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GROUP ALGEBRAS OF VECTOR-VALUED FUNCTIONS

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Let G be a compact group, $1 \leqq p < \infty$, and A be a Banach algebra. Define $B^p(G, A)$ to be the set of all functions, $f: G \to A$, such that $\int_G \|f(x)\|^p\, dx < \infty$. Similarly define $C(G, A)$ to be the set of all continuous functions from G to A. These sets form Banach algebras under the usual operations and convolution multiplication. This paper studies general properties of these algebras and in particular the inheritance of properties, such as structure, from the image algebra A. The techniques used, in part, involve certain topological tensor products, and the discussion is generalized to the context of more general topological tensor products.

In [9] it is seen that when A is an H^*-algebra, a natural generalization of $L^2(G)$, $B^2(G, A)$ is also an H^*-algebra, and the structure is determined accordingly. In this paper we consider natural generalizations of $L^p(G)$, viz., dual and annihilator algebras. In § 1 we review the basic properties of these algebras and discuss some new results. In § 2 we discuss basic properties of the algebras $B^p(G, A)$ and $C(G, A)$. In § 3 we discuss topological tensor products and relate them to our present problems. Finally in § 4 we examine questions concerning the structure of the above algebras and more generally determine the structure of a suitably normed tensor product of semisimple annihilator Banach algebras.

1. Preliminary ideas.

For a subset, S, of an algebra, A, we will let $\mathscr{L}(S) = \{a \in A : a \cdot S = (0)\}$ and $\mathscr{R}(S) = \{a \in A : S \cdot a = (0)\}$. if $\mathscr{L}(S) = \mathscr{R}(S)$, we will denote this set by $\mathscr{A}(S)$. $\mathscr{L}(S)$ and $\mathscr{R}(S)$ are respectively called the *left* and *right annihilator* of S, and $\mathscr{A}(S)$ is called the *annihilator* of S.

DEFINITION. A topological algebra, A, is said to be an *annihilator algebra* if for an arbitrary closed left ideal $I \subseteqq A$ and closed right ideal $J \subseteqq A$ the following conditions are satisfied:

(a) $\mathscr{R}(I) = (0)$ if and only if $I = A$,

(b) $\mathscr{L}(J) = (0)$ if and only if $J = A$.

A is said to be a *dual algebra* if

(*a*) $\mathscr{L}(\mathscr{R}(I)) = I$

(b) $\mathscr{R}(\mathscr{L}(J) = J$.

It is easily seen that every dual algebra is an annihilator algebra It has recently been shown however [1] that the converse is not in general true. For semisimple Banach algebras though, this is still an open question.

We will restrict our discussion to the consideration only of semisimple annihilator and dual Banach algebras. "Semisimplicity" is taken to mean the vanishing of the Jacobson radical. We shall also have occasion to discuss strong semisimplicity, i.e., the vanishing of the strong radical, which is the intersection of all modular maximal ideals of A.

The structure of semisimple annihilator and dual Banach algebras is well-known and is as follows (cf. [13], [2]): every semisimple annihilator Banach algebra, A, is the direct topological sum of all of its minimal closed ideals. Each of these minimal closed ideals is a topologically simple (i.e., (0) is the only proper closed ideal) annihilator algebra. If A is a dual algebra, these minimal closed ideals are also dual algebras. Every topologically simple, simisimple annihilator Banach algebra is continuously isomorphic to an algebra of operators on a reflexive Banach space. In addition, this algebra of operators contains the algebra of all bounded operators of finite rank as a uniformly dense subalgebra. Whether or not a distinction must be made between topologically simple, semisimple annihilator Banach algebras and topologically simple, semisimple dual Banach algebras seems to be still an open question.

In [9] it is shown that for an H^*-algebra, A, the following conditions are equivalent:

(1) Every minimal closed ideal of A is finite-dimensional.

(2) A is strongly semisimple.

(3) A is completely continuous (i.e., all left and right regular representation operators on A are compact.)

We now note that many of these implications hold in a more general context.

First of all, Kaplansky shows ([12], p. 699) that if A is a semisimple, completely continuous Banach algebra, then any minimal closed ideal that A may possess must be finite-dimensional. For semisimple annihilator Banach algebras, we can prove the following related result:

PROPOSITION 1. Let A be a semisimple annihilator Banach algebra. If every minimal closed ideal of A is finite-dimensional, then A is completely continuous.

Proof. Let $\{I_\alpha\}$ be the collection of minimal closed ideals of A. Under the given conditions, the left (right) regular representation operator on A corresponding to each element of $\sum_\alpha \oplus I_\alpha$ is seen to be of finite rank. Also since $\sum_\alpha \oplus I_\alpha$ is dense in A and the left (right) regular representation is continuous, the left (right) regular representation operator corresponding to any element of A is the uniform limit of operators of finite rank and thus is compact.

In this paper we will have occasion to refer to some classes of Banach algebras that have been studied but apparently have not been designated in any special way. Therefore we make the following definition:

DEFINITION. A Banach algebra is said to be a *D^*-(I^*-) algebra* if it is a semisimple dual (annihilator) Banach algebra with a continuous involution satisfying the condition:

$$a^*a = 0 \quad \text{implies} \quad a = 0 .$$

It should be remarked that several very important concrete algebras are D^*-algebras. For example, the algebra of all compact operators on a Hilbert space is seen to be a D^*-algebra if involution is defined by the usual adjoint relation ([14], p. 283). Also if G is a compact group and $1 \leqq p < \infty$, then it is easily seen (cf. [12], p. 699) that $L^p(G)$ and $C(G)$ (with convolution multiplication) are D^*-algebras if involution is defined as usual.

PROPOSITION 2. If A is a D^*-algebra, then A is strongly semisimple if and only if every minimal closed ideal of A is finite-dimensional.

Proof. Assume A is strongly semisimple, and let I be a minimal closed ideal of A. I is known to be a topologically simple D^*-algebra ([14], pp. 99, 100, and 267). Since I is an ideal of a strongly semisimple algebra, I is also strongly semisimple. But I being topologically simple requires that (0) be the only proper closed ideal. Thus (0) must be a modular maximal ideal of I, i.e., in particular I must have an identity element. This is true only if I is finite-dimensional ([14], p. 268).

Conversely assume that every minimal closed ideal of A is finite-dimensional. Let I be any such ideal. Since A is a semisimple dual algebra, $\mathscr{L}(I) = \mathscr{R}(I) = \mathscr{A}(I)$ (([14], p. 99). We will show first that $M = \mathscr{A}(I)$ is a modular maximal ideal of A. As above, I is a topologically simple D^*-algebra, and thus I being finite-dimensional implies I has an identity ([14], p. 268). Also it is true that $I \cap M =$ (0) and $I \oplus M$ is dense in A ([14], p. 99). Thus if φ_0 is the restriction to I of the canonical mapping $\varphi : A \rightarrow A/M$, φ_0 is continuous and one-to-one. Thus $\varphi_0(I)$ is dense in A/M. In fact, since I is finite-dimensional, $\varphi_0(I) = A/M$. As a result, since I has an identity,

A/M must also have an identity, i.e., M is modular. That M is also maximal follows directly from the fact that A is dual and I is a minimal closed ideal in A.

Now if $\mathfrak{R}_s$ is the strong radical of A, then $\mathfrak{R}_s$ is the intersection of all modular maximal ideals of A, which in turn is contained in the intersection of all modular maximal ideals of A that are annihilators of minimal closed ideals of A. Thus since the direct sum of all minimal closed ideals is dense in A, a short calculation (cf. [13], p. 317) shows that $\mathscr{A}(\mathfrak{R}_s) = A$. Therefore since A is a dual algebra, $\mathfrak{R}_s = (0)$.

