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SEMIGROUP ALGEBRAS THAT ARE GROUP ALGEBRAS

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If S is a finite semigroup, and if K is a field, under what conditions is there a group G such that the semigroup algebra KS is isomorphic to the group algebra KG?

The following theorems are proved:

1. Let S have odd order n, and let K be either a real number field or $GF(q)$, where q is a prime less than any prime divisor of n. If $KS \cong KG$ for a group G, then S is a group.

2. Let K be a cyclotomic field over the rationals, and let G be an abelian group. Then $KG \cong KS$ for a semigroup S that is not a group if and only if for some prime p and some positive integer k, K contains all p^kth roots of unity and the cyclic group of order p^k is a direct factor of G.

3. Let S be a commutative semigroup of order n, and let $K = GF(p)$, where p is a prime not exceeding the smallest prime dividing n. If $KS \cong KG$ for a group G, then S is a group.

The semigroup ring of a semilattice is also considered.

1. Preliminary remarks.

The basic definitions and concepts involving semigroups that are used here can be found in [2].

For related literature, see [5], [6], [7], [9], [10], and § 5.2 in [2].

Let S be a finite semigroup and let K be a field. The *semigroup algebra* KS is the free algebra on S; that is S forms a K-basis for KS and multiplication in KS is induced by that in S.

If S has a zero element z, let K_0S denote the *contracted semigroup algebra* of S. We see that K_0S is an algebra that has the nonzero members of S as a basis, with multiplication $\circ$ determined by

$$s \circ t = st \text{ if } st \neq z \quad \text{and} \quad s \circ t = 0 \text{ if } st = z;\ t \in S\backslash\{z\}\ .$$

If J is an ideal in S, let S/J denote the Rees quotient semigroup of S modulo J.

It is easy to verify that if J is an ideal in S, then the factor algebra KS/KJ is isomorphic to the contracted algebra of S/J. Also, if S has a zero, then $K_0S/K_0J \cong K_0(S/J)$. [2, p. 160].

If A is an algebra over K, we denote by A_k the algebra of $k \times k$ matrices over A, where k is a positive integer.

By a *nongroup* we mean a semigroup that is not a group.

$GF(q)$ denotes the Galois field with q elements.

2. Odd order semigroups.

Let S be a finite semigroup, and let $\varnothing \subset J_1 \subset J_2 \subset \cdots \subset J_k = S$ be a principal series for S. Suppose that K is a field such that KS is semisimple. Then by [2, pp. 161-162], each J_i/J_{i-1} is 0-simple, $i = 2, \cdots, k$, and

$$KS \cong KJ_1 \oplus K_0(J_2/J_1) \oplus \cdots \oplus K_0(J_k/J_{k-1}) .$$

According to M. Teissier (see [2, p. 165]), J_1 is a group. Also, for each $i = 2, \cdots, k$, there is a group H_i such that $K_0(J_i/J_{i-1}) \cong (KH_i)_{k_i}$, the algebra of $k_i \times k_i$ matrices over KH_i, for some positive integer k_i. This is due to W. D. Munn; see [2, p. 162]. Each KH_i, being semisimple, has K as a direct summand. It follows that each $K_0(J_i/J_{i-1})$ has K_{k_i} as a simple direct summand. It is well known that the group algebra KG is semisimple if and only if the characteristic of K does not divide the order of G. Thus we have

THEOREM 2.1. *Let G be a finite group of order n, and let K be a field whose characteristic does not divide n. Suppose that $KG \cong K \oplus \sum_{i=1}^{t} (D_i)_{k_i}$, where each D_i is a division algebra properly containing K. If S is a semigroup such that $KS \cong KG$, then S is a group.*

If n is odd, and if K contains no n-th roots of unity except 1, then it follows from [1] that the hypothesis of the theorem holds. Hence we have the following special case.

