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THE SPECTRAL RADIUS OF AVERAGING OPERATORS

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This paper is concerned with the properties of certain operators which act on rearrangement invariant spaces of functions. The spectral radius of such an operator is determined precisely in terms of the spectral radius of the translation operator E_s. A new inequality is obtained for the norm of the iterates of the averaging operator P in terms of the norm of E_s.

We consider further properties of the operators P_a and Q_a which were exploited in [2] (with a slightly different notation).

If f is a measurable function defined on $R^+ = [0, \infty)$, then $P_a f$ and $Q_a f$ are given by the following:

$$P_a f(t) = \int_0^1 s^{-a} f(st) ds = t^{a-1} \int_0^t s^{-a} f(s) ds \tag{1}$$

$$Q_a f(t) = \int_1^\infty s^{a-1} f(st) ds = t^{-a} \int_t^\infty s^{a-1} f(s) ds \ , \tag{2}$$

whenever the defining integrals exist a.e.

We define the translation operator E_s by

$$E_s f(t) = f(st), \qquad 0 < s < \infty, \qquad t \in R^+ \ . \tag{3}$$

Thus, in some sense which need not be made precise here,

$$P_a = \int_0^1 s^{-a} E_s ds, \quad \text{and} \quad Q_a = \int_1^\infty s^{a-1} E_s ds \ . \tag{4}$$

Now, suppose that X is a rearrangement invariant Banach space as defined in [1]. It was shown in [2] that it is important to know whether or not the operators P_a and Q_a define continuous mappings of X into itself. In terms of the norm of E_s, which we write as $h(s) = \| E_s \|$, a sufficient condition in order that $\| P_a \| < \infty$ is that

$$\int_0^1 s^{-a} h(s) ds < \infty \ , \tag{5}$$

and a bound for the norm is given by

$$\| P_a \| \leqq \int_0^1 s^{-a} h(s) ds \ . \tag{6}$$

A similar expression holds for $\| Q_a \|$. This is a consequence of Theorem 3.1 of [1]. Here, we will always adopt the convention that if P_a is not a bounded operator from all of X into itself then $\| P_a \| = \infty$.

For the special case $a = 0$, it was shown in [1], that (5) is a necessary condition for $\| P_0 \| < \infty$, and in fact

$$\| P_0 \| \leqq \int_0^1 h(s)ds \leqq 2\sqrt{2} \, \| P_0 \| \, . \tag{7}$$

Our first object here is to show that (5) is a necessary condition for $\| P_a \| < \infty$ for all real a. This is Theorem 1. Notice we need only consider P_a for $0 \leqq a < 1$, since if $a \geqq 1$ P_a is not defined for all of X, since X contains characteristic functions of intervals, and so in order that $\| P_a \| < \infty$ for any such X, we must have

$$\int_0^1 s^{-a} ds < \infty \, .$$

Another question we consider is the determination of conditions under which equality holds in (6), since a classical result of Hardy, Littlewood and Polya shows that when $X = L^p$ equality does hold ([3], p. 227). We show in Theorem 2 that if we consider the spectral radius of the operators involved, rather than the norm, then we do have

$$r(P_a) = \int_0^1 s^{-a} r(E_s) ds \, , \tag{8}$$

where $r(T)$ denotes the spectral radius of an operator T. As a corollary of (8) we obtain conditions under which equality holds in (6) which includes the case $X = L^p$.

A final result, which may be useful when equality does not hold in (6), is given in Theorem 3. This improves the estimate in (7) and gives a similar estimate for the iterates of P_0.

Although the results will be stated only for the operators P_a, analogous results hold for Q_a. For, if X' is the associate of X, then it can be shown that the norm of P_a as an operator in X is equal to the norm of Q_a as an operator in X'. This is because Q_a is the "transpose" of P_a. Combining this with the fact that if $h'(s)$ is the norm of E_s in X', then $h'(s) = s^{-1}h(s^{-1})$, we can obtain results for Q_a by change of variable from those for P_a.