COROLLARY 1. *If A is a D^*-algebra, then A is strongly semisimple if and only if A is completely continuous.*

COROLLARY 2. *Commutative D^*-algebras are strongly semisimple.*

Proof. This result follows immediately from the above proposition and ([14], p. 268).

COROLLARY 3. *If G is a compact group and $1 \leqq p < \infty$, then $L^p(G)$ and $C(G)$ (with convolution multiplication) are strongly semisimple and completely continuous.*

Proof. The above proposition and ([12], p. 700) provide a new proof to these well-known results.

2. The generalized group algebras, $B^p(G, A)$ and $C(G, A)$. For the rest of this paper G will denote a compact topological group with Haar measure, m, normalized so that $m(G) = 1$. Also A will denote a Banach algebra over the complex number field. In this section we will study interrelations and general properties of algebras that are natural generalizations of $L^p(G)$ $(1 \leqq p < \infty)$ and $C(G)$, viz., $B^p(G, A)$ and $C(G, A)$.

DEFINITION. For G, A, and p as above, we define $B^p(G, A)$ to be the space of all equivalence classes (modulo null functions) of measurable functions $f: G \rightarrow A$ such that $\int_G \| f(x) \|^p \, dx < \infty$. We also define $C(G, A)$ to be the space of all continuous functions from G to A.

We will make the usual abuse of the language and speak of the functions in $B^p(G, A)$ rather than of the equivalence classes they represent.

If we define addition and scalar multiplicaion on the above spaces to be the pointwise operations, and for $f \in B^p(G, A)$ we define

$$\| f \|_p = \left[\int_G \| f(x) \|^p \, dx \right]^{1/p},$$

and for $f \in C(G, A)$ we define $\| f \|_\infty = \sup \{\| f(x) \| : x \in G\}$, then $B^p(G, A)$ and $C(G, A)$ are Banach spaces. Moreover if multiplication is defined on all of these spaces by convolution, then by a straight-forward argument they are all seen to be Banach algebras. In the course of verifying this one also notes that for $1 \leqq q \leqq p < \infty$, $B^p(G, A)$ is continuously isomorphic with an ideal of $B^q(G, A)$, and $C(G, A)$ is continuously isomorphic with an ideal of $B^p(G, A)$ for all $1 \leqq p < \infty$.

Furthermore, since G is a compact group and multiplication is defined by convolution, we have the following result.

PROPOSITION 3. A is isometrically isomorphic with an ideal of $C(G, A)$ or of $B^p(G, A)$ $(1 \leqq p < \infty)$.

COROLLARY. *If for some compact group, G, and some $1 \leqq p < \infty$, $B^p(G, A)$ or $C(G, A)$ is semisimple, then A is also semisimple.*

It is conjectured that the converse is also true, i.e., if A is a semisimple Banach algebra and if G is any compact group, then $B^p(G, A)$ $(1 \leqq p < \infty)$ and $C(G, A)$ are also semisimple. In § 4 of this paper this result will be proven in some special cases. In [16] this point is discussed in detail but at present no general proof nor counter-example is known.

We can however somewhat reduce the problem of semisimplicity via the following result.

PROPOSITION 4. Let G be a compact group and A be a Banach algebra. If $1 \leqq p < \infty$ and $1 \leqq q < \infty$, then $B^p(G, A)$ is semisimple if and only if $B^q(G, A)$ is semisimple. Similarly $B^p(G, A)$ is semisimple if and only if $C(G, A)$ is semisimple.

Proof.

LEMMA 1. *If G, A, and p are as above, and $\{u_\alpha\}$ is an approximate identity of $L^p(G)$ (consisting say of continuous functions), then $\lim_\alpha u_\alpha * f = f$ for every $f \in B^p(G, A)$.*

Proof. It is seen *mutatis mutandis* as in the case of scalar-valued functions that $u_\alpha * f \in B^p(G, A)$ for every $f \in B^p(G, A)$. [N.B.

In the future, such results will be carried over from the scalar-valued case to the vector-valued case without special mention.] Now consider $f \in B^p(G, A)$ of the form

$$f = \sum_{i=1}^{n} f_i(\cdot)a_i \qquad [a_i \in A, f_i \in L^p(G), (f_i(\cdot)a_i)(x) = f_i(x)a_i] .$$

For such f it is seen that $\lim_\alpha(u_\alpha * f - f) = 0$, but also the collection of such f is dense in $B^p(G, A)$ (since for example it contains the simple functions). As a result, $\lim_\alpha(u_\alpha * f - f) = 0$ for every $f \in B^p(G, A)$.

Notation. When we are considering annihilators of subalgebras, and there may be some confusion about the algebra relative to which the annihilator is being taken, we will use the following notation: if $X \subseteqq Y$, then $\mathscr{R}_Y(X)$ is the right annihilator of X in Y. A similar definition holds for $\mathscr{L}_Y(X)$ and $\mathscr{A}_Y(X)$.

LEMMA 2. *Let G and p be as above, and let A be a Banach algebra such that $\mathscr{R}_A(A) = (0)$ (or $\mathscr{L}_A(A) = (0)$). Then if I is any nonzero ideal in $B^p(G, A)$, $I \cap C(G, A) \neq (0)$.*

Proof. Let $g \neq 0$ be an element of I. If $C(G) * g = (0)$ then taking $\{u_\alpha\} \subseteqq C(G)$ to be an $L^p(G)$ approximate identity, we have $u_\alpha * g = 0$ for every α. But by Lemma 1, $\lim_\alpha u_\alpha * g = g$, i.e., $g = 0$, which is a contradiction. Thus there is an $f \in C(G)$ such that $f * g \neq 0$. Note also that $f * g \in C(G, A)$.

Now for any $a \in A$, let fa be defined by $(fa)(x) = f(x) \cdot a$, and similarly define ag and $a(f * g)$. Note that $fa \in C(G, A)$ and $ag \in B^p(G, A)$. Also we see that $a(f * g) = (fa) * g = f * (ag)$. Now if for every $a \in A$, $a(f * g) = 0$, then since $f * g \in C(G, A)$, $A \cdot (f * g)(x) = (0)$ for every $x \in G$. But then since $\mathscr{R}_A(A) = (0)$, $(f * g)(x) = 0$ for every $x \in G$, i.e., $f * g = 0$ which is a contradiction. Thus there must be an $a \in A$ such that $a(f * g) \neq 0$. Finally, since I is an ideal in $B^p(G, A)$, $(fa) * g \in I$, and also

$$f * (ag) \in C(G) * B^p(G, A) \subseteqq C(G, A) .$$

Thus $0 \neq a(f * g) \in I \cap C(G, A)$.

Now to prove Proposition 4. Assume $p > q$. Thus $B^p(G, A)$ is an ideal of $B^q(G, A)$, and as a result, if $B^q(G,A)$ is semisimple, then $B^p(G, A)$ is semisimple. To prove the converse, we note that since $B^p(G, A)$ is an ideal of $B^q(G, A)$, it is always true that

$$\mathfrak{R}_p = B^p(G, A) \cap \mathfrak{R}_q$$

(where $\mathfrak{R}_k$ is the radical of $B^k(G, A)$). Thus if $B^p(G,A)$ is semisimple, $(0) = B^p(G, A) \cap \mathfrak{R}_q$. Also in this case A, as an ideal of $B^p(G, A)$, is semisimple, and thus $\mathscr{R}_A(A) = (0)$. Therefore by Lemma 2, if $\mathfrak{R}_q \neq (0)$, then $(0) \neq C(G, A) \cap \mathfrak{R}_q \subseteqq B^p(G, A) \cap \mathfrak{R}_q = (0)$. Thus $\mathfrak{R}_q$ must equal (0).