COROLLARY 2.2. *Let K be a field of real numbers, and let S be a semigroup of odd order. If $KS \cong KG$ for some group G, then S is itself a group.*

COROLLARY 2.3. *Let S be a semigroup of order n, and let $K = GF(p^m)$, where p is a prime such that no prime divisor of n divides $p(p^m - 1)$. If $KS \cong KG$ for some group G, then S is a group.*

A CONSTRUCTION 2.4. Suppose that A is an algebra over K such that $A = A_0 \oplus A_1 \oplus \cdots \oplus A_t$, for ideals A_i. Suppose further that $A_0 = KS_0$ for a semigroup S_0, and that for each $i = 1, \cdots, t$, A_i is either KS_i or K_0S_i' for a semigroup S_i or a semigroup $S_i' = S_i \cup 0$ with zero, respectively.

Let $S = S_0 \cup \{x + e_0 \colon x \in \bigcup_{i=1}^{t} S_i\}$, where e_0 is an idempotent in S_0. Since $A_iA_j = (0)$ for $i \neq j$, we see that S is a semigroup. Since $S_0 \cup S_1 \cup \cdots \cup S_t$ is a basis for A, we have that $A = KS$. Since S_0 is an ideal in S, S is not a group.

This construction follows that in the proof of Theorem 5.30 in [2]. In that case $S_0 = \{e_0\}$ and A_i is a full matrix algebra, for $i > 0$.

We now see that the hypothesis that n is odd is needed in 2.2. For let D denote the dihedral group of order 8, and let K be a field of characteristic $\neq 2$. Then $KD \cong K \oplus K \oplus K \oplus K \oplus K_2$. By 2.4 there is a nongroup S such that $KS \cong KD$. If K has characteristic 2, there is no such S. In fact, if G is a p-group, if K is a field of characteristic p, and if $KS \cong KG$, then S is a group. For in this case KG has no idempotents except 0 and 1; thus $KG \cong KS$ forces S to have exactly one idempotent which must be an identity. (Notice that a zero element in S is not the zero of KS). Thus the finite semigroup S is a group.

Another example is of interest here. Let $G = S_3$, the symmetric group on 3 letters, and let K have characteristic $\neq 3$. Then $KG \cong KC \oplus K_2$, where C is the group of order 2. Thus, as before, $KG \cong KS$ for some nongroup S.

In examining examples we use the fact that the matrix algebra K_m is a contracted semigroup algebra. This raises the question: What are the semigroups S such that $K_0S \cong K_m$? From Theorem 5.19 and Corollary 3.12 in [2] we get the following answer.

Let P be a nonsingular $m \times m$ matrix over K all of whose entries are either 0 or 1. Let $\{E_{ij}\}$ be the usual m^2 matrix units; $E_{ij}E_{kr} = \delta_{jk}E_{ir}$. Let $U(P)$ denote the multiplicative semigroup of matrices consisting of the zero matrix and all matrices of the form PE_{ij}; $1 \leqq i, j \leqq m$. If S is a semigroup with zero, then $K_0S \cong K_m$ if and only if $S \cong U(P)$ for some such nonsingular P. Moreover, $U(P) \cong U(P')$ if and only if P and P' have the same number of entries equal to one. We see that there are exactly $m^2 - 2m + 2$ nonisomorphic semigroups $U(P)$. Note also that $U(P) \cong U(P')$ if and only if there is a nonsingular matrix T such that $T^{-1}U(P)T = U(P')$.

3. Commutative semigroup algebras.

Let G be an abelian group of order n, and let K be a field whose characteristic does not divide n. Then according to [8], we have

$$KG \cong \bigoplus \sum a_d K(\zeta_d) ; \tag{1}$$

summation is over divisors of n, ζ_d is a primitive d-th root of unity, and $a_dK(\zeta_d)$ indicates $K(\zeta_d)$ as a direct summand a_d times. Further $a_d = n_d/v_d$, where n_d is the number of elements of order d in G and $v_d = \deg(K(\zeta_d)/K)$.

If there are groups $G_1, \cdots, G_m$, with $m > 1$, such that $KG \cong KG_1 \oplus \cdots \oplus KG_m$, then by 2.4 there is a nongroup S such that $KS \cong KG$. By Theorem 5.21 in [2], we see that the converse holds.

