SECOND NOTE ON DAVID HARRISON'S
THEORY OF PREPRIMES

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Some classes of partially ordered rings are studied by
means of representations in $C(X)$, where X is a compact
Hausdorff space. The first theorem generalizes the characte-
rizations, due, respectively, to Harrison and the author, of
the subrings of the real field, and the subrings of $C(X)$.
Among the many consequences proved are the following:

1. If A is a simple ring (no two-sided ideals) partially
ordered so that (a) if n is a positive integer and $nx \geq 0$, then
$x \geq 0$; and (b) for all x in A there exists an integer n exceed-
ing x; then A is a commutative field.

2. Let F be a field, let P be a conic prime in F whose
intersection P_c in the center C of F is a primitive (AC) cone
in C. Then P_c is an Archimedean order in C.

3. The compact Hausdorff space X admits a base of power
at most $2^{\aleph_0}$ of open-and-closed sets if and only if $C(X)$ contains
a dense subfield.

We prove three theorems and many corollaries on infinite pre-
primes with Archimedean conditions. Theorem 1 generalizes the main
result of our first note on preprimes, Theorem 2 relates two Archimedean
conditions for infinite *primes*, and Theorem 3 is concerned with infinite
preprimes in a *field*. The corollaries to Theorem 1 generalize Hilbert's
Theorem that an Archimedean ordered field is commutative; these tend
to confirm (but see also the examples) our conjecture that any primitive
(AC) conic prime in a field is an order. An application is given in the
corollary to Theorem 3 which characterizes a large class of totally
disconnected compact Hausdorff spaces by a property (that makes no
reference to partial order or preprimes) of the rings of continuous real
functions on the spaces. A rather large collection of examples is given
to show that conditions in the theorems are not superfluous. For
furnishing us with encouragement, ideas and unpublished notes we are
very grateful to David Harrison, whose influence shows (we hope)
throughout this note.

The best historical account we can give of the problems studied
here is a chronological list of theorems needed later. See § 2 b, or our
previous note for the definitions. Although Theorem 02 is not precisely
equivalent to any statement in the first three references it is easily
deduced by methods of each of them; it is proved in the fourth refer-
ence by a routine Zorn's lemma argument. We denote by X a compact
Hausdorff space and by $C(X)$ the partially ordered topological ring (or

real algebra) of all continuous real functions on X, with the topology of uniform convergence and with $f \geqq g$ if and only if $f(x) \geqq g(x)$ holds for all x in X. The theorems are stated in our terminology; Stone added a completeness condition to the hypotheses of Theorem 03 but noted that it was not really essential to the method of proof. We denote by N the set of all positive integers.

THEOREM 01. (*Hilbert* [9].) *If a field contains an Archimedean order then it is commutative.*

THEOREM 02. (*Artin* [1]; *Artin-Schreier* [2]; *Baer* [3]; *cf.* [4].) *Let P be a conic division preprime containing all squares in the commutative field F. Then P is the intersection of all the orders in F that contain P.*

THEOREM 03. (*Stone* [11]). *Let A be a commutative ring with unit 1 such that for all n in N there exists x in A with $nx = 1$. Let $\langle A, P \rangle$ be a Stone ring, where P contains all squares in A. Then $\langle A, P \rangle$ is order-isomorphic with a subring of $C(X)$.*

THEOREM 04. (*Kadison* [10].) *Every complete Archimedean ordered algebra is a $C(X)$.*

THEOREM 05. (*Dubois* [4].) *If P is a primitive (AC) division cone in a commutative field F then P is an intersection of orders in F.*

THEOREM 06. (*Harrison* [7].) *A prime is finite if and only if it is a subring.*

THEOREM 07. (*Harrison* [7].) *Let P be a prime in the commutative ring A, and let u be a unit of A belonging to P. Then u^{-1} belongs to P if and only if $-u$ does not belong to P.*

THEOREM 08. (*Harrison* [7]; *see also* [5].) *If A is partially ordered by a real infinite Archimedean conic prime then A is order-isomorphic with a subring of the real field R.*

THEOREM 09. (*Dubois* [5]; *uniqueness is conjectured by Harrison* [8].) *Let $\langle A, P \rangle$ be a Stone ring. Then there exists a unique compact Hausdorff space $X = X_A$ satisfying: The ring $C(X)$ contains a subring S which is order-isomorphic with $\langle A, P \rangle$ and whose ring S_N of fractions with respect to N is dense in $C(X)$.*