It should be remarked that one of the initial motivations for this paper was the determination of necessary and sufficient conditions for the algebras $B^p(G, A)$ and $C(G, A)$ to be annihilator or dual algebras. As an immediate corollary to Proposition 3, we see that if for some compact group G and some $1 \leqq p < \infty$, $B^p(G, A)$ or $C(G, A)$ is a semisimple dual Banach algebra, then A is a semisimple dual Banach algebra ([14], p. 100). It has recently been shown [1] that not every closed ideal of an annihilator algebra need be an annihilator algebra, however it can be shown (cf. [16]) that if $B^p(G, A)$ or $C(G, A)$ is a semisimple annihilator Banach algebra, then A must also be such.

The sufficiency of these conditions is discussed in § 4.

3. Topological tensor products. In this section we will discuss certain topological tensor products. Since, as we will see, $B^p(G, A)$ and $C(G, A)$ are expressible as topological tensor products, such tensor products present a direction for generalization of our discussion. Also the realization that the generalized group algebras are topological tensor products provides a formalistic motivation for a determination of their structure.

We use A' to denote the dual Banach space of a Banach algebra, A. For $a \in A$, $a' \in A'$, $\langle a, a' \rangle$ denotes the action of a and a' on each other.

DEFINITION. If A and B are Banach algebras, and $\{a_i\}_{i=1}^n \subseteqq A$, $\{b_i\}_{i=1}^n \subseteqq B$, then these sequences give rise to a bounded complex-valued bilinear form, T, defined on $A' \times B'$ by

$$T(a', b') = \sum_{i=1}^{n} \langle a_i, a' \rangle \langle b_i, b' \rangle .$$

We will symbolically denote T by $\sum_{i=1}^n a_i \otimes b_i$ and write $T \sim \sum_{i=1}^n a_i \otimes b_i$. (We may sometimes simplify the notation and speak of the "tensor", $\sum_{i=1}^n a_i \otimes b_i$.) The *algebraic tensor product* of A and B, denoted $A \otimes B$, is the vector space of all bilinear forms of the above type.

This definition of the algebraic tensor product of the algebras A and B agrees with the usual definition of the algebraic tensor product ([4], p. 5).

We can next introduce a well-defined multiplication in $A \otimes B$ by the formula: if

$$T_1 \sim \sum_{i=1}^{m} a_i \otimes b_i\ , \qquad T_2 \sim \sum_{j=1}^{n} c_j \otimes d_j\ ,$$

then define

$$T_1 \cdot T_2 \sim \sum_{i=1}^{m} \sum_{j=1}^{n} a_i c_j \otimes b_i d_j\ .$$

DEFINITION. If α is a norm on the tensor product, $A \otimes B$, we say that α is a *cross-norm* if for every tensor, T, with representative of the form $a \otimes b$, $\alpha(T) = \|a\| \cdot \|b\|$. We say that α is *compatible with multiplication* if for every $T_1, T_2 \in A \otimes B$, $\alpha(T_1 \cdot T_2) \leqq \alpha(T_1) \cdot \alpha(T_2)$.

Since in general $A \otimes B$ may not be complete with respect to a given cross-norm, α, (cf. [4], p. 8), we let $A \otimes^{\alpha} B$ denote the normed linear space, $A \otimes B$ supplied with the α norm, and let $A \otimes_{\alpha} B$ denote the completion of $A \otimes^{\alpha} B$ with respect to α.

We can now extend our definition of multiplication to $A \otimes_{\alpha} B$ by taking limits. It is easily seen that if α is compatible with multiplication on $A \otimes B$ it is also so on $A \otimes_{\alpha} B$, and thus in this case $A \otimes_{\alpha} B$, is a Banach algebra.

We are now in a position to relate topological tensor products to our discussion of the generalized group algebras. In fact for a given compact group, G, and some $1 \leqq p < \infty$, let M be a closed subalgebra of $L^p(G)$. Also for a given Banach algebra, A, let N be a closed subalgebra of A. Now consider $M \otimes N$ with the "p-norm" defined by

$$\| T \|_p = \left[\int_G \| \sum_{i=1}^{k} m_i(x) n_i \|^p \, dx \right]^{1/p}$$

(where $T \sim \sum_{i=1}^{k} m_i \otimes n_i$).

We see as a result of the following proposition that $\|\cdot\|_p$ is a well-defined norm.

PROPOSITION 5. For G, A, and p as above, $L^p(G) \otimes A$ is isomorphic to a dense subalgebra of $B^p(G, A)$.

Proof. If $T \sim \sum_{i=1}^{n} f_i \otimes a_i$ is an element of $L^p(G) \otimes A$, let $\varphi(T) = \sum_{i=1}^{n} f_i(\cdot) a_i$ (where $(\sum_{i=1}^{n} f_i(\cdot) a_i)(x) = \sum_{i=1}^{n} f_i(x) a_i$, and $\varphi(T)$ is of course defined modulo null functions). Note that for every T, $\varphi(T) \in B^p(G, A)$, and φ is linear. Thus to show φ is well-defined, i.e., independent of the choice of representative for T, it suffices to show that $T = 0$ implies $\varphi(T) = 0$ (independent of the representative for

T). Now ([4], p. 4) shows that $0 = T \sim \sum_{i=1}^{n} f_i \otimes a_i$ if and only if one of the sets $\{f_i\}, \{a_i\}$ is linearly dependent. Say $\{f_i\}$ is linearly dependent and $\sum_{i=1}^{n-1} \lambda_i f_i = f_n$ (for some complex numbers, λ_i). Then it is seen that $T \sim \sum_{i=1}^{n-1} f_i \otimes (a_i + \lambda_i a_n)$ and also that $\sum_{i=1}^{n} f_i(\cdot)a_i = \sum_{i=1}^{n-1} f_i(\cdot)(a_i + \lambda_i a_n)$, i.e., T can be represented using one less term, and the image of T is invariant under this change of representative. Continuing in this manner we may reduce to a representative for T of the form $u \otimes v$ with one of u or v equal to 0, and without changing $\varphi(T)$. As computed with this last representative, $\varphi(T) = 0$.

A straightforward computation shows that φ is multiplicative, and an application of the above criterion for representatives of the zero tensor shows that φ is one-to-one. That the image of φ is dense in $B^p(G, A)$ follows from the fact that it contains the simple functions.

Next we note that the "p-norm" as we have defined it on $L^p(G) \otimes A$ is just the norm inherited by the above isomorphism. Thus the "p-norm" is a well-defined norm on $L^p(G) \otimes A$. In fact, it is easily seen to be a cross-norm on $L^p(G) \otimes A$.

As an immediate result we have

PROPOSITION 6. $L^p(G) \otimes_p A$ is isomorphic and isometric to $B^p(G, A)$.

The above discussion allows us to settle immediately the problem of determining necessary and sufficient conditions for $B^p(G, A)$ ($1 \leqq p < \infty$) and $C(G, A)$ to be strongly semisimple.

PROPOSITION 7. $B^p(G, A)$ (or $C(G, A)$) is strongly semisimple if and only if A is strongly semisimple.

Proof. Assume first A is strongly semisimple. Since for $1 < p < \infty$, $B^p(G, A)$ and $C(G, A)$ are ideals of $B^1(G, A)$, it suffices to show that $B^1(G, A)$ is strongly semisimple.

Grothendieck shows ([7], p. 59) that $B^1(G, A) = L^1(G) \otimes_\gamma A$, where γ is the greatest cross-norm. (For our purposes γ is distinguished by the fact that for Banach algebras, A and B, γ may be defined as a cross-norm on $A \otimes B$, compatible with multiplication and such that if α is any other cross-norm on $A \otimes B$, $\gamma \geqq \alpha$.) Actually Grothendieck discusses only the case of Banach spaces, but his discussion of the above result is easily extended to Banach algebras. Grothendieck ([7], p. 185) also shows that $L^1(G)$ satisfies the condition of approximation. Finally, Gelbaum shows ([3], p. 538) that $B^1(G, A)$ is thus strongly semisimple.