2. Preliminary lemmas.

Recall that the spectral radius of an operator T is the smallest number r such that the disc $\{\lambda : |\lambda| \leqq r\}$ contains the spectrum of T. It is easily shown that if $\| T \|$ is the norm of T as a mapping from a space to itself, and $r(T)$ is the spectral radius of T, then

$$r(T) \leqq \| T \| \, .$$

It is important to notice that since we will be considering the same operator applied to many spaces that the spectrum, spectral radius and norm of the operator all depend on the space in which it acts. However, to avoid the constant repetition of hypotheses, it should be understood that whenever the norm or spectral radius of an operator appears, the operator is considered as a mapping from a given rearrangement invariant Banach space X to itself. That is, all the norms appearing in any given theorem refer to the same space but the theorems apply to all rearrangement invariant spaces.

LEMMA 1. *Let E_s be defined as in* (3); *let X be a rearrangement invariant space; let $h(s)$ denote the norm of E_s, and $r(E_s)$ denote the spectral radius of E_s as a mapping from X into itself. Then*

(i) $h(st) \leqq h(s)h(t)$

(ii) *$h(s)$ is nonincreasing in s, and $sh(s)$ is nondecreasing in s;*

(iii) $\min(s^{-1}, 1) \leqq h(s) \leqq \max(s^{-1}, 1), \quad 0 < s < \infty.$

(iv) *Let $\theta(s)$ denote the ratio $[-\log h(s)/\log s]$ If $\alpha = \inf_{s<1} \theta(s)$ and $\beta = \sup_{s>1} \theta(s)$, then $\alpha = \lim_{s\to 0+} \theta(s)$ and $\beta = \lim_{s\to\infty} \theta(s)$, and $0 \leqq \beta \leqq \alpha \leqq 1$*

(v) $r(E_s) = \max(s^{-\alpha}, s^{-\beta})$. *where α and β are as in* (iv).

Proof. Parts (i) to (iv) are proved in [1].

For part (v), we use Gelfand's formula for the spectral radius. Namely

$$r(T) = \lim_{n\to\infty} \| T^n \|^{1/n}$$

where T^n is the nth iterate of T. (See [4], p. 263.) Thus,

$$r(E_s) = \lim_{n\to\infty} \| E_s^n \|^{1/n} = \lim_{n\to\infty} h(s^n)^{1/n} = \begin{cases} s^{-\alpha}, & \text{if } 0 < s \leqq 1 \\ s^{-\beta}, & \text{if } 1 \leqq s < \infty \end{cases} \tag{9}$$

For by (iv), given $\varepsilon > 0$, $\exists\delta(\varepsilon) > 0$ such that for $0 < s < \delta(\varepsilon) \leqq 1$ we have

$$s^{-\alpha} \leqq h(s) \leqq s^{-\alpha-\varepsilon} .$$

But, if $s < 1$, $s^n < \delta(\varepsilon)$ for n sufficiently large so that

$$s^{-\alpha} \leqq h(s^n)^{1/n} \leqq s^{-\alpha-\varepsilon}$$

and this proves (9) for $0 < s < 1$. If $s > 1$, a similar argument applies. If $s = 1$, E_s is the identity so $h(1) = 1$.

LEMMA 2. *Let P_a be defined as in* (1), *and suppose X is a rearrangement invariant space for which $\| P_a \| < \infty$. Then, for all $f \in X$, if P_a^n denotes the* nth *iterate of P_a,*

$$P_a^n f(t) = \int_0^1 s^{-a} \frac{1}{(n-1)!} \left(\log \frac{1}{s}\right)^{n-1} f(st) ds . \tag{10}$$

Also, if λ is complex with $|\lambda| < r(P_a)^{-1}$, and I is the identity operator, then $(I - \lambda P_a)^{-1}$ exists and is a bounded operator from X into itself.

The following formula holds for real λ, with $|\lambda| < r(P_a)^{-1}$

$$P_{a+\lambda} = P_a(I - \lambda P_a)^{-1} . \tag{11}$$

Proof. Let T denote the operator defined by the right member of (10), and suppose $\| T \| < \infty$. Then

$$\begin{aligned} P_a T f(t) &= \int_0^1 u^{-a} du \int_0^1 v^{-a} \frac{1}{(n-1)!} \left(\log \frac{1}{v}\right)^{n-1} f(tuv) dv \\ &= \int_0^1 du \int_0^u s^{-a} \frac{1}{(n-1)!} \left(\log \frac{u}{s}\right)^{n-1} f(ts) \frac{ds}{u} \\ &= \int_0^1 s^{-a} \frac{1}{(n-1)!} f(st) ds \int_0^1 \left(\log \frac{u}{s}\right)^{n-1} \frac{du}{u} \\ &= \int_0^1 s^{-a} \frac{1}{n!} \left(\log \frac{1}{s}\right)^{n} f(st) ds, \quad \text{almost all } t \in R^+ , \end{aligned} \tag{12}$$

where the second line follows by the change of variable $s = uv$, and the interchange of order of integration follows from Fubini's theorem for almost all $t \in R^+$.

