \qquad \Phi(y) = \begin{cases} y^{2n-1} , & y \leqq -1 \\ y^3 , & -1 \leqq y \leqq 0 \\ y^{2n-1} , & 0 \leqq y \leqq 1 \\ y^3 , & 1 \leqq y , \end{cases}$$

with

$$b(t) = a(t) = \sum_{k=2}^{n} a_k(t) .$$

Waltman [7] considers the equation $y'' + c(t)f(y) = 0$, with $c(t) \geqq 0$, $f(0) = 0$, $f(y) \neq 0$ for $y \neq 0$, $f'(y) \geqq 0$ and continuous, and, for some $p > 1$, $\liminf |f(y)| \cdot |y|^{-p} \neq 0$ for $y \to \pm\infty$. Noting that the last condition implies that

$$\int_\alpha^\infty [f(u)]^{-1} du < \infty , \qquad \int_{-\alpha}^{-\infty} [f(u)]^{-1} du < \infty ,$$

for some $\alpha > 0$, we can obtain Waltman's Theorem 2 by setting $\Phi(y) = \Psi(y) = f(y)$, $a(t) = b(t) = c(t)$ in our Theorem.

We now establish a sufficient condition for the nonoscillation of (1) analogous to Atkinson's condition (3). Our restrictions on $f(t, x)$ are more severe than in Theorem 1, though they are still global rather than local.

THEOREM 2. *Let* $f(t, x)$ *be continuous on* S, *with* $f_t(t, x)$ *defined and continuous on* S, *and such that* $f(t, 0) = 0$, $xf_t(t, x) \leqq 0$ *and* $xf(t, x) > 0$ *for* $x \neq 0$. *Furthermore, assume that* $y(t) \equiv 0$ *is the only solution of* (1) *in the class* $\mathscr{T}$ *such that* $y(\tau) = y'(\tau) = 0$ *for any*

$\tau \in [0, \infty)$. *Finally, assume that for* $0 \leqq t < \infty$, $0 \leqq x < \infty$, *we have*

$$f(t, x) \leqq a(t)\Phi(x) ,$$

where $a(t)$ *is locally integrable,* $\Phi(x)$ *is nondecreasing and such that, for some* $B \geqq 0$, *we have*

$$\Phi(xy) \leqq \chi(x)\Phi(y) \qquad \text{for} \quad 0 < x < \infty , \quad B < y < \infty ,$$

with

$$\limsup_{n \to 0+} x^{-1}\chi(x) < \infty .$$

Then

$$\int_0^\infty \Phi(t)a(t)dt < \infty \Longrightarrow \textit{equation (1) is nonoscillatory.}$$

Proof. For any solution from the class $\mathscr{F}$, $y(t)$, defined on some interval, $[T, \infty)$, we define

$$V(t) = y'^2 + 2\int_0^{y(t)} f(t, u)\, du \geqq 0 \qquad \text{on} \quad [T, \infty) .$$

Then

$$V'(t) = 2\int_0^{y(t)} f_t(t, u)du \leqq 0 \qquad \text{on} \quad [T, \infty) .$$

Thus $V(t)$ is bounded above, and hence so is $|y'(t)|$, say $|y'(t)| \leqq M$ for $t \in [T, \infty)$. Suppose $y(t)$ is a solution from $\mathscr{F}$ that oscillates at $t = \infty$, and select a sequence of points $s_k \to \infty$ at which $y(s_k) = 0$, $y'(s_k) > 0$. This is possible because no zero of $y(t)$ can be a zero of $y'(t)$, hence one of two consecutive zeros must be of the type desired. Let t_k be the first zero of $y'(t)$ on $t \geqq s_k$, and note that $y(t)$ is positive and increasing while $y'(t)$ is positive and decreasing, on (s_k, t_k),

Since

$$0 \leqq y'(s_k) = \int_{s_k}^{t_k} f(u, y(u))du \leqq \int_{s_k}^{t_k} a(u)\Phi(y(u))du ,$$

and

$$0 \leqq y(t) = \int_{s_k}^{t_k} y'(u)du \leqq y'(s_k)t \qquad \text{for} \quad s_k \leqq t \leqq t_k ,$$

we have, from the monotonicity of Φ,

$$\begin{aligned} (8) \qquad 0 \leqq y'(s_k) &\leqq \int_{s_k}^{t_k} a(u)\Phi(y'(s_k)u)du \\ &\leqq \int_{s_k}^{t_k} a(u)\Phi(Mu)du \leqq \chi(M)\int_{s_k}^{t_k} a(u)\Phi(u)du , \end{aligned}$$

for k large enough to make $s_k \geqq B$. Since, by hypothesis,

$$\int_0^\infty a(u)\Phi(u)du < \infty ,$$

the upper bound in (8) goes to zero as $k \to \infty$, hence $y'(s_k) \to 0$. From (8) it also follows that

$$0 \leqq y'(s_k) \leqq \chi(y'(s_k))\int_{s_k}^{t_k} a(u)\Phi(u)du ,$$

thus

$$1 \leqq [y'(s_k)]^{-1}\chi(y'(s_k)) \int_{s_k}^{t_k} a(u)\Phi(u)du ,$$

which yields the desired contradiction.

Theorem 2 is related somewhat to a result of Das' [3, Th. 3]. It should be remarked that Das apparently overlooks the necessity of requiring that the trivial solution be the only solution in $\mathscr{F}$ such that $y(\tau) = y'(\tau) = 0$ at a point in $[0, \infty)$. If something to this effect is not assumed, the existence of a sequence, $\{t_k\}$, $t_k \to \infty$, $y(t_k) = 0$, $y'(t_k) > 0$, is not guaranteed.

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UNIVERSITY OF ALBERTA

EDMONTON, ALBERTA, CANADA