Since A is an ideal of all of the generalized group algebras, the converse is immediate.

Next let M be a closed subalgebra of $L^p(G)$ and N be a closed subalgebra of A. $M \otimes N$ can be isomorphically embedded in $L^p(G) \otimes A$ in the following manner. Let I be the subalgebra of $L^p(G) \otimes A$ generated by all tensors having a representative of the form $m \otimes n$ (where $m \in M$ and $n \in N$). For $T \in I$, let T' be the restriction of the bilinear form T to $M' \times N'$. The map ψ: $T \rightarrow T'$ is seen to be a well-defined isomorphism from I onto $M \otimes N$. Since the "p-norm" as defined previously on $M \otimes N$ is the same as the norm inherited from I by this correspondence, the "p-norm" on $M \otimes N$ is a well-defined norm on $M \otimes N$. Note also that the p-norm on $L^p(G) \otimes A$ is an extension of the p-norm of $M \otimes N$, i.e., for $T \in M \otimes N \subseteqq L^p(G) \otimes A$, $\| T \|_p$ is independent of viewing $T \in M \otimes N$ or $T \in L^p(G) \otimes A$.

DEFINITION. A norm, α, on the tensor product, $A \otimes B$, of two Banach algebras is of *local character* if for any closed subalgebras $M \subseteqq A$, $N \subseteqq B$, α on $A \otimes B$ is an extension of α on $M \otimes N$, i.e., $M \otimes_\alpha N \subseteqq A \otimes_\alpha B$.

Thus the p-norm is of local character. There are norms that are not always of local character. In fact the greatest cross-norm is such a norm ([15], p. 53). Thus while $L^1(G) \otimes_\gamma A = L^1(G) \otimes_1 A$, it may not be true that $M \otimes_\gamma N = M \otimes_1 N$ for all closed subalgebras $M \subseteqq L^1(G)$ and $N \subseteqq A$.

In addition to the above remarks, since the p-norm is compatible with multiplication, we see that $M \otimes_p N$ is a closed (left, right) ideal in $L^p(G) \otimes_p A$ if M and N are closed (left, right) ideals in $L^p(G)$ and A respectively.

The above analysis can also be carried out for $C(G) \otimes A$ and $C(G, A)$, i.e., $C(G) \otimes A$ can be isomorphically mapped onto a dense subalgebra of $C(G, A)$. The norm on $C(G) \otimes A$ that equals the inherited sup-norm of $C(G, A)$ is the so called least cross-norm, λ, ([7], p. 90). Thus $C(G) \otimes_\lambda A$ is isomorphic and isometric with $C(G,A)$. The λ-norm can be defined on the algebraic tensor product, $A \otimes B$, of two Banach algebras as follows: let $T \in A \otimes B$, i.e., T is a bilinear form from $A' \times B'$ to the complex numbers and of a special type. $\| T \|_\lambda$ is defined to be the norm of T as a bilinear form. λ is seen [15] to be of local character, and in our case λ is compatible with multiplication, although this is not always true ([5], p. 80)

DEFINITION. A cross-norm, α, on the tensor product, $A \otimes B$, of two Banach spaces is said to be *ordinary* if $\lambda \leqq \alpha \leqq \gamma$ on $A \otimes B$.

PROPOSITION 8. The p-norm is an ordinary cross-norm on $L^p(G) \otimes A$.

Proof. Since the p-norm is a cross-norm, $\|\cdot\|_p \leqq \gamma$ for all p. To show $\|\cdot\|_p \geqq \lambda$, we note first that if $p = 1$, $\|\cdot\|_1 = \gamma \geqq \lambda$. If $1 < p < \infty$, let $T \sim \sum_{i=1}^n f_i \otimes a_i$ be an element of $L^p(G) \otimes A$ and $h_T = \sum_{i=1}^n f_i(\cdot)a_i$ be the corresponding element of $B^p(G, A)$. Now let $g \in L^{p'}(G)$ $(1/p + 1/p' = 1)$ and $a' \in A'$. Then $g(\cdot)a' \in B^{p'}(G, A') \subseteqq [B^p(G, A)]'$. Therefore

$$\begin{aligned} | T(g, a') | &= \left| \sum_{i=1}^n \langle f_i, g \rangle \langle a_i, a' \rangle \right| \\ &= \left| \sum_{i=1}^n \left(\int_G f_i(x)\overline{g(x)}dx \right) \langle a_i, a' \rangle \right| \\ &= \left| \int_G \left\langle \sum_{i=1}^n f_i(x)a_i, \overline{g(x)}a' \right\rangle dx \right| \\ &= \left| \int_G \langle h_T(x), \overline{g(x)}a' \rangle \, dx \right| \\ &= | \langle h_T, \overline{g(\cdot)}a' \rangle | \leqq \| h_T \|_p \| \overline{g(\cdot)}a' \|_{p'} \\ &= \| T \|_p \| g \|_{p'} \| a' \|_{A'} \, . \end{aligned}$$

As a result

$$\| T \|_\lambda \leqq \| T \|_p \, .$$

4. **Structure theorems.** In this section we will study the structure inherited by $B^p(G, A)$ and $C(G, A)$ from A, and more generally the structure inherited by certain topological tensor products from the component algebras. The type of result we would like could perhaps be modeled on the following results from [9]: a suitably normed tensor product, $A \otimes_\sigma B$, of H^*-algebras is again an H^*-algebra, and the minimal closed ideals of the tensor product are naturally associated with the tensor product of minimal closed ideals of A and B. [N.B. H^*-algebras are D^*-algebras, and their structure is accordingly known and is relevant to the present discussion (cf. [14], pp. 272-276)]. In particular, $B^2(G, A)$ is an H^*-algebra if and only if A is an H^*-algebra, and the minimal closed ideals of $B^2(G, A)$ are associated with pairs of minimal closed ideals of $L^2(G)$ and A. The difficulty in directly carrying over the techniques used in proving the above results to our present context of dual and annihilator algebras arises from the fact that H^*-algebras have more or less global defining characterizations, but the presently known characterizations of dual and annihilator algebras require some prior knowledge of the structure of the algebras. Thus, for example, even though we know $B^p(G, A) = L^p(G) \otimes_p A$, in trying to show directly that $B^p(G, A)$ is a semisimple dual (annihilator) Banach algebra if A is such, we immediately run into the following problems. If, say M is a closed right

ideal in $L^p(G)$ and N is a closed right ideal in A, then $M \otimes_p N$ is a closed right ideal in $L^p(G) \otimes_p A$. One might hope that $\mathscr{L}(M \otimes_p N) = \mathscr{L}(M) \otimes_p \mathscr{L}(N)$, but in fact in general

$$\begin{aligned}\mathscr{L}(M \otimes_p N) &\supseteqq \{\mathscr{L}(M) \otimes_p A, L^p(G) \otimes_p \mathscr{L}(N)\} \\ &\supsetneqq \mathscr{L}(M) \otimes_p \mathscr{L}(N) .\end{aligned}$$

Thus one does not have a good hold on the annihilators of ideals. To make things even worse, the above example indicates that not every closed one sided ideal of the tensor product need even be expressible as the tensor product of closed ideals, e. g., consider $\mathscr{L}(M \otimes_p N)$ above!

As a result, one soon abandons the attempt to classify the algebras under consideration and approaches the problems of structure directly.

Unless otherwise noted, for the remainder of this section we will let A and B be semisimple annihilator Banach algebras with collections of minimal closed ideals $\{M_\alpha\}$ and $\{N_\beta\}$ respectively. c will denote a cross-norm on $A \otimes B$ that is compatible with multiplication and of local character. For a Banach space, X, $\mathscr{B}(X)$ will denote the Banach algebra of bounded linear operators on X. We may also have occasion to use terminology that has common though perhaps not universal currency. We will attempt to follow the terminology of [14].