Thus given the abelian group G, the semigroups S such that $KS \cong KG$ are precisely those commutative semigroups S such that

(i) S is the disjoint union of groups, $G_1, \cdots, G_s$; and

(ii) $KG \cong KG_1 \oplus \cdots \oplus KG_s$.

By Theorem 4.11 in [2] all semigroups satisfying (i) can be determined. Also, since all finite groups of order less than n, and their corresponding numbers n_d, can be determined, we can use formula (1) to check condition (ii).

Note that if K contains a primitive p^k-th root of unity, and if the cyclic group $C(p^k)$ of order p^k is a direct factor of G, then condition (ii) holds. For in this case $KC(p^k) \cong p^k K$, so that

$$K(C(p^k) \times H) \cong KC(p^k) \otimes KH \cong p^k K(H) .$$

In the following case the converse holds.

Let Q denote the rational field. To avoid trivialities, when we write $K(\zeta_d)$ we assume that d is either odd or divisible by 4.

THEOREM 3.1. *Let $K = Q(\zeta)$, where ζ is a primitive m-th root of unity, and let G be an abelian group. There is a nongroup S such that $KS \cong KG$ if and only if there is a prime p and a positive integer k such that K contains all the p^k-th roots of unity and $C(p^k)$ is a direct factor of G.*

Proof. We just observed the sufficiency of the condition.

Suppose conversely that

$$KG \cong KG_1 \oplus \cdots \oplus KG_s , \qquad s > 1 . \tag{2}$$

Assume that each group algebra KG_i is indecomposable as a direct sum of group algebras. Then for each i, either $G_i = 1$, or KG_i is the direct sum of fields $K(\zeta_d)$, not all equal to K.

Suppose that q is a prime dividing the order of G_i; then q divides the order n of G. For there is some power q^a of q such that $K < K(\zeta_{q^a}) = K(\zeta_d)$ for a divisor d of n. (Otherwise, using the remarks preceding the theorem, KG_i would be decomposable.) Thus $K < Q(\zeta_t) = K(\zeta_{q^a}) = K(\zeta_d)$, where $t = [m, q^a] = [m, d]$, the least common multiple. Since q^a does not divide m, we have that q divides d.

Suppose now that our condition fails, and let $p_1, \cdots, p_r$ be the distinct prime divisors of n. Then for each i, there is a positive integer n_i such that $p_i^{n_i}$ does not divide m and $C(p_i^{n_i})$ is a subgroup of every nontrivial cyclic direct factor of the p_i-Sylow subgroup of G. Choose each n_i to be the smallest such integer. We may assume without loss of generality that $p_i^{n_i-1}$ divides m.

In (2), think of KG and each KG_i being expressed as in (1). Now delete all fields $K(\zeta_d)$ for which $([m, d], n)$ exceeds $p_1^{n_1} \cdots p_r^{n_r}$. On the left of (2) we have left the group algebra of a subgroup of G whose p_i-Sylow subgroup is of type $(p_i^{n_i}, \cdots, p_i^{n_i})$. On the right, after possibly some further decomposition, we have a like situation. We may thus assume that for each i, the p_i-Sylow subgroup P_i of G is of type $(p_i^{n_i}, \cdots, p_i^{n_i})$, with say k_i factors; and for each G_j, the p_i-Sylow subgroup P_i^j of G_j is either trivial or of type $(p_i^{n_i}, \cdots, p_i^{n_i})$, with say k_{ij} factors. Take $k_{ij} = 0$ in case $P_i^j = 1$.

Using (1), we have that

$$KP_i \cong a_i K \oplus b_i K(\zeta_d) , \tag{3}$$

where $d = p_i^{n_i}$, $a_i = p_i^{(n_i-1)k_i}$, and

$$b_i = (p_i^{n_i k_i} - p_i^{(n_i-1)k_i})/\delta_i ; \qquad \delta_i = \deg (K(\zeta_d)/K) .$$

Similarly

$$KP_i^j \cong a_{ij} K \oplus b_{ij} K(\zeta_d) , \tag{4}$$

where $a_{ij} = p_i^{(n_i-1)k_{ij}}$ and

$$b_{ij} = (p_i^{n_i k_{ij}} - p_i^{(n_i-1)k_{ij}})/\delta_i .$$

For some pair α, β we have $k_\alpha > k_{\alpha\beta}$. Otherwise some G_j would be isomorphic to G.