Proof. (*of uniqueness*). Let X and X' be compact Hausdorff spaces, let S and S' be subrings of $C(X)$ and $C(X')$, respectively, such that S_N and S'_N are dense in $C(X)$ and $C(X')$ and let φ be an order-isomorphism of S onto S'. The canonical map $\varphi_N: S_N \to S'_N$ is also an order-isomorphism. The density conditions imply that φ_N extends uniquely to a bicontinuous order-isomorphism of $C(X)$ onto $C(X')$. By [6; Th. 4.9] X and X' are homeomorphic.

2. New theorems and definitions.

a. *Statements of the theorems corollaries.* In these theorems we are using 'Archimedean' in a different sense from the previous note. We say that P is *Archimedean*, where P is an infinite preprime, when for every p in P, there exists an integer n with $n - p$ in P. The definitions are given after the statements of the theorems.

THEOREM 1. *Let P be an infinite N-cancellation preprime in the ring A, let $B = B_P$, $J = J_P$.*

A. c.f. *Harrison* [8]. *If P is Archimedean then P^* is an infinite Archimedean N-cancellation (AC) preprime. If P is full Archimedean then P^* is also full Archimedean.*

B. *If P is full Archimedean then $\langle A/J, P^*/J \rangle$ is a Stone ring. Hence A/J is commutative.*

C. *Let $P_1 = P \cap B$, considered as a preprime in B. Then $\langle B/J, P_1^*/J \rangle$ is a Stone ring.*

D. *If P is Archimedean then $\langle (P - P)/J, P^*/J \rangle$ is a Stone ring.*

COROLLARY 1. *Let P be an N-cancellation infinite preprime in the ring A. Each of the following conditions implies that A is commutative:*

A. *For every x in A, $1 + x^2$ is invertible, and P is an (AW) cone containing all squares in A.*

B. *A is a field and P is an (AW) cone containing all squares in A.*

C. *P is a full-infinite (AW) cone, and for every $p \in P$, $1 + p$ has an inverse in P.*

D. *A is a field, P is a primitive (AW) cone containing $(1 + p)^{-1}$ for every p in P.*

E. *A is a simple ring and P is full Archimedean.*

COROLLARY 2. *Let F be a field, let P be a conic prime in F whose intersection P_C in the center C of F is a primitive (AC) cone in C. Then P_C is an Archimedean order in C.*

COROLLARY 3. *Let F be a field with center C, let P be an (AC) cone in F, and for each x in F let $P_x = P \cap C(x)$, where $C(x)$ is the extension field of C by x. Suppose for each x, P_x is a full-infinite prime in $C(x)$. Then P is an Archimedean order in F.*

THEOREM 2. *If P is an infinite N-cancellation (GAC) preprime in the ring A then P contains no large elements. If P is a nonzero prime, if $P \cap (-P) = \{0\}$ and if P contains no large elements, then P is a (GAC) N-cancellation cone.*

THEOREM 3. *Let F be a field, let P be a full Archimedean N-cancellation cone in F. Then $\langle F, P \rangle$ is a Stone ring, P is a division cone, F is isomorphic (algebraically) with a subfield of the reals, and the space X_F of Theorem 09 is totally disconnected with a base of power not exceeding $c = 2^{\aleph_0}$ of open-and-closed sets. Conversely, if X is a totally disconnected compact Hausdorff space with a base of power at most c of open-and-closed sets then $C(X)$ contains a dense subfield.*

COROLLARY. *Let X be a compact Hausdorff space. Then X admits a base of power at most c of open-and-closed sets if and only if $C(X)$ has a dense subfield.*

b. *Definitions.*

Let P be an infinite preprime in the ring A with unity 1; that is P contains zero and one and is closed under addition and multiplication. P is *conic* (or a *cone*) when $P \cap (-P) = \{0\}$. We denote by N the set of all positive integers. We say P is an *N-cancellation preprime* when from nx in P, where $n \in N, x \in A$, follows $x \in P$. P is full or *full-infinite* (or *real-infinite*) whenever $P - P = \{p_1 - p_2; \ p_1, p_2 \in P\}$ is equal to A. We define sets:

$$P^* = \{x \in A; \ \text{for all } n \in N, \ 1 + nx \in P\},$$
$$B_P = \{x \in A; \ \text{for some } n \in N, \ n \pm x \in P\},$$
$$J_P = \{x \in A; \ \text{for all } n \in N, \ 1 \pm nx \in P\}.$$

