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SOME PROPERTIES OF SEQUENCES, WITH AN APPLICATION TO NONCONTINUABLE POWER SERIES

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For a real sequence $f = \{f(n)\}$ and positive integer N, let F^N denote the sequence of N-tuples $\{(f(n+1), \cdots, f(n+N))\}$. A functional equation method due to Kemperman is used to obtain a sufficient condition on s in order that s^N have an independent N-tuple among its cluster points. If a bounded s has the latter property, and if $g = rs$, where $r(n) \to \infty$ and $r(n+1)/r(n) \to 1$ as $n \to \infty$, then there is a subsequence S of the sequence of positive integers such that, for almost all real α, the restriction of αg^N to S is uniformly distributed (mod 1) **in the N-cube.**

Let F be an analytic function whose Maclaurin series has bounded coefficients $\{a_n\}$ which satisfy the additional requirement

$$\lim_{M\to\infty} \inf_{0 \leqq k < \infty} \sum_{n=k}^{k+M} |a_n| = \infty .$$

If $a_n = |a_n| \exp\{2\pi i f(n)\}$, then the density (mod 1) of f^N for each N is sufficient in order that F have the unit circle as a natural boundary. Hence, a metric result for noncontinuable series is obtained from the results for sequences.

1. Notation.

For x real, let $((x)) = x - [x]$, and $e(x) = \exp(2\pi i x)$. $h_1, \cdots, h_N$ will denote an N-tuple of integers, not all of which are zero. The sequence of nonnegative integers will be denoted by Z, and subsequences of Z by S_1, S_2, etc. For a real sequence f, we denote by Δf the sequence $\{f(n+1) - f(n)\}$ and

$$\Delta^{j+1} f = \Delta(\Delta^j f) , \; (j = 1, 2, \cdots)$$

2. The property (PN).

DEFINITION. A bounded sequence s of real numbers will be said to have *property* (PN) if there is an independent N-tuple among the cluster points of s^N. In other words, s has property (PN) if there is a subsequence S of Z such that for every N-tuple $h_1, \cdots, h_N$ of integers not all zero, there holds

$$\lim_{n\to\infty} |h_1 s(n+1) + \cdots + h_N s(n+N)| > 0 , \qquad (n \in S) . \tag{2.1}$$

We shall be interested in sequences s of the following form:

$$(2.2) \qquad s(n) = \varphi(\psi(n)) , \qquad (n \in Z) ,$$

where φ is a function of period 1 with at most a nowhere dense set of points of discontinuity, and ψ has the property (QN).

(QN) There exists a subsequence S_1 of Z such that

(2.3) (i) $\Delta^j \psi(n)$ converges (mod 1) for $n \to \infty$, $n \in S_1$ $(j = 2, \cdots, N)$

(ii) $\{(((\psi(n)), ((\Delta\psi(n))): n \in S_1\}$ is not nowhere dense.

THEOREM 2.1. *Let s be of the form* (2.2), *where φ and ψ have the properties listed above. Then either s has property* (PN), *or else φ agrees on some interval $I \subset [0, 1]$ with a polynomial of degree N-2 at most.*

Proof. Under the conditions on φ and ψ, it is possible to obtain a subsequence S_2 of S_1 and an open disk D in the plane such that

(2.4) (i) $\lim_{n\to\infty} \Delta^j \psi(n) = \tau_j$ (mod 1), $(n \in S_2)$, $(j = 2, \cdots, N)$,

(ii) $\{(((\psi(n)), ((\Delta\psi(n))): n \in S_2\}$ is dense in D,

(iii) for every (τ_0, τ_1) in D, and every p, $1 \leqq p \leqq N$, the point

$$\tau_0 + p\tau_1 + \sum_{j=2}^{p} \binom{p}{j} \tau_j$$

is a point of continuity for φ.

For each (τ_0, τ_1) in D, a subsequence $S_3 = S_3(\tau_0, \tau_1)$ of S_2 can be chosen so that the corresponding subsequence of (2.4 (ii)) converges to (τ_0, τ_1). In this case, as $n \to \infty$, $n \in S_3$, one has for every $h_1, \cdots, h_N$,

$$\lim_{n\to\infty} \sum_{p=1}^{N} h_p s(n + p) = \lim_{n\to\infty} \sum_{p=1}^{N} h_p \varphi\Big(\psi(n) + p\Delta\psi(n) + \sum_{j=2}^{p} \binom{p}{j} \Delta^j \psi(n)\Big)$$

so that

$$(2.5) \qquad \lim_{n\to\infty} \sum_{p=1}^{N} h_p s(n + p) = \sum_{p=1}^{N} h_p \varphi\Big(\tau_0 + p\tau_1 + \sum_{j=2}^{p} \binom{p}{j} \tau_j\Big), \qquad (n \in S_3) .$$

Suppose now that s does not have property (PN). Then for each (τ_0, τ_1) in D, there is an N-tuple $h_1, \cdots, h_N$ such that the right hand member of (2.5) is zero. Hence D is a countable union of closed sets

$$F = F(h_1, \cdots, h_N) = \{(\tau_0, \tau_1) \in D: (2.5) \text{ vanishes}\} .$$