Now use formulas (3) and (4) and the fact that $K(A \times B) \cong KA \otimes KB$ to count the number of summands on each side of (2) that are isomorphic to K. We obtain

$$\prod_{i=1}^{r} p_i^{(n_i-1)k_i} = \sum_{j=1}^{s} \prod_{i=1}^{r} p_i^{(n_i-1)k_{ij}} . \tag{5}$$

Let $f = p_\alpha^{n_\alpha}$; use (3) and (4) to count all summands on each side of (2) isomorphic to $K(\zeta_f)$. Then add the terms in (5) to each side of the resulting equation, getting

$$\prod_{i \neq \alpha} p_i^{(n_i-1)k_i} \cdot p_\alpha^{n_\alpha k_\alpha} = \sum_{j=1}^{s} \prod_{i \neq \alpha} p_i^{(n_i-1)k_{ij}} \cdot p_\alpha^{n_\alpha k_{\alpha j}} . \tag{6}$$

Multiplying (5) by $\prod_{i=1}^{r} p_i^{k_i}$, we have

$$\prod_{i=1}^{r} p_i^{n_i k_i} = \sum_{j=1}^{s} \prod_{i=1}^{r} p_i^{n_i k_{ij} + k_i - k_{ij}} . \tag{7}$$

Multiplying (6) by $\prod_{i \neq \alpha} p_i^{k_i}$, we have

$$\prod_{i=1}^{r} p_i^{n_i k_i} = \sum_{j=1}^{s} \prod_{i \neq \alpha} p_i^{n_i k_{ij}} \cdot p_\alpha^{n_\alpha k_{\alpha j}} . \tag{8}$$

But $k_\alpha > k_{\alpha\beta}$, so that (7) and (8) cannot both hold. This contradiction completes the proof.

COROLLARY 3.2. *Let G be a finite abelian group such that $QG \cong QS$ for a nongroup S. Then $C(2)$ is a direct factor of G.*

REMARK 3.3. Let S be a commutative semigroup of order $2m$, where m is odd. If $QS \cong QG$ for a group G, then either $S \cong G$ or S is the disjoint union of two copies of the group H, where $G = C(2) \times H$.

Proof. Suppose that $QS \cong QG$. Let $G = C(2) \times H$, where H has order m. According to [8], QG completely determines G. Hence if S is a group, then $S \cong G$.

QG has two simple direct summands isomorphic to Q. Thus if S is not a group, $QS \cong QG_1 \oplus QG_2$ for groups G_1 and G_2. It is clear that the orders of G_1 and G_2 have the same prime divisors, and those are the prime divisors of m. Let p be one of these primes, and let P, P_1 and P_2 be the p-Sylow subgroups of H, G_1 and G_2, respectively. Then we have

$$Q(C(2) \times P) \cong QP_1 \oplus QP_2 \,. \tag{9}$$

This leads to an equation $2p^a = p^b + p^c$, which implies $b = c = a$. Thus P, P_1 and P_2 all have the same order p^a. By induction on the exponent p^e of P we see that $P \cong P_1 \cong P_2$. If $e = 1$, then P, P_1 and P_2 are all elementary abelian of the same order, hence isomorphic. Suppose $e > 1$. Deleting direct summands $Q(\zeta_{p^e})$ from both sides of (9) we have

$$Q(C(2) \times P') \cong QP_1' \oplus QP_2' \,,$$

where $P' = \{x \in P: x^{p^e} = 1\}$. As before, P', P_1' and P_2' have the same order; and by induction $P' \cong P_1' \cong P_2'$. From (9), and the fact that P, P_1 and P_2 have the same order, the three groups have the same number of elements of order p^e. Thus $P \cong P_1 \cong P_2$.