Elements of B_P are called *bounded elements*, those of J_P are *infinitesimal*. Observe that $P \cap (-P) \subset J_P = P^* \cap (-P^*) \subset B_P$. If P is an N-cancellation preprime (more generally, if $2x \in P$ implies $x \in P$) then B_P is a subring and J_P is an ideal in B_P; see [4; § 2]. P is (AC), or *Archimedean in the sense of Clifford*, when $P^* = P$, is (AW), or *Archimedean in the weak sense*, when $J_P = \{0\}$. From the observation just above we see that an (AW) preprime is conic. Call P *Archi-*

medean whenever $P \subset B_P$. In our first note we used "Archimedean" to mean what we are now calling "full Archimedean"; thus a *Stone ring* is a pair (A, P) where P is an infinite conic preprime which is full, Archimedean, and (AC). To say that P is a full Archimedean preprime is equivalent to the condition $A = B_P$. An element p (necessarily in P) is *large* if $p - n$ belongs to P for all n in N. P is (GAC), where G stands for 'generalized', provided for each p in P, the set $\{x;$ for all n in $N, p + nx \in P\}$ is contained in P (therefore equal to P). In the special case that A is a *field* (commutative or not) the preprime P in A is *primitive* if it is not contained in any proper subfield of A (this is weaker than full, for if P is full then it is contained in no proper sub*ring* of A), and P is a *division preprime* if it contains the reciprocal of each of its nonzero members. In the general ring situation again, we call P an *order* if it is conic and $P \cup (-P) = A$. Our Theorem 2.1 of [4] becomes: *If P is a division cone in the commutative field A then P is primitive if and only if A is the field of quotients of B_P.* For an order P in a field A all these Archimedean-like conditions are equivalent to the condition that P contains no large elements. If P is full Archimedean in A then we denote by P/J the set of all cosets y in A/J $(J = J_P)$ such that y contains a member of P.

3. Proof of Theorem 1 and its corollaries. Let P be an (infinite) N-cancellation preprime in a ring A. [For example, let O be a non-Archimedean order in a field F, and let p be a large element. Then p and $-p^{-1}$ belong to O^* but their product -1 does not. Thus P^* is not always closed under multiplication.] We state without proof some easy properties of P^*: (1) for each n_1 and n_2 in N, $P^* = \{x \in A;$ for all $n \in N, n_1 + n(n_2 x) \in P\}$; (2) -1 is not in P^*; (3) P^* contains P and is additively closed; (4) if P^* is a preprime then $B_{P^*} = B_P$, $J_{P^*} = J_P$ and P^* is an infinite N-cancellation (AC) preprime, which is conic if and only if P is (AW). These are due to Harrison [8].

A. Next suppose that P is Archimedean, still assuming that P is an infinite N-cancellation preprime. We have to prove that P^* is an Archimedean N-cancellation preprime. By (2) and (3), proof that P^* is a preprime requires only closure under multiplication. Let x and y be members of P^*. By (2) and (3) again, $x + y$ belongs to P^* and so $1 + x + y$ is in P. By the Archimedean property of P there exists m in N with $m - 1 - x - y$ in P. Let n be arbitrary in N. Then $1 + nx, 1 + ny$ and $n - 1$ all belong to P and consequently $n(1 + x + y + nxy) = n - 1 + (1 + nx)(1 + ny)$ is also in P. From the left side of this equality cancel n and then add $m - 1 - x - y$ to get $m + nxy$ in P. By (1) above, xy belongs to P. If x is any member of P^*

then $1 + x$ is in P and there exists m in N with $m - 1 - x$ in P. This implies that $m - x$ belongs to P, hence to P^*; hence P^* is Archimedean. If P is full Archimedean then P^* is clearly also full Archimedean since $B_{P^*} = B_P = A$. The rest of part A is included in (4) above, now that we have shown that P^* is a preprime.

