PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 1, 1968

A NOTE ON QUASI-FROBENIUS RINGS

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Morita and Curtis proved independently that if A is a quasi-Frobenius ring and $P^\vee_a$ finitely generated, projective, faithful, left A-module, then the ring of endomorphisms $B = \mathrm{End}_A(P)$ is quasi-Frobenius and P is a finitely generated, projective, faithful, left B-module. It also turns out that $A \cong \mathrm{End}_B(P)$. We prove a theorem implying that every quasi-Frobenius ring can be represented as such a ring of endomorphisms.

In fact the following holds:

THEOREM. *If A is a quasi-Frobenius ring there is a Frobenius ring B such that $B/\mathrm{Rad}(B)$ is the product of a finite number of (not necessarily commutative) fields and a finitely generated, projective, faithful, left B-module P such that $A \cong \mathrm{End}_B(P)$. If B' if another Frobenius ring such that $B'/\mathrm{Rad}(B')$ is the product of a finite number of fields and P' a finitely generated, projective, faithful, left B'-module such that $A \cong \mathrm{End}_{B'}(P)$ then there is a semi-linear isomorphism of the B-module P into the B'-module P'.*

We note the results mentioned above appear in [2, pp. 405–406].

Proof. Let A_s be A considered as a left A-module. Let $A_s = E_1 + \cdots + E_n$ (direct) where each E_i is nonzero and indecomposable, and so has a simple socle. Consider the equivalence relation $E_i \cong E_j$ on the set $\{E_1, E_2, \cdots, E_n\}$. Note $E_i \cong E_j$ if and only if $S_i \cong S_j$ where S_i is the socle of E_i for each i. Choose one representative from each equivalence class and let P be their direct sum. Then we easily see that P is a finitely generated, projective, faithful, left A-module. Let $B = \mathrm{End}_A(P)$. Then by Morita and Curtis' result, B is a quasi-Frobenius ring and P is a finitely generated, projective, faithful, left B-module. We claim that if we show $B/\mathrm{Rad}(B)$ is the product of a finite number of fields then it will follow that B is Frobenius. For in this case $B/\mathrm{Rad}(B)$ is the direct sum of a finite number of simple pair-wise nonisomorphic left B-modules. But since B is quasi-Frobenius each simple left B-module is isomorphic to a submodule of B [2, p. 401, Corollary 58.13]. But to show $B/\mathrm{Rad}(B)$ is a product of fields we only need note that $B/\mathrm{Rad}(B) \cong \mathrm{End}_A(T)$ where T is the socle of P. But by the construction of P, T is the direct sum of a finite number of pair-wise nonisomorphic simple left A-modules so $\mathrm{End}_A(T)$ is the product of a finite number of fields. But now as remarked above, $A \cong \mathrm{End}_B(P)$ and P is a finitely generated, projective, faithful, left B-module.

Now suppose $A \cong \mathrm{End}_{B'}(P')$ where B' is a Frobenius ring with $B'/\mathrm{Rad}(B')$ the product of a finite number of fields and that P' is a finitely generated, projective, faithful, left B'-module. Then P' is a finitely generated, projective, faithful, left A-module and $B' \cong \mathrm{End}_A(P')$. But then since A is quasi-Frobenius, $P' \cong \bigoplus_{i=1}^{m} E_{k_i}$ where $1 \leqq k_i \leqq n$ for each $i = 1, 2, \cdots, m$ [2, p. 401, Corollary 58.13]. But P' is a faithful left A-module so it's easy to see that for each $j, 1 \leqq j \leqq n$, $E_{ki} \cong E_j$ for some $i, 1 \leqq i \leqq m$. But now if T' is the socle of P' (as a left A-module), $B'/\mathrm{Rad}(B') \cong \mathrm{End}_A(T')$. But $B'/\mathrm{Rad}(B')$ is the product of a finite number of fields so we see that T' is the direct sum of a finite number of pair-wise nonisomorphic simple left A-modules. Thus $P \cong P'$ (as left A-modules). But then

$$B \cong \mathrm{End}_A(P) \cong \mathrm{End}_A(P') \cong B' \quad \text{and}$$

we easily see that there is a semi-linear isomorphism from the B-module P to the B'-module P'.

We note that if A is a simple ring (i.e. left Artinian, without radical and having no nontrivial two sided ideals) we get the usual representation of A as the ring of matrices over a field (i.e. the endomorphism ring of a finite dimensional vector space) since in this case B is a field.

Bibliography

1. C. W. Curtis, *Quasi-Frobenius rings and Galois theory*, Illinois J. Math. **3** (1959), 134–144.

2. C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience Publishers, New York, New York. 1962.

3. K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Science reports of the Tokyo Kyoiku Daigaku, **6** (1958), 83–142.

Received February 28, 1967.

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