The fact that $T = P_a^n$ now follows easily by induction from (12) since it is true for $n = 1$.

The statement that $I - \lambda P_a$ has a bounded inverse in X for $|\lambda| < r(P_a)^{-1}$ is standard, and in fact

$$(I - \lambda P_a)^{-1} = \sum_{n=0}^{\infty} \lambda^n P_a^n , \tag{13}$$

where the series converges in operator norm. (See [4], p. 262.)

Thus, if $f \in X$,

(14)
$$\begin{aligned} &P_a(I - \lambda P_a)^{-1}f(t) \\ &= \sum_{n=0}^{\infty} \lambda^n (P_a^{n+1})f(t) \\ &= \sum \lambda^n \int_0^1 s^{-a}\frac{1}{n!}\left(\log\frac{1}{s}\right)^n f(st)ds \\ &= \int_0^1 s^{-a}f(st)\left\{\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left(\log\frac{1}{s}\right)^n\right\}ds\ , \end{aligned}$$

for almost all $t \in \boldsymbol{R}^+$,

$$= \int_0^1 s^{-a-\lambda}f(st)ds = P_{a+\lambda}f(t)\ .$$

Here equation (10) has been used in the second line and the interchange of order of summation and integration follows by Fubini's theorem.

LEMMA 3. *Let K be a measurable nonnegative function on R^+, and let T be defined by the following expression for every f for which the integral exists a.e.*

$$Tf(t) = \int_0^\infty K(s)f(st)ds\ .$$

If X is a rearrangement invariant space, and if $\|T\| < \infty$, considered as a mapping from X to itself, then

$$\max\left(h(s)\int_0^s K(t)dt,\ sh(s)\int_s^\infty K(t)\frac{dt}{t}\right) \leqq \|T\|\ .$$

Proof. The fact that

$$h(s)\int_0^s K(t)dt \leqq \|T\|$$

was proved in [1]. To establish the other inequality notice that if T' is the transpose of T then

$$T'f(t) = \int_0^\infty K\left(\frac{1}{s}\right)f(st)\frac{ds}{s}\ .$$

And, if $h'(s)$ is the norm of E_s in X', then $h'(s) = s^{-1}h(s^{-1})$. Thus,

$$h'(s)\int_0^s K\left(\frac{1}{t}\right)\frac{dt}{t} \leqq \|T'\|_{X'}\ .$$

But using the fact that $\|T'\|_{X'} = \|T\|_X$ and changing variables proves that

$$sh(s)\int_s^\infty K(t)\frac{dt}{t} \leqq ||\, T \,|| \ .$$

3. Main results.

THEOREM 1. *Suppose that* P_a, E_s *are defined by* (1). *Then*

$$||\, P_a \,|| < \infty \ \textit{if and only if} \ \int_0^1 s^{-a}h(s)ds < \infty \ .$$

Proof. We need only consider $0 \leqq a < 1$, and $||\, P_a \,|| < \infty$. Let $\varepsilon > 0$ satisfy $\varepsilon < r(P_a)^{-1}$ and $\varepsilon < 1 - a$. Then, from Lemma 2,

$$P_{a+\varepsilon} = P_a(I - \varepsilon P_a)^{-1} \quad \text{and} \quad ||\, P_{a+\varepsilon} \,|| < \infty \ .$$

Applying Lemma 3 to $T = P_{a+\varepsilon}$,

$$h(s)\int_0^s t^{-a-\varepsilon}dt \leqq ||\, P_{a+\varepsilon} \,|| \ ,$$

and thus

(15) $$h(s) \leqq cs^{-1+a+\varepsilon} \ ,$$

where $c = (1 - a - \varepsilon)\, ||\, P_{a+\varepsilon} \,|| < \infty$.

Hence, we have

$$\int_0^1 h(s)s^{-a}ds \leqq c\int_0^1 s^{\varepsilon-1}ds < \infty \ .$$

The converse follows from the inequality (6) stated earlier. To prove the next theorem we need a result which would be obvious if the entities involved were positive numbers rather than positive operators.