Theorem 3.1 fails for arbitrary finite extensions of Q. For let $K = Q(\sqrt{3})$, and let $G = C(12)$. Notice that

$$K(\zeta_3) = K(\zeta_4) = K(\zeta_6) = K(\zeta_{12}) = Q(\sqrt{3}, i) \,.$$

Using this we see that

$$KG \cong KG_1 \oplus KG_2 \,,$$

where $G_1 = C(3)$ and $G_2 = C(3) \times C(3)$.

Theorem 3.1 also fails for the prime fields $GF(p), p$ a prime. To see this, let $K = GF(5)$. Then $KC(8) \cong KC(2) \oplus KC(6)$. Here $K(\zeta_4) = K$ and $K(\zeta_3) = K(\zeta_6) = K(\zeta_8) \cong GF(25)$.

THEOREM 3.4. *Let K be a field of characteristic $p \neq 0$; let*

$G = P \times H$, where P is a p-group and H is an abelian group of order prime to p. Then $KG \cong KS$ for a nongroup S if and only if $KH \cong KT$ for a nongroup T.

Proof. If $KH \cong KT$, and if T is a nongroup, then $S = P \times T$ is a nongroup, and $KG \cong KS$.

Conversely, suppose that S is a nongroup and that $R = KS \cong KG$. Let $KH \cong K_1 \oplus \cdots \oplus K_r$ for fields K_i. Then $R = R_1 \oplus \cdots \oplus R_k$, where $R_i \cong K_i \otimes KP$. The R_i are the indecomposable components of R. As a ring, R_i is isomorphic with K_iP. Thus every element in R_i is either nilpotent or a unit. Let $\pi_1, \cdots, \pi_k$ be the projections of R onto the R_i. Let $X = \{1, \cdots, k\}$.

Let $X_1 = \{i \in X: \pi_i(s)$ is a unit in R_i for all $s \in S\}$. Then $X_1 \neq \varnothing$; otherwise the element $s_1 \cdot s_2 \cdot \cdots \cdot s_n$, the product of all members of S, would be the zero element of R. Let $G_1 = \{s \in S: \pi_j(s) = 0$ for $j \notin X_1\}$. Then G_1 is a group, KG_1 is an ideal in R, and $KG_1 = \sum R_i$ $(i \in X_1)$. Also $R = KG_1 \oplus K_0U$, where $U = \{\rho_1(s): s \in S, s \notin G_1\}$; $\rho_1 = \sum \pi_j$ $(j \notin X_1)$. Fix $j \notin X_1$, and choose $t \in S$ such that $\pi_j(t)$ is a unit in R_j. There is such an element; for if not, R_j would be nilpotent. Let $X_2 = \{i \in X: i \notin X_1$ and $\pi_i(t)$ is a unit in $R_i\}$. Suppose $X \neq X_1 \cup X_2$. Let $\eta = \sum \pi_i$ $(i \in X_2)$ and $\rho_2 = \sum \pi_j$ $(j \notin X_1 \cup X_2)$, and let $G_2' = \{\eta(s): s \notin G_1$ and $\rho_2(s) = 0\}$ and $G_3' = \{\rho_2(s): s \notin G_1$ and $\rho_2(s) \neq 0\} \cup \{0\}$. Note that $0 \in G_2'$.

We have $R = KG_1 \oplus K_0G_2' \oplus K_0G_3'$; $KG_1 = \sum R_i$ $(i \in X_1)$, $K_0G_2' = \sum R_i$ $(i \in X_2)$, and $K_0G_3' = \sum R_i$ $(i \notin X_1 \cup X_2)$.