Some F, then, must contain an open subdisk D_1, with center

(τ_0', τ_1'). That is, there exists an N-tuple $h_1, \cdots, h_N$ of integers not all zero with the property that for all sufficiently small positive h and k,

$$\sum_{p=1}^{N} h_p \varphi\Big(h + pk + \tau_0' + p\tau_1' + \sum_{j=2}^{p} \binom{p}{j} \tau_j\Big) = 0 .$$

The assertion of the theorem follows upon taking

$$\varphi_p(x) = h_p \varphi\Big(x + \tau_0' + p\tau_1' + \sum_{j=2}^{p} \binom{p}{j} \tau_j\Big)$$

in the following lemma, a weak version of one due to Kemperman [4, p. 41]. The proof is included for completeness.

LEMMA. *Let $a > 0$, and let $\varphi_1, \cdots, \varphi_N$ be real functions, with φ_j defined and continuous on $I_j = (-(j+1)a, (j+1)a)$, $(j = 1, \cdots, N)$. Suppose that for all x, y in $(-a, a)$, there holds*

$$\sum_{j=1}^{N} \varphi_j(x + jy) = 0 . \tag{2.6}$$

Then φ_j is equal on I_j to a polynomial of degree N-2 at most.

Proof. We may suppose that $N \geqq 2$ (the case $N = 1$ is trivial), and that the lemma holds for $N - 1$. Let $0 < b < a$, and let $I_j' = (-(j+1)b, (j+1)b)$.

Next, we choose and keep fixed a number h, $0 < h < \min(b, a - b)$. For this h, and $j = 1, \cdots, N$, let

$$\tilde{\varphi}_j(x) = \varphi_j(x + (1 - j/N)h) - \varphi_j(x), \qquad (x \in I_j') .$$

We note that each $\tilde{\varphi}_j$ is continuous, and $\tilde{\varphi}_N \equiv 0$. Moreover, if x, y are in $(-b, b)$, then $x, y, x + h$, and $y - h/N$ are in $(-a, a)$. Thus, for all x, y in $(-b, b)$, we have

$$\sum_{j=1}^{N-1} \tilde{\varphi}_j(x + jy) = \sum_{j=1}^{N} \varphi_j(x + h + j(y - h/N)) - \sum_{j=1}^{N} \varphi_j(x + jy) = 0 .$$

The induction hypothesis implies that, for $j = 1, \cdots, N-1, \tilde{\varphi}_j$ is a polynomial of degree $N - 3$ at most on I_j'. Hence φ_j is, on I_j', the sum of a polynomial of degree $N - 2$ at most and a function of period $(1 - j/N)h$. But such a representation is given for every sufficiently small positive h, which, with the continuity of φ_j, implies that φ_j is a polynomial of degree $N - 2$ at most on I_j', $(1 \leqq j \leqq N - 1)$. From the arbitrariness of b, φ_j is such a polynomial on I_j. Finally, (2.6) shows that φ_N is also such a polynomial on I_N.

In a previous paper [1], results of *v. d.* Corput were used to obtain various sufficient conditions on a real sequence ψ in order that ψ satisfy condition (I):

(I) *There exists a sequence* S *such that* $\lim \Delta^j\psi(n)$ $(n \in S)$ *exists for all* $j \geqq r$, *while* $\{(\psi(n), \Delta\psi(n), \cdots, \Delta^{r-1}\psi(n)): n \in S\}$ *is uniformly distributed* (mod 1) *in the r-dimensional unit cube.*

(I) clearly implies that ψ has property (QN) for every $N \geqq 2$. The reader is referred to the paper for details and proofs.

3. A metric result for uniform distribution in the N-cube.

THEOREM 3.1 *Let* $g = \{g(n): n \in Z\}$ *be a sequence of real numbers. Let there exist a subsequence* S_0 *of* Z *such that, for every N-tuple* $h_1, \cdots, h_N$ *of integers not all zero there holds*

$$\lim \left| \sum_{p=1}^{N} h_p g(n+p) \right| = \infty\,, \qquad \text{as } n \to \infty\,, \quad n \in S_0\,. \tag{3.1}$$

Then there exists a subsequence S *of* S_0 *such that, for almost all real* α, *the sequence* $(\alpha g^N) \mid S$ *is uniformly distributed* (mod 1) *in the N-cube.*

Proof. Let the set of all such N-tuples be ordered, with, say, $h_1', \cdots, h_N'$ as the first. Let a subsequence $S_1 \subset S_0$ be taken such that

$$\sum_{p=1}^{N} h_p'\{g(n+p) - g(m+p)\}$$

is either greater than 1 for every n, m in S_1, with $n > m$, or else is less than -1 for every such n and m. Successively extracting subsequences $S_1 \supset S_2 \supset \cdots$ in this way, and then using a diagonal procedure, one finally obtains a sequence S such that, for *every* N-tuple $h_1, \cdots, h_N$, there is an $m_0 = m_0(h_1, \cdots, h_N)$ such that one has either

$$\sum_{p=1}^{N} h_p\{g(n+p) - g(m+p)\} \geqq 1 \tag{3.2}$$

for all n *and* m *in* S *with* $n > m \geqq m_0$
or else

$$\sum_{p=1}^{N} h_p\{g(n+p) - g(m+p)\} \leqq -1 \tag{3.3}$$

for all such n *and* m.