PROPOSITION 9. If $\Re$ is the radical of $A \otimes_c B$, then $\Re = \mathscr{L}(A \otimes_c B) = \mathscr{R}(A \otimes_c B)$.

Proof. It obviously suffices to show $\Re \subseteqq \mathscr{L}(A \otimes_c B)$ and $\Re \subseteqq \mathscr{R}(A \otimes_c B)$. The same argument works for both cases so consider the first one. As a semisimple annihilator algebra, A has minimal left ideals and each of these is of the form Ae_γ where $e_\gamma{}^2 = e_\gamma$ and $e_\gamma Ae_\gamma$ is isomorphic to the complex numbers ([14], pp. 97 and 45). Similarly B has minimal left ideals each of which is of the form Bf_δ where $f_\delta{}^2 = f_\delta$ and $f_\delta Bf_\delta$ is isomorphic to the complex numbers. Now for each pair (γ, δ), since c is of local character, $Ae_\gamma \otimes_c Bf_\delta$ may be identified with $(A \otimes_c B)(e_\gamma \otimes f_\delta)$ where $(e_\gamma \otimes f_\delta)^2 = e_\gamma \otimes f_\delta$, and $(A \otimes_c B)(e_\gamma \otimes f_\delta)$ is seen to be isomorphic to the complexes. It follows then ([14], p. 46) that $Ae_\gamma \otimes_c Bf_\delta$ is a minimal left ideal of $A \otimes_c B$. Thus the left regular representation of $A \otimes_c B$ on $Ae_\gamma \otimes_c Bf_\delta$ is irreducible, and its kernel is $\mathscr{L}(Ae_\gamma \otimes_c Bf_\delta)$. But since A and B are semisimple annihilator algebras, it is seen ([14], p. 100) that $\overline{\sum_\gamma Ae_\gamma} = A$ and $\overline{\sum_\delta Bf_\delta} = B$ where the sums are taken over all minimal left ideals. Therefore

$$\overline{\sum_{(\gamma,\delta)}(Ae_\gamma \otimes_c Bf_\delta)} = A \otimes_c B ,$$

and letting

$$I = \bigcap_{(\gamma, \delta)} \mathscr{L}(Ae_\gamma \otimes_c Bf_\delta) ,$$

we have as a result, $I \subseteqq \mathscr{L}(A \otimes_c B)$. But by definition, $\mathfrak{R} \subseteqq I$.

COROLLARY. *If A is a semisimple annihilator Banach algebra, then $B^p(G, A)$ $(1 \leqq p < \infty)$ and $C(G, A)$ are semisimple.*

Proof. It is readily seen that $\mathscr{L}_A(A) = (0)$ implies $\mathscr{L}(B^p(G, A)) = (0)$ and similarly for $C(G, A)$ (cf. [8], p. 24).

We see from § 3 that for every (α, β), $M_\alpha \otimes_c N_\beta$ is a closed ideal of $A \otimes_c B$, and since $\sum_\alpha \oplus M_\alpha$ is dense in A and $\sum_\beta \oplus N_\beta$ is dense in B, $\sum_{(\alpha, \beta)}(M_\alpha \otimes_c N_\beta)$ is dense in $A \otimes_c B$.

PROPOSITION 10. If for every (α, β), $M_\alpha \otimes_c N_\beta$ is a minimal closed ideal of $A \otimes_c B$ and $\mathscr{L}(A \otimes_c B) = (0)$, then in fact every minimal closed ideal of $A \otimes_c B$ is of the form $M_{\alpha'} \otimes_c N_{\beta'}$ for some (α', β').

Proof. Let I be any minimal closed ideal of $A \otimes_c B$. If $I \cap (M_\alpha \otimes_c N_\beta) = (0)$ for all (α, β), then $I \cdot (M_\alpha \otimes_c N_\beta) = (0)$ for all (α, β), but then $I \cdot (A \otimes_c B) = (0)$, which is impossible by hypothesis. Thus $I \cap (M_{\alpha'} \otimes_c N_{\beta'}) \neq (0)$ for some (α', β'). By minimality, $I = M_{\alpha'} \otimes_c N_{\beta'}$ then.

Thus the problem of determining the structure of $A \otimes_c B$ essentially reduces to determining when ideals of the form $M_\alpha \otimes_c N_\beta$ are minimal closed ideals of $A \otimes_c B$, and in this case to then find a concrete representation for these ideals. Since for every (α, β), M_α and N_β are topologically simple, semisimple annihilator Banach algebras, the first of the above problems would be solved if it could be determined when a suitably normed tensor product of such algebras is topologically simple.

In the following rather simple case both of the problems stated above are easily solved. Also in light of § 1, there are some immediate applications.

PROPOSITION 11. If for every (α, β) M_α and N_β are finite dimensional then:

(1) $M_\alpha \otimes_c N_\beta$ is a minimal closed ideal of $A \otimes_c B$.

(2) $M_\alpha \otimes_c N_\beta$ is a finite-dimensional dual algebra.

(3) $A \otimes_c B$ is an annihilator algebra if and only if it is semisimple.

(4) In the event that A and B have approximate identities, $A \otimes_c B$ is a dual algebras if and only if it is semisimple.

Proof. As a finite-dimensional simple annihilator algebra, $M_\alpha = \mathscr{B}(X_\alpha)$ for some X_α. Similarly $N_\beta = \mathscr{B}(X_\beta)$. Therefore, $M_\alpha \otimes_c N_\beta$ is isomorphic and homeomorphic with the finite-dimensional, simple, dual algebra $\mathscr{B}(X_\alpha \otimes^c Y_\beta)$.

The necessity statements in (3) and (4) follow directly from Proposition 9 and the definitions of "dual" and "annihilator algebras". The sufficiency statement in (3) follows from (2), the remark preceding Proposition 10, and ([14], p. 106). The sufficiency statement in (4) follows from these same results and the additional fact that since A and B each have an approximate identity and c is compatible with multiplication, $A \otimes_c B$ has an approximate identity.

Notation. If $I \subseteqq B^p(G, A)$ let $[I]$ denote the ideal generated by I. If $N \subseteqq L^p(G)$ and $M \subseteqq A$ let $N(\cdot)M = \{f(\cdot)a\colon f \in N, a \in M\}$.

COROLLARY 1. *If $\{N_\beta\}$ is the collection of minimal closed ideals of $L^p(G)$ and if all minimal closed ideals, M_α, of a semisimple annihilator Banach algebra, A, are finite-dimensional, then:*

(1) *$B^p(G, A)$ is a semisimple annihilator Banach algebra.*

(2) *Every ideal of $B^p(G, A)$ of the form $[N_\alpha(\cdot)M_\beta]$ is a minimal closed ideal of $B^p(G, A)$ and a finite dimensional dual algebra.*

(3) *Every minimal closed ideal of $B^p(G, A)$ is of the above form.*

(4) $B^p(G, A) = \overline{\sum_{(\alpha, \beta)} \oplus ([N_\alpha(\cdot)M_\beta])}$.

If, in addition, A has an approximate identity, then $B^p(G, A)$ is a dual Banach algebra.

COROLLARY 2. *If A is a semisimple annihilator Banach algebra, then $B^p(G, A)$ is completely continuous if and only if A is completely continuous.*

COROLLARY 3. *If A is any Banach algebra with an approximate identity, then $B^p(G, A)$ is a strongly semisimple D^*-algebra if and only if A is a strongly semisimple D^*-algebra.*

Similar statements can be made for $C(G, A)$.