B. Let P be a full Archimedean N-cancellation preprime in A. From what we just proved we may assume that P is also (AC). Put $J = J_P$, $B = B_P$. Let x be an element of P and suppose $x - y$ belongs to J. Then for all n in N, $2 + ny = 1 + nx + (1 + n(y - x))$ belongs to P. By (1), and the (AC) condition on P, y is also in P. Hence a coset belongs to P/J if and only if all its members belong to P. It follows that -1 is not in P/J, while closure under addition and multiplication is obvious. Thus P/J is an (infinite) preprime. If $n - x$ belongs to P, for x in A and n in N then $n - (x + J)$ belongs to P/J. Hence P/J is full Archimedean. If $1 + n(x + J)$ belongs to P/J for all n then $1 + nx \in P$ for all n. Since P is (AC) this implies x in P and so $x + J$ belongs to P/J; thus P/J is (AC). The same argument shows that if $1 \pm n(x + J)$ belongs to P/J for all n then $1 + nx$ belongs to P for all n, x is thus in J, $x + J = 0$. So P/J is (AW) and is therefore a cone. We have shown that $\langle A/J, P/J \rangle$ is a Stone ring, completing the proof of part B.

C. Since $P_1 = P \cap B$ is full Archimedean in B the assertion C is an immediate consequence of part A.

D. This is deduced from part B by observing that an Archimedean P is full Archimedean in the ring $P - P$.

Proof of Corollary 1. A. Since for any x, $1 + x^2$ is invertible we may write $x = x(1 + x^2)^{-1}(1 + x^2)$; putting $a = x(1 + x^2)^{-1}$, $b = (1 + x^2)^{-1}$, we have $x = ab^{-1}$. Theorem $1C$ guarantees that B is commutative. Hence the proof will be complete if we show that each of a and b belongs to B. Since P contains all squares, $b = (1 + x^2)^{-1} = (1 + x^2)^{-2} + x^2(1 + x^2)^{-2}$ is positive which implies positivity of $1 - b = x^2(1 + x^2)^{-1}$. Thus $1 \pm b \in P$, $b \in B$. Also $4[1 \pm x(1 + x^2)^{-1}] = (1 + x^2)^{-1}(4x^2 \pm 4x + 4) = (1 + x^2)^{-1}[(2x \pm 1)^2 + 3] \in P$. By cancellation we see that $1 \pm a \in P$, $a \in B$.

B. Immediate from part A.

C. As in part A, the representation $p = [p(1 + p)^{-1}][(1 + p)^{-1}]^{-1}$ has the form ab^{-1}, with a and b in B, provided p belongs to P; in fact, $0 \leqq (1 + p)^{-1} \leqq 1$, $0 \leqq p(1 + p)^{-1} \leqq 1$ is valid. This shows that the elements of P commute with each other. By the full-infinite property, any two elements x and y can be written $x = p - q$, $y = p' - q'$, p, q, p', q' all in P. But then certainly $xy = yx$. Hence A is commutative.

D.   The argument of part C shows that if M is the smallest subring of A containing P then M is commutative, and so, therefore, is its field of quotients.   Since P is primitive this last field is A.

E.   The hypotheses imply that $A = B$.   Since J is an ideal in the simple ring B, and 1 is not in J, J is null.   That is, P is (AW) so Theorem 1C applies to show that $A = B$ is commutative.

Proof of Corollary 2.   For each s in the center of F, let

$$P\langle s \rangle = \{p + qs;\ p, q, qs^2 \text{ belong to } P\}\ .$$

Then simple computations show that $P\langle s \rangle$ is an additively and multi-plicatively closed subset of F containing P.   We show that for every s in C, *at least one of $P\langle s \rangle$ and $P\langle -s \rangle$ is equal to P*.   Suppose both contain P properly so that -1 belongs to both.   From membership of -1 in $P\langle s \rangle$ follows $-1 = p + qs$ for some p, q in $P, q \neq 0$.   Hence $s = -q^{-1}(1 + p) = -(1 + p)q^{-1}$.   Since also -1 belongs to $P\langle -s \rangle$, we get $-s = -v^{-1}(1 + u) = -(1 + u)v^{-1}$ for some u, v in P, $v \neq 0$. Combining these we find $(1 + u)q = -v(1 + p)$, so $(1 + u)q$ belongs to $P \cap (-P)$, hence is zero.   But q is not zero so $u = -1$, contrary to $u \in P$.