By a well-known result of Weyl [6, p. 348], either condition (3.2) or (3.3) implies that, for almost all real α, the sequence

$$\alpha \sum_{p=1}^{N} h_p g(n+p) \qquad (n \in S) \tag{3.4}$$

is uniformly distributed (mod 1). There being only countably many N-tuples, it follows that, for almost all α, (3.4) is uniformly distributed (mod 1) for every N-tuple $h_1, \cdots, h_N$. But this shows [2, p. 66] that for almost all α the sequence $(\alpha g^N) \mid S$ is uniformly distributed (mod 1) in the N-cube.

It is easy to see that if $\theta > 1$ is a transcendental number and $g(n) = \theta^n$, then Theorem 3.1 is applicable. The next result shows the less obvious fact that Theorem 3.1 also applies if, for instance, $g(n) = n^3 \log n \sin n^2$.

THEOREM 3.2. *Let* $g = \{g(n): n \in z\}$ *be of the form*

$$g(n) = r(n)s(n) , \qquad n \in Z , \tag{3.5}$$

where s *has property* (PN), *while*

$$\lim r(n) = \infty , \qquad \lim (r(n + 1)/r(n)) = 1 . \tag{3.6}$$

Then there is a subsequence S_0 of Z *such that* (3.1) *holds for every* N-*tuple* $h_1, \cdots, h_N$ *of integers not all zero.*

Proof. For $p = 1, 2, \cdots, N$, it follows from (3.6) that

$$r(n + p) = r(n)(1 + 0(1)) , \quad \text{as } n \to \infty .$$

Therefore we have

$$g(n + p) = r(n)s(n + p)(1 + o(1)) , \quad \text{as } n \to \infty , \quad p = 1, \cdots, N. \tag{3.7}$$

Since s has property (PN), there exists a subsequence S_0 of Z such that

$$\lim_{n\to\infty} \mid h_1 s(n + 1) + \cdots + h_N s(n + N) \mid > 0 , \qquad (n \in S_0) \tag{2.1}$$

for all N-tuples $h_1, \cdots, h_N$ of integers not all zero. But (3.6), (3.7), and (2.1) imply (3.1).

4. An application to noncontinuable power series. Perry [5] has proved that, for every real sequence $f = \{f(n): n \in Z\}$, there exists a sequence of moduli $\{\mid a_n \mid: n \in Z\}$ such that the power series

$$\sum_{n=0}^{\infty} \mid a_N \mid e(f(n))z^n \tag{4.1}$$

has radius of convergence 1 and the analytic function it represents can be continued analytically across a semicircle of the unit circle. However, if the additional requirements

$$\mid a_n \mid = 0(1) \qquad \text{as } n \to \infty \tag{4.2}$$

and

$$\lim_{N\to\infty} \inf_{0\leq k<\infty} \sum_{n=k+1}^{k+N} |a_n| = \infty \tag{4.3}$$

are imposed, then there are conditions on f sufficient that (4.1) represent a function with $|z| = 1$ as its natural boundary. Some such conditions were given in [1]. Theorem 4 gives a metric result in this direction.

THEOREM 4. *Let* $\{|a_n|: n \in Z\}$ *satisfy* (4.2) *and* (4.3). *Let* g *be a real sequence which, for each* N, *satifies the hypothesis of Theorem* 3.1. *For each real* α, *let*

$$F_\alpha(z) = \sum_{n=0}^{\infty} |a_n| e(\alpha g(n)) z^n, \qquad |z| < 1 . \tag{4.4}$$

Then the set of α *for which* F_α *can be continued across an arc of the unit circle has measure zero.*

Example. $\sum e(\alpha n \sin n^2) z^n$ has $|z| = 1$ as its natural boundary for almost all α.

For $N = 2, 3, \cdots$, let A_N be the set of those real α for which αg^N is dense (mod 1) in the unit N-cube.

By Theorem 3.1, A_N contains almost all α, and it follows that almost all α are in A_N for *every* N. For each such α, and each $z_0 = e(\theta_0)$, there holds

$$\limsup_{k\to\infty} \left| \sum_{k+1}^{k+N} a_n e(\alpha g(n) + n\theta_0) \right| \geqq \liminf_{k\to\infty} \sum_{k+1}^{k+N} |a_n| . \tag{4.5}$$

In view of (4.3), (4.5) shows that the partial sums of the series in (4.4) are unbounded at z_0. By (4.2) and a well-known theorem of Fatou [3, p. 391], it follows that z_0 is a singularity for F.

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