One might hope that since the minimal closed ideals of $L^p(G)$ are well-known, we could get almost as good results without requiring that A have finite-dimensional minimal closed ideals. In fact, this is the case. We notice (cf. § 1) that the minimal closed ideals of $L^p(G)$ are topologically simple I^*-algebras. These have been studied ([14], pp. 267–270), and for convenience we present a summary of the results most pertinent to our discussion. If M is such an algebra, then there is a family of "matrix units" $\{e_{\alpha\beta}\}$, in M with the properties:

(1) $e^*_{\alpha\beta} = e_{\beta\alpha}$.

(2) $e_{\alpha\beta}\, e_{\gamma\delta} = \delta_{\beta\gamma}\, e_{\alpha\delta}$ (where $\delta_{\beta\gamma}$ is the Kronecker delta).

(3) $\{e_{\alpha\alpha}\}$ is a family of pairwise orthogonal, minimal Hermitian idempotents.

(4) $Me_{\alpha\alpha}$ is a minimal left ideal of M, and $e_{\alpha\alpha} M$ is a minimal right ideal of M for every α. In addition, $\sum_\alpha Me_{\alpha\alpha}$ and $\sum_\alpha e_{\alpha\alpha} M$ are dense in M.

(5) The set of finite linear combinations of $\{e_{\alpha\beta}\}$ is dense in M.

(6) For every $m \in M$,

$$e_{\alpha\alpha}\, m\, e_{\beta\beta} = \lambda_{\alpha\beta}(m) e_{\alpha\beta}$$

(where $\lambda_{\alpha\beta}(m)$ is a complex number).

By means of such matrix units, a generalization of the technique used to show that the ring of $n \times n$ complex matrices is simple could be used to determine conditions under which the tensor product of two topologically simple I^*-algebras is topologically simple. This technique may be further refined to prove the following proposition.

PROPOSITION 12. Let M be a topologically simple I^*-algebra, N be a topologically simple, semisimple Banach algebra (not necessarily an annihilator algebra), and let c be a cross-norm on $M \otimes N$ that is compatible with multiplication and of local character, then the following conditions are equivalent:

(1) $M \otimes_c N$ is topologically simple.

(2) $M \otimes_c N$ is semisimple.

(3) $\mathscr{L}(M \otimes_c N) = \mathscr{R}(M \otimes_c N) = (0)$.

Proof. Assume $M \otimes_c N$ is topologically simple and let $\mathfrak{R}$ be its radical. Since $\mathfrak{R}$ is a closed ideal, $\mathfrak{R} = M \otimes_c N$ or $\mathfrak{R} = (0)$. However, since M is semisimple, there must be at least one element, m, of M that is not topologically nilpotent (i.e., $\lim_k \| m^k \|^{1/k} \neq 0$). Similarly there is a nontopologically nilpotent $n \in N$. Since c is a cross-norm, $T \sim m \otimes n$ is not topologically nilpotent, and thus $T \notin \mathfrak{R}$ therefore, $\mathfrak{R}$ must be (0).

That (2) implies (3) is obvious, and thus it only remains to show that (3) implies (1). Therefore, let I be a nonzero closed ideal of $M \otimes_c N$. Notice that if $\{e_{\alpha\beta}\}$ is the set of matrix units in M and if n_γ varies freely through N, for every (α, γ), $(M \otimes_c N)\ (e_{\alpha\alpha} \otimes n_\gamma)$ can be isomorphically and isometrically identified with $Me_{\alpha\alpha} \otimes_c Nn_\gamma$, which in turn can be embedded in $M \otimes_c N$ since c is of local character. Thus by an argument similar to that used in the statement just preceding Proposition 10, $\sum_{(\alpha,\gamma)}(M \otimes_c N)(e_{\alpha\alpha} \otimes n_\gamma)$ is seen to be dense in $M \otimes_c N$. Similarly $\sum_{(\alpha,\gamma)}(e_{\alpha\alpha} \otimes n_\gamma)(M \otimes_c N)$ is dense in $M \otimes_c N$

also. Now if $0 \neq T \in I$, by the two preceding statements and the hypothesis that $\mathscr{L}(M \otimes_c N) = \mathscr{R}(M \otimes_c N) = (0)$, there must be indices α, β and $n_1, n_2 \in N$ such that

$$0 \neq (e_{\alpha\alpha} \otimes n_1) \cdot T \cdot (e_{\beta\beta} \otimes n_2) .$$

Next let $\{T_i\}$ be a sequence of simple tensors ($T_i \sim \sum_{k=1}^{K_i} m_{ik} \otimes n_{ik}$) such that $\lim_i T_i = T$. It is readily seen that for each index, i, $(e_{\alpha\alpha} \otimes n_1) \cdot T_i \cdot (e_{\beta\beta} \otimes n_2) = e_{\alpha\beta} \otimes n_i$ for some $n_i \in N$. In fact, it is then seen that $0 \neq (e_{\alpha\alpha} \otimes n_1) \cdot T \cdot (e_{\beta\beta} \otimes n_2) = e_{\alpha\beta} \otimes n_0$ for some $n_0 \in N$. Now let $(0) \neq K_{\alpha\beta} = \{n: e_{\alpha\beta} \otimes n \in I\} \subseteqq N$. $K_{\alpha\beta}$ is seen to be a closed ideal, and thus by topological simplicity, $K_{\alpha\beta} = N$, i.e., $e_{\alpha\beta} \otimes n \in I$ for every $n \in N$.

Now consider any pair of indices (γ, δ). There must be some $n, n' \in N$ such that $0 \neq e_{\gamma\beta} \otimes nn' = (e_{\gamma\alpha} \otimes n)(e_{\alpha\beta} \otimes n')$, since otherwise $nn' = 0$ for every $n, n' \in N$, i.e., $N^2 = (0)$, which contradicts the semisimplicity of N. Letting now $(0) \neq K_{\gamma\beta} = \{n: e_{\gamma\beta} \otimes n \in I\}$, we see by the argument used above, $K_{\gamma\beta} = N$. Repeating these arguments once more, we see that $e_{\gamma\delta} \otimes n \in I$ for every $n \in N$. Since the set of finite linear combinations of the elements $e_{\gamma\delta}$ is dense in M and c is a cross-norm, it is seen directly that finite linear combinations of elements of the form $e_{\gamma\delta} \otimes n$ is dense in $M \otimes_c N$. Thus since I is a closed ideal in $M \otimes_c N$, it follows that $I = M \otimes_c N$.

As an immediate application we see that if A is an I^*-algebra and B is a semisimple annihilator Banach algebra, then the structure of $A \otimes_c B$ is determined in the event that $A \otimes_c B$ is semisimple. In fact, in this case for every (α, β), $M_\alpha \otimes_c N_\beta$ is semisimple also and thus is a minimal closed ideal of $A \otimes_c B$. In particular, the structure of $B^p(G, A)$ and $C(G, A)$ is thus known when A is a semisimple annihilator Banach algebra. In fact in this case each $B^p(G, A)$, for example, is the topological direct sum of its minimal closed ideals. Each of these minimal closed ideals is of the form $\overline{[M(\cdot)N]}$, where M is a minimal closed ideal of $L^p(G)$ and N is a minimal closed ideal of A (cf. the notation introduced in Proposition 11).

Similar results may be stated when the non-I^*-algebra above is not necessarily an annihilator algebra but is a semisimple, topologically simple Banach algebra. However, if say A is such an algebra, it has not yet been shown that $B^p(G, A)$ is semisimple.