LEMMA 4. *Let* P_a *be defined by* (1), *and let* $\lambda > 0$ *be a positive number. Then,*

$$||\, P_{a+\lambda} \,|| < \infty \ \textit{if and only if} \ \sum_{n=0}^\infty \lambda^n \,||\, P_a^{n+1} \,|| < \infty \ .$$

Proof. If $||\, P_{a+\lambda} \,|| < \infty$, then $||\, P_{a+\lambda+\varepsilon} \,|| < \infty$ for some $\varepsilon > 0$ (and we may assume that $a + \lambda + \varepsilon < 1$ as in Theorem 1). This follows from Lemma 2. Thus, from Lemma 3,

$$h(s)\int_0^s s^{-a-\lambda-\varepsilon}ds \leqq ||\, P_{a+\lambda+\varepsilon} \,|| \ ,$$

so

$$h(s) \leqq cs^{-1+a+\lambda+\varepsilon}, \qquad \text{where } c < \infty .$$

Thus, by the expression (10) for P_a^n, and the standard inequality contained in Theorem 3.1 of [1], i.e.

$$\left\| \int_0^\infty K(s) E_s ds \right\| \leqq \int_0^\infty K(s) \| E_s \| ds ,$$

we have

$$\begin{aligned} \| P_a^n \| &\leqq \int_0^1 h(s) s^{-a} \frac{1}{(n-1)!} \left(\log \frac{1}{s} \right)^{n-1} ds \\ &\leqq C \int_0^1 s^{-1+\lambda+\varepsilon} \frac{1}{(n-1)!} \left(\log \frac{1}{s} \right)^{n-1} ds = \frac{C}{(\lambda+\varepsilon)^n} . \end{aligned} \tag{16}$$

Thus

$$\sum_{n=0}^\infty \lambda^n \| P_a^{n+1} \| < \infty ,$$

since it is majorized by a convergent geometric series.

THEOREM 2. *Let P_a, E_s be defined as in* (1), (3) *and let* $r(P_a)$, $r(E_s)$ *denote the spectral radii of these operators as mappings from a rearrangement invariant space X into itself, then*

$$r(P_a) = \int_0^1 s^{-a} r(E_s) ds .$$

Proof. By ([4], p. 262—Theorem 5.2-C),

$$\begin{aligned} [r(P_a)]^{-1} &= \sup \left\{ \lambda \in \boldsymbol{C} : \sum_{n=0}^\infty |\lambda|^n \| P_a^{n+1} \| < \infty \right\} \\ &= \sup \left\{ \lambda > 0 : \sum_{n=0}^\infty \lambda^n \| P_a^{n+1} \| < \infty \right\} . \end{aligned} \tag{17}$$

But, by Lemma 4,

$$\begin{aligned} &\sup \left\{ \lambda > 0 : \sum_{n=0}^\infty \lambda^n \| P_a^{n+1} \| < \infty \right\} \\ &\qquad = \sup \{ \lambda > 0 : \| P_{a+\lambda} \| < \infty \} = \lambda_0, \quad \text{say.} \end{aligned} \tag{18}$$

However, by Theorem 1, $\| P_{a+\lambda} \| < \infty$, if and only if

$$\int_0^1 s^{-a-\lambda} h(s) ds < \infty .$$

Thus since $r(E_s) \leqq h(s)$, and by Lemma 1, $r(E_s) = s^{-\alpha}$ if $0 \leqq s \leqq 1$, we have

$$\frac{1}{1-\alpha-a-\lambda}=\int_0^1 s^{-a-\lambda}r(E_s)ds<\infty\ , \tag{19}$$

provided $\|P_{a+\lambda}\|<\infty$.

Conversely, if

$$\int_0^1 s^{-a-\lambda-\alpha}ds<\infty\ ,$$

then $a+\lambda+\alpha<1$, and so there is an $\varepsilon>0$ such that

$$\int_0^1 s^{-a-\lambda-\alpha-\varepsilon}ds<\infty\ .$$

But this implies

$$\int_0^1 s^{-a-\lambda}h(s)ds<\infty$$

by using part (iv) of Lemma 1.