If p is any nonzero member of P_C then $-p$ is outside P so by Theorem 07, p^{-1} belongs to P, and in fact p^{-1} belongs to P_C.   Thus P_C is a division cone in C.   Since P_C is also a primitive (AC) cone Theorem 05 applies to guarantee that P_C contains all squares in C, whence $P\langle s \rangle = \{p + qs;\ p, q \in P\}$.   For every s in $C, s = 0 + 1s$ belongs to $P\langle s \rangle$.   The italicized statement of the first paragraph shows that for every s, either s or $-s$ belongs to P_C (s in C); this means that P_C is an order in C.   The (AC) condition implies that P_C is an Archimedean order in C.

Proof of Corollary 3.   Let $x \in F$.   The hypotheses imply that P_x is a full-infinite (AC) conic prime in $C(x)$.   By Corollary 2, P_x is an Archimedean order in $C(x)$ ($C(x)$ is commutative).   If x is any member of F, there accordingly exists an integer n with $n - x$ in $P_x \subset P$.   This implies that P is itself Archimedean.   Moreover, either x or $-x$ belongs to P_x, and P is therefore an order.

4.   Proof of Theorem 2.   The first claim is trivial, since if $p > n$ holds for all n in N then $p + n(-1) \in P$ for all n so, by (GAC), $-1 \in P$, contradiction.   Now let P be a prime with $P \cap (-P) = (0)$, $P \neq \{0\}$.   Then P is not finite, since finite primes are additive sub-groups according to Theorem 06.   Hence 1 belongs to P, and $n \cdot 1$ for $n \in N$, cannot be zero, since $n \cdot 1 = 0$ implies $-1 = (n - 1) \cdot 1 \in P$.   Let P' be the set of all x in A such that for some n in N, $nx \in P$.   This

is obviously closed under addition and multiplication. From -1 in P' follows $n(-1) \in P$ for some n. But $n \cdot 1 \in P$ so $n \cdot 1 = 0$, contrary to what was just proved. So P' is a preprime which contains the prime P. Thus $P' = P$ and we have proved that P is an N-cancellation cone. That is, *if P is a nonzero prime and $P \cap (-P) = (0)$, then P is infinite and is an N-cancellation cone.* Now suppose further that there are no large elements in P. In order to deduce (GAC) it is necessary and sufficient to show that the set P_1 of all x such that for some p in P, $p + nx \in P$ holds for all n in N, is equal to P. If x belongs to P then $0 + nx$ belongs also to P for all n so P is contained in P_1. Next we show that P_1 is a preprime and the proof is then complete, since P is a prime. Let x and y be members P_1, say $p + nx$, $q + ny$ belong to P for all n and p and q belong to P. For all n, $p + q + n(x + y)$ belongs to P, $p + q$ belongs to P, so $x + y$ is in P_1. Also members of P are $pq + nxq = (p + nx)q$, $pq + npy = p(q + ny)$ and $pq + n(xq + py) + n^2xy = (p + nx)(q + ny)$, for all n in N; putting $n = 2$ in the first two expressions we see that $2(pq + xq + py) = (pq + 2xq) + (pq + 2py)$ belongs to P. By cancellation, $pq + xq + py$ is in P. Also $npq + n(xq + py) + n^2xy \geqq pq + n(xq + py) + n^2xy$, the last belonging to P. Cancelling n we get $pq + xq + py + nxy$ in P. We just showed that P contains $pq + xq + py$, and n is arbitrary. Hence xy belongs to P_1. Finally, if -1 belonged to P_1 then for some p in P we would have $p + n(-1) \in P$ for all n, which would imply $p > n$ for all n, and this asserts that p is large, contrary to our hypothesis that P contains no large elements. This completes the proof that P_1 is a preprime and the theorem is proved.