LEMMA. *If A is a semisimple, topologically simple Banach algebra, then $B^p(G, A)$ $(1 \leqq p < \infty)$ and $C(G, A)$ are semisimple.*

Proof. Since A is semisimple, $\mathscr{L}_A(A) = \mathscr{R}_A(A) = (0)$ which implies $\mathscr{L}(B^p(G, A)) = \mathscr{R}(B^p(G, A)) = (0)$. Now let $\{N_\beta\}$ be the collection of minimal closed ideals of $L^p(G)$. By minimality, $N_\beta \cdot N_\gamma = (0)$ if $\beta \neq \gamma$, and thus $(N_\beta \otimes_p A)\cdot(N_\gamma \otimes_p A) = (0)$. Now if for some β', $0 \neq T \in \mathscr{L}_{N_{\beta'} \otimes_p A}(N_{\beta'} \otimes_p A)$, then by the above orthogonality relation, $T\cdot(N_\gamma \otimes_p A) = (0)$ for all $\gamma \neq \beta'$. As a result, $T\cdot\sum_\beta(N_\beta \otimes_p A) = (0)$, and thus $T\cdot B^p(G, A) = 0$, which is impossible. Thus in fact for all β, $\mathscr{L}_{N_\beta \otimes_p A}(N_\beta \otimes_p A) = (0)$. Similarly for right annihilators. Thus by Proposition 12, $N_\beta \otimes_p A$ is semisimple for every β. This implies that $B^p(G, A)$ is also semisimple, since if $\mathfrak{R}$ is the radical of $B^p(G, A)$ and if $\mathfrak{R} \neq (0)$, then by the technique used in the proof of Proposition 10, $\mathfrak{R} \cap (N_{\beta'} \otimes_p A) \neq (0)$ for some β'. But also $\mathfrak{R} \cap (N_{\beta'} \otimes_p A)$ is the radical of $N_{\beta'} \otimes_p A$, which must be (0).

The above result is useful in trying to extend our discussion beyond a consideration of annihilator algebras.

We have now essentially solved the problem of determining the structure of $B^p(G, A)$ and $C(G, A)$, and we have determined the minimal closed ideals of these algebras. To complete our study we have only to now find a concrete representation for these minimal closed ideals. Such a representation is demonstrated as part of the following generalization of the results discussed in Proposition 12.

PROPOSITION 13. Let M and N be topologically simple, semisimple, annihilator Banach algebras and let c be an ordinary cross-norm on $M \otimes N$ that is compatible with multiplication and of local character, then $M \otimes_c N$ is topologically simple if and only if it is semisimple.

Proof. We saw in Proposition 12 how topological simplicity implies semisimplicity. For the converse we need the following lemma.

LEMMA. *Let P_1 and P_2 be primitive Banach algebras with dense socles. Let X_1, X_2 be minimal left ideals of P_1, P_2 respectively. Finally let c be an ordinary cross-norm on $P_1 \otimes P_2$ that is compatible with multiplication and of local character. Then the radical of $P_1 \otimes_c P_2$ is the kernel of the left regular representation of $P_1 \otimes_c P_2$ on $X_1 \otimes_c X_2$.*

Proof. As in Proposition 9 we see that $X_1 \otimes_c X_2$ is a minimal left ideal of $P_1 \otimes_c P_2$, and thus the left regular representation is irreducible. Therefore the radical of $P_1 \otimes_c P_2$ must be contained in the kernel of this representation.

Conversely, we note first that the left regular representation of P_i on X_i is a faithful, continuous, strictly dense representation, ([14], p. 68). Let

$$\psi_1 : P_1 \leftrightarrow \mathscr{B}_1 \subseteqq \mathscr{B}(X_1)$$

and

$$\psi_2\colon P_2 \leftrightarrow \mathscr{B}_2 \subseteqq \mathscr{B}(X_2)$$

denote these representations. By means of ψ_1 and ψ_2, $P_1 \otimes^c P_2$ may be algebraically identified with $\mathscr{B}_1 \otimes \mathscr{B}_2$. In fact, let

$$\mu'\colon P_1 \otimes^c P_2 \to \mathscr{B}_1 \otimes \mathscr{B}_2$$

be defined by

$$\mu' \left(\sum_{i=1}^{n} p_{1i} \otimes p_{2i}\right) = \sum_{i=1}^{n} \psi_1(p_{1i}) \otimes \psi_2(p_{2i}) .$$

It is easily seen that μ' is a well-defined isomorphism onto $\mathscr{B}_1 \otimes \mathscr{B}_2$. Next let

$$\theta'\colon \mathscr{B}_1 \otimes \mathscr{B}_2 \to \mathscr{B}(X_1 \otimes_c X_2)$$

be defined by

$$\left[\theta' \left(\sum_{i=1}^{n} T_{1i} \otimes T_{2i}\right)\right]\left(\sum_{j=1}^{m} x_{1j} \otimes x_{2j}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} T_{1i}(x_{1j}) \otimes T_{2i}(x_{2j}) .$$

Since for all $1 \leqq i \leqq n$, $T_{1i} = \psi_i(p_{1i})$ and $T_{2i} = \psi_2(p_{2i})$, the above expression may be written as

$$(*) \qquad \begin{aligned} &\left[\theta' \left(\sum_{i=1}^{n} T_{1i} \otimes T_{2i}\right)\right]\left(\sum_{j=1}^{m} x_{1j} \otimes x_{2j}\right) \\ &\quad = \left(\sum_{i=1}^{n} p_{1i} \otimes p_{2i}\right)\cdot\left(\sum_{j=1}^{m} x_{1j} \otimes x_{2j}\right) . \end{aligned}$$

Thus since c is compatible with multiplication, $\theta'(\sum_{i=1}^{n} T_{1i} \otimes T_{2i})$ is seen to be bounded on $X_1 \otimes^c X_2$ and thus may be extended by continuity to a bounded operator on all of $X_1 \otimes_c X_2$. We will also denote this extended operator by $\theta'(\sum_{i=1}^{n} T_{1i} \otimes T_{2i})$. Gil de Lamadrid ([6], p. 360) shows that θ' as thus defined is a well-defined algebraic isomorphism from $\mathscr{B}_1 \otimes \mathscr{B}_2$ into $\mathscr{B}(X_1 \otimes_c X_2)$. By means of these two algebraic isomorphisms, $\mathscr{B}_1 \otimes \mathscr{B}_2$ may inherit either of two norms, viz., the c-norm from $P_1 \otimes^c P_2$ via μ', or the operator norm from $\mathscr{B}(X_1 \otimes_c X_2)$ via θ'. By means of the relation (*) above we see that for $\sigma \in \mathscr{B}_1 \otimes \mathscr{B}_2$, $\|\sigma\|_{op} \leqq \|\sigma\|_c$. Thus, if we complete $\mathscr{B}_1 \otimes^c \mathscr{B}_2$ to $\mathscr{B}_1 \otimes_c \mathscr{B}_2$, μ' can be extended to the isometric and isometric map

$$\mu\colon P_1 \otimes_c P_2 \to \mathscr{B}_1 \otimes_c \mathscr{B}_2 ,$$

and θ' can be extended by continuity to the well-defined (although not necessarily one-to-one) map

$$\theta\colon \mathscr{B}_1 \otimes_c \mathscr{B}_2 \to \mathscr{B}(X_1 \otimes_c X_2) .$$

Now let $\varphi = \theta \circ \mu$. φ is readily seen to be the left regular representation of $P_1 \otimes_c P_2$ on $X_1 \otimes_c X_2$. We now wish to show that if $\mathfrak{R}$ is the radical of $P_1 \otimes_c P_2$ then $\ker \varphi \subseteqq \mathfrak{R}$. We will extend a technique given in ([14], p. 103).