Hence $\|P_{a+\lambda}\|<\infty$ if and only if $(1-\alpha-a-\lambda)^{-1}<\infty$. Hence,

$$\lambda_0=\sup\{\lambda\colon (1-\alpha-a-\lambda)^{-1}<\infty\}=1-\alpha-a\ . \tag{20}$$

Putting together (17), (18) and (20), we obtain finally that

$$r(P_a)=\frac{1}{\lambda_0}=\frac{1}{1-\alpha-a}=\int_0^1 s^{-a-\alpha}ds \tag{21}$$
$$=\int_0^1 s^{-a}r(E_s)ds\ .$$

COROLLARY. *If* $h(s)=s^{-\alpha}$ *for* $0\leqq s\leqq 1$, *then*

$$\|P_a\|=\int_0^1 s^{-a}h(s)ds\ . \tag{22}$$

Proof. By (6),

$$\|P_a\|\leqq\int_0^1 s^{-a}h(s)ds\ .$$

But, by assumption $h(s)=s^{-\alpha}=r(E_s)$ for $0\leqq s\leqq 1$. Thus, from the theorem,

$$r(P_a)=\int_0^1 s^{-a-\alpha}ds=\int_0^1 s^{-a}h(s)ds \tag{23}$$

and, since $r(P_a)\leqq\|P_a\|$, we have

$$\int_0^1 s^{-a}h(s)ds=r(P_a)\leqq\|P_a\|\leqq\int_0^1 s^{-a}h(s)ds\ , \tag{24}$$

which proves (22).

The final result involves the operator P_0 and its iterates. We shall now write $P \equiv P_0$.

THEOREM 3. *Let K_n be defined by,*

$$P^n f(t) = \int_0^1 K_n(s) f(st) ds . \tag{25}$$

we have

$$\| P^n \| \leqq \int_0^1 K_n(s) h(s) ds \leqq \left(2 + \log\binom{2n}{n}\right) \| P^n \| , \tag{26}$$

where the norms of the operators involved are as mappings from a rearrangement invariant space X into itself.

Proof. By Lemma 2,

$$K_n(s) = \frac{1}{(n-1)!}\left(\log \frac{1}{s}\right)^{n-1} \chi_{[0,1]}(s) . \tag{27}$$

Thus, we have

$$\begin{aligned} s\int_s^\infty K_n(t)\frac{dt}{t} &= s\int_s^1 \frac{1}{(n-1)!}\left(\log \frac{1}{s}\right)^{n-1} \frac{ds}{s} \\ &= \frac{s}{n!}\left(\log \frac{1}{s}\right)^n . \end{aligned} \tag{28}$$

Thus, if $\| P^n \| < \infty$, then by Lemma 3, we have

$$s\left(\log \frac{1}{s}\right)^n h(s) \leqq \| P^n \| \, n! \tag{29}$$

For convenience, denote $C_n = \| P^n \|$. So, if $C_n < \infty$, we have $C_{2n} \leqq C_n^2 < \infty$.

Thus, applying (29) for n and $2n$, and part (iii) of Lemma 1, we obtain

$$h(s) \leqq \min\left(\frac{1}{s}, \frac{n!\, C_n}{s\left(\log \frac{1}{s}\right)^n}, \frac{(2n)!\, C_{2n}}{s\left(\log \frac{1}{s}\right)^{2n}}\right) . \tag{30}$$

Thus, if t_1, and t_2 are such that

$$\left(\log \frac{1}{t_1}\right)^n = n!\, C_n, \quad \text{and} \quad \left(\log \frac{1}{t_2}\right)^n = \frac{(2n)!\, C_{2n}}{n!\, C_n} , \tag{31}$$

then

$$
\begin{aligned}
(32) \qquad & \int_0^1 K_n(s)h(s)ds \\
&\leqq \int_0^{t_2} \frac{1}{(n-1)l}\left(\log\frac{1}{s}\right)^{-n-1}(2n)!\,C_{2n}\frac{ds}{s} \\
&\quad + \int_{t_2}^{t_1} \frac{1}{(n-1)!}\left(\log\frac{1}{s}\right)^{-1} n!\,C_n\frac{ds}{s} \\
&\quad + \int_{t_1}^{1} \frac{1}{(n-1)!}\left(\log\frac{1}{s}\right)^{n-1}\frac{ds}{s} \\
&= C_n + C_n \log\binom{2n}{n}\frac{C_{2n}}{C_n^2} + C_n \\
&\leqq C_n\left(2 + \log\binom{2n}{n}\right),
\end{aligned}
$$

where we have integrated and used the definitions (31). This proves (26) since the left hand inequality is Theorem 3.1 of [1].

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