5. Proof of Theorem 3. Since F is a field, the ideal J in $F = B_P$ is null, and $\langle F, P^* \rangle$ is a Stone ring by Theorem 1B. Identify F with a subfield of $C(X)$, $X = X_F$. Then F is dense in $C(X)$, P is dense in P^*. Since F is a field, every nonzero member of F vanishes nowhere in X. If f is a nonzero member of P^* then, by compactness of X, there exists m in N such that $f(x) > 1/m$ holds for all x in X. Since P^* is the set of all nonnegative functions in F, $f - 1/m$ belongs to P^* and hence $f = (f - 1/m) + 1/m$ belongs to P. This shows that $P^* = P$. Clearly P^* is a division cone in F. Let x_0 be fixed in X and define φ, $F \to R$ = reals, by $\varphi(f) = f(x_0)$. This is a field monomorphism which establishes an isomorphism of F with the subfield $\varphi(F)$ of R. Let x_1 and x_2 be distinct points of X. The subring F, as a dense subring of $C(X)$ containing the rational constant functions, separates points in X, since X is a compact Hausdorff space. Hence there exists a function f in F with $f(x_1) \neq f(x_2)$, and so there is a rational r between $f(x_1)$ and $f(x_2)$. But $f - r$ is a continuous function on X

which has opposite signs at x_1 and x_2, but has no zeros. Thus no connected subset of X contains both x_1 and x_2, and we have proved that X is totally disconnected. By compactness, X has a base of open-and-closed sets [6; Th. 16.17]. Let T be such a base. For every open set G in T, the characteristic function of G is continuous so we see that the power of $C(X)$ is at least as great as the power of T. But $C(X)$, as completion of the metric space F, has power at most $c^{\aleph_0} = c$, since $|F| \leq c$ (F is a subfield of R). We have shown that every base of open-and-closed sets has power at most c.

Now for the converse, let T be a base of power at most c of open-and-closed sets in a compact Hausdorff space X. Let the members of T be indexed by ordinals:

$$G_1, G_2, \cdots, G_\xi, \cdots; (\xi < \varLambda)$$

where $\varLambda$ is the least ordinal with the same power as T, and let

$$t_{11}, t_{12}, t_{21}, t_{22}, \cdots, t_{\xi 1}, t_{\xi 2}, \cdots; (\xi < \varLambda)$$

be a well-ordered algebraically independent sequence of transcendental reals. For each ξ define f_ξ by the formula: $f_\xi(x) = t_{\xi 1}$ if x belongs to G_ξ, $f_\xi(x) = t_{\xi 2}$ if x is not in G_ξ. These functions are continuous. Let D be the subring of $C(X)$ generated by all the f_ξ and the rational-valued constant functions. To show that D is dense in $C(X)$ we need only show that D separates points in X and apply one form of the Stone-Weierstrass theorem. If $x_1 \neq x_2$ then for some ξ, x_1 belongs to G_ξ and x_2 does not. Then $f_\xi(x_1) = t_{\xi 1}, f_\xi(x_2) = t_{\xi 2} \neq t_{\xi 1}$. Thus D separates points. It remains only to show that no nonzero member of D has any zeros and then put F equal to the field of quotients of D in $C(X)$. Let f be a member of D. Then f is a polynomial in the f_ξ with rational coefficients. An equality $f(x) = 0$, for x in X, implies an algebraic relation among the $f_\xi(x)$. But for any x, the $f_\xi(x)$ are algebraically independent, as distinct entries in the well ordered list $t_{11}, t_{12}, \cdots$. Hence if $f(x) = 0$ then all the rational coefficients for f are zero, $f = 0$.

Proof of the corollary. Obvious in view of Theorem 3 and the uniqueness assertion of Theorem 09.

6. **Examples.** We recall briefly Hilbert's construction [9; § 33]. Let F be a field (not necessarily commutative), let P be an infinite preprime in F, let σ be an automorphism of F such that $\sigma(x)$ belongs to P if and only if x belongs to P. In the additive group of all formal Laurent series $\sum a_k t^k$ with only finitely many negative powers, the usual multiplication is altered as follows: instead of assuming that

the indeterminate t commutes with every element of F we make $t^k a = \sigma^k(a) t^k$ for all integers k, and all a in F. The resulting structure is a field $H = H(F, \sigma)$ which is noncommutative unless F is commutative and σ is the identity. In H set P_1 equal to the set of all series whose lowest coefficient belongs to P and denote by P_2 the set of all series whose lowest coefficient belongs to P. Then always P_2 is properly contained in P_1, and both are infinite preprimes. If P is conic then so are P_1 and P_2; if P is an order then P_1 is an order but P_2 is never an order (nor even a prime). Suppose now that P is a cone. Then neither P_1 nor P_2 can be Archimedean while P_1 can never be (AC): $1 + n(-t)$ belongs to P_1 for all n but $-t$ is not in P_1. In case P is an order and σ is not the identity, then P_1 gives a (non-Archimedean) ordering of the noncommutative field H. This is Hilbert's example. Note further that if P is full then so are P_1 and P_2. In the examples below Q is the field of all rational numbers.