Let $\tau \in \ker \varphi$, then $\mu(\tau) = \sigma$ is an element of $\ker \theta$. Take $\{\sigma_n\} \subseteqq \mathscr{B}_1 \otimes^c \mathscr{B}_2$ such that $\| \sigma_n - \sigma \|_c \to 0$. Note that then $\| \sigma_n \|_{op} \to 0$. Next notice that every bounded operator of rank 1 on X_1 can be written in the form $f_1(\cdot)x_1$ where $f_1 \in X_1'$, $x_1 \in X_1$. Similarly every bounded operator of rank 1 on X_2 can be written in the form $f_2(\cdot)x_2$, where $f_2 \in X_2'$, and $x_2 \in X_2$. Now if we let

$$\psi_1(S_1) = \mathscr{F}_1 \subseteqq \mathscr{B}_2$$

and

$$\psi_2(S_2) = \mathscr{F}_1 \subseteqq \mathscr{B}_2$$

(where S_i is the socle of P_i), then it is seen ([14], pp. 68 and 65) $\mathscr{F}_i$ is precisely the ideal consisting of all of those bounded operators of finite rank on X_i that are in $\mathscr{B}_i$. Thus taking $f_1(\cdot)x_1, g_1(\cdot)y_1 \in \mathscr{F}_1$ and $f_2(\cdot)x_2, g_2(\cdot)y_2 \in \mathscr{F}_2$ we see by direct computation that for every n

$$\begin{aligned} &\| (f_1(\cdot)x_1 \otimes f_2(\cdot)x_2)\cdot\sigma_n\cdot(g_1(\cdot)y_1 \otimes g_2(\cdot)y_2) \|_c \\ &\quad = \| (f_1(\cdot)f_2(\cdot)[\theta(\sigma_n)(y_1 \otimes y_2)])\cdot(g_1(\cdot)g_2(\cdot)(x_1 \otimes x_2)) \|_c \end{aligned}$$

where, since by hypothesis $c \geqq \lambda$, $f_1(\cdot)f_2(\cdot)$, $g_1(\cdot)g_2(\cdot)$ are elements of $[X_1 \otimes_c X_2]'$ (cf. [13], p. 43). Thus the above expression equals

$$\begin{aligned} &| f_1(\cdot)f_2(\cdot)[\theta(\sigma_n)(y_1 \otimes y_2)]|\cdot\|g_1(\cdot)g_2(\cdot)(x_1 \otimes x_2) \|_c \\ &\quad \leqq \| f_1(\cdot)f_2(\cdot) \|_\infty\cdot\| \theta(\sigma_n)(y_1 \otimes y_2) \|_c\cdot\| g_1(\cdot)g_2(\cdot)(x_1 \otimes x_2)\|_c \\ &\quad \leqq \| f_1(\cdot)f_2(\cdot) \|_\infty\cdot\| \theta(\sigma_n)\|_{op}\cdot\| y_1 \otimes y_2\|_c\cdot\| g_1(\cdot)g_2(\cdot)(x_1 \otimes x_2)\|_c \, . \end{aligned}$$

But by hypothesis $\| \theta(\sigma_n) \|_{op} \to 0$, and thus the above inequality yields that

$$\|(f_1(\cdot)x_1 \otimes f_2(\cdot)x_2)\cdot\sigma_n\cdot(g_1(\cdot)y_1 \otimes g_2(\cdot)y_2)\|_c \to 0$$

also, i.e.,

$$(f_1(\cdot)x_1 \otimes f_2(\cdot)x_2)\cdot\sigma\cdot(g_1(\cdot)y_1 \otimes g_2(\cdot)y_2) = 0 \, .$$

But $f_1(\cdot)x_1$, $g_1(\cdot)y_1$ and $f_2(\cdot)x_2$, $g_2(\cdot)y_2$ were arbitrary bounded operators of rank 1 in $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively. Thus it must be true that

$$(\mathscr{F}_1 \otimes \mathscr{F}_2)\cdot\sigma\cdot(\mathscr{F}_1 \otimes \mathscr{F}_2) = (0) \, .$$

Also by hypothesis, S_1 is dense in P_1 and S_2 is dense in P_2, therefore it follows that $S_1 \otimes S_2$ is dense in $P_1 \otimes P_2$. Thus, as a result of the method we have used to define the c-norm on $\mathscr{B}_1 \otimes \mathscr{B}_2$, it must also be true that $\mathscr{F}_1 \otimes \mathscr{F}_2$ is dense in $\mathscr{B}_1 \otimes_c \mathscr{B}_2$. Therefore

$$(\mathscr{B}_1 \otimes_c \mathscr{B}_2)\cdot\sigma\cdot(\mathscr{B}_1 \otimes_c \mathscr{B}_2) = (0) \, .$$

Applying μ^{-1} to this expression we see that

$$(P_1 \otimes_c P_2) \cdot \tau \cdot (P_1 \otimes_c P_2) = (0) .$$

However, this means $\tau \in \mathfrak{R}$ ([10], p. 304).

Now to use this lemma to prove the rest of Proposition 13. M and N, as topologically simple, semisimple annihilator Banach algebras, are primitive Banach algebras with dense socles ([14], pp. 100-1). Therefore, since by hypothesis $M \otimes_c N$ is semisimple, the left regular representation, φ, of $M \otimes_c N$ on a minimal left ideal of the form $X_1 \otimes_c X_2$ is faithful and irreducible. In fact, φ is also strictly dense ([14], p. 68).

Now let I be a nonzero closed ideal of $M \otimes_c N$. $\varphi(I)$ is thus a nonzero ideal of $\varphi(M \otimes_c N)$. The socle of $\varphi(M \otimes_c N)$ consists of all operators on $X_1 \otimes_c X_2$ that are of finite rank and contained in $\varphi(M \otimes_c N)$, and it is contained in every nonzero ideal of $\varphi(M \otimes_c N)$ ([14], p. 65). Therefore in the notation of the above lemma,

$$(0) \neq \varphi(S_1 \otimes S_2) = \mathscr{F}_1 \otimes \mathscr{F}_2 \subseteq \text{socle}\, \varphi \{M \otimes_c N) \subseteq \varphi(I) .$$

Thus $S_1 \otimes S_2 \subseteq I$, but $S_1 \otimes S_2$ is dense in $M \otimes_c N$, and therefore so is I. Since I is closed, however, $I = M \otimes_c N$ then.

We, therefore, have completely determined the structure of a suitably normed tensor product, $A \otimes_c B$, of two semisimple annihilator Banach algebras in the event $A \otimes_c B$ is semisimple. In fact, in this case, $A \otimes_c B$ is the topological direct sum of its minimal closed ideals, each of which is of the form $M_\alpha \otimes_c N_\beta$. In turn each of these minimal closed ideals has a faithful, continuous, strictly dense representation on a Banach space, and the subalgebra consisting of those bounded operators of finite rank that are contained in the image of $M_\alpha \otimes_c N_\beta$ is dense in the image of $M_\alpha \otimes_c N_\beta$.

Since we have already noted that the p-norm is an ordinary cross-norm, the above representation of the minimal closed ideals may be carried out for $B^p(G, A)$ and $C(G, A)$.

In fact, we can make a further comment about the above representation in the event that one of the algebras has finite-dimensional minimal closed ideals (as is the case for $B^p(G, A)$ and $C(G, A)$). Then, since c is an ordinary cross-norm, the Banach space mentioned above is reflexive (cf. [15], pp. 137, 51, 141 in that order).

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Added in proof: With regards to the question raised following the first definition in § 1 as to the existence of a semisimple Banach algebra that is an annihilator algebra but not a dual algebra, such an example has been given by B. E. Johnson (Newcastle upon Tyne). In fact his example is also commutative. The topologically simple case is still an open question. Some of the results in § 1 follow also from results in *Modular Annihilator Algebras* (Canadian J. of Math. 18 (1966), 566–578) by B. Barnes. The conjecture following Proposition 3 concerning inheritance of semisimplicity by $B^p(G, A)$ and $C(G, A)$ has been answered in the affirmative by this author and others. The author's proof will soon appear in the Proceedings of the American Mathematical Society.

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