EXAMPLE 1. Take $F = Q(\sqrt{2})$, let P be the intersection of the two distinct orders in F: $P = \{a + b\sqrt{2}; a \pm b\sqrt{2}$ are positive reals$\}$, and let $\sigma(a + b\sqrt{2}) = a - b\sqrt{2}$. Then H is noncommutative and P_2 is a full-infinite (AC) N-cancellation cone in H.

EXAMPLE 2. The set P of all $n \times n$ matrices with every entry a nonnegative real is a full-infinite (AC) conic N-cancellation prime in the noncommutative simple ring A of all real $n \times n$ matrices. It is not Archimedean.

EXAMPLE 3. The set P of all positive reals in the field U_R of all real quaternions is an Archimedean N-cancellation (AC) conic prime in a noncommutative field. It is not full-infinite.

EXAMPLE 4. The set P of all lower triangular $n \times n$ matrices containing the zero matrix and all those with strictly positive real entries on the diagonal is an Archimedean full-infinite N-cancellation cone in the noncommutative ring of all real lower triangular $n \times n$ matrices. P is not (AW) since every nilpotent element is infinitesimal.

EXAMPLE 5. (*Dubois* [4].) Let F be a commutative field, let P be an N-cancellation cone in F with $J_P \neq \{0\}$, and let θ be integral over B_p of degree n. In $F(\theta)$,

$$P(\theta) = \{p(1 + \sum_{i=1}^{n-1} a_i \theta^i); p \in P, a_i \in J_P\}$$

is an N-cancellation cone. If P is full-infinite or a division cone then

so is $P(\theta)$. For the special case of the following example, more is
true:

EXAMPLE 6. (*Harrison* [7].) Let P be a non-Archimedean order
in the field F, let $\theta = i$, $i^2 = -1$. Then $P(i)$ is a *prime* in $F(i)$.
Example 8 shows that P can be a prime while $P(\theta)$ is not a prime.
But the following is plausible: with the notation of Example 5, let P
be an order and suppose $F(\theta)$ is not formally real (i.e., suppose $F(\theta)$
cannot be ordered). Then $P(\theta)$ is (we conjecture) a prime.

EXAMPLE 7. With F and P as in Example 6, let U_F be the field
of quaternions over F, with generators $1, k_1, k_2, k_3, k_i^2 = -1$, and let

$$P' = \{p(1 + \textstyle\sum a_i k_i); a_i \in J_P\} \ .$$

Then P' is a full-infinite N-cancellation conic division prime in U_F.

Note. By Theorem 07, for a prime in a field the division and
conic properties are equivalent.

EXAMPLE 8. Let $F = Q(x)$, where x is an indeterminate and let
P be the order containing x as an infinitesimal. The cone $P(\sqrt{2})$ of
of Example 5 is not a prime since it is properly contained in the
order P_0 in $Q(x, \sqrt{2})$ which induces the usual (real) ordering of $Q(\sqrt{2})$
and contains x as an infinitesimal.

An immediate corollary of Theorem 05 is that *any primitive* (AC)
conic division prime in a commutative field is an Archimedean order.
Corollary 2 generalizes this as follows: *Let P be a primitive* (AC) *conic
prime in a commutative field F. Then P is an Archimedean order
in F*.

EXAMPLE 9. Let F be the (commutative) field of all formal
Laurent series in x, with coefficients in Q and let P be the set of all
such series with all coefficients nonnegative. Then P is a primitive
(in fact a full-infinite)(AC) cone in F. But $(1 + x^2)^{-1}$ is a sum of
squares that is not in P. Thus the division condition cannot be deleted
from Theorem 05.

EXAMPLE 10. Let F and P be the same as in Example 9, let
$Q(x)$ be the subfield of all rational functions in F, and let $S =
P \cap Q(x)$. Then S is a full-infinite (AC) cone in $Q(x)$. But $Q(x)$ is not
the field of quotients of B_S; in fact $B_S = Q$.

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