We continue this procedure until we have

$$R = KG_1 \oplus K_0G_2' \oplus \cdots \oplus K_0G_m' ,$$

with $m > 1$, where the set X is partitioned into disjoint subsets $X_1, \cdots, X_m$; $K_0G_q' = \sum R_j$ $(j \in X_q)$ and for each $q \geqq 1$, either $G_q' = G_q \cup 0$ for a group G_q, or $K_0G_q' \cong R_j$ for some j, and G_q' is not a group with zero. Suppose that the former holds for $q = 1, \cdots, w$, and that X_q is a singleton for $q > w$. Let N be the radical of R, and for each q, let N_q be the radical of K_0G_q'. If $q > w$, then $K_0G_q'/N_q \cong K$. For since $K_0G_q' \cong R_j$ has no nontrivial idempotents, it follows that G_q' has at most two idempotents. If G_q' has only one idempotent, then R_j is nilpotent. This is not the case. Thus G_q' has exactly two idempotents, the 0 and 1 in R_j. Thus G_q' is the disjoint union of a nilpotent semigroup Z and a group V. Clearly $K_0Z \subset N_q$. Thus there is a homomorphism μ of $KV \cong K_0G_q'/K_0Z$ onto $K_j \cong R_j/\mathrm{Rad}\, R_j$. The normalized units of finite order in $K_j \otimes KP$ have order a power of p. Thus V is a p-group (perhaps trivial). Thus the kernel of μ is the radical of KV and $K_j \cong K$.

According to Deskins [4], $R/N \cong KH$ and $KG_q/N_q \cong KH_q$ for $q \leqq w$, where H_q is the p-complement of G_q. Thus

$$KH \cong KH_1 \oplus \cdots \oplus KH_w \oplus K \oplus \cdots \oplus K .$$

This completes the proof.

COROLLARY 3.5. *Let S be a commutative semigroup of order n, and let $K = GF(p)$, where p is the smallest prime dividing n. If $KS \cong KG$ for a group G, then S is a group.*

COROLLARY 3.6. *Let $K = GF(2)$. If S is a commutative semigroup, and if $KS \cong KG$ for some group G, then S is a group.*

Note that $GF(2)$ and transcendental extensions of $GF(2)$ are the only fields K for which Corollary 3.6 will hold. For if K contains $GF(2^t)$, and if G is the cyclic group of order $2^t - 1$, then $KG \cong \sum K$. If K has characteristic $\neq 2$, then $KC(2) \cong K \oplus K$.

THEOREM 3.7. *Let K be the real number field, and let S be a commutative nongroup of order n. Then there is a group G such that $KS \cong KG$ if and only if the following conditions hold:*

(i) *n is even;*

(ii) *S is the disjoint union of group $G_1, \cdots, G_m$;*

(iii) *If 2^{e_i} is the number of elements x in G_i such that $x^2 = 1$, then $\sum_{i=1}^m 2^{e_i}$ is a power of 2 dividing n.*

Proof. The necessity of the conditions follows from the fact that if G is an abelian group, then $GK \cong aK \oplus bL$, where $a - 1$ is the number of elements of G of order 2, and L is the complex field.

Conversely, suppose the conditions hold, and let $\sum_{i=1}^m 2^{e_i} = 2^e$. Let $n = 2^e \cdot 2^f \cdot m$, with m odd; let $G = C(2) \times \cdots \times C(2) \times C(2^{f+1}) \times H$, where there are $e - 1$ factors $C(2)$ and H is any abelian group of order m. Then clearly $KS \cong KG$.

4. **Semilattices.** A semigroup in which every element is idempotent is called a *band*. A commutative band is a (lower) semilattice under the ordering: $e \leqq f$ if $e = ef$. Conversely, any semilattice is a commutative band under the operation $e \cdot f = e \wedge f$.

If S is a semilattice, and if R is a commutative ring with identity, then the semigroup ring RS has an identity. ([6, Th. 7.5]). Corresponding to Theorem 5.27 in [5] we have

THEOREM 4.1. *Let S be a semilattice of order n. Then RS is* *the direct sum of n copies of R and R_0S is the direct sum of $n-1$ copies of R.*

Proof. The theorem is trivial for $n = 1$. If $n = 2$, and $S = \{z, e\}$, with $ez = ze = z$, then $R_0S = Re$ and $RS = Rz \oplus R(e - z)$, so the theorem holds.

Suppose that $n > 2$ and proceed inductively. Choose $f \in S$ such that f is neither the zero of S nor the identity of S, in case there is one. Let $J = Sf$. Then $RS = (RS)f \oplus RS(1 - f) \cong RJ \oplus R_0(S/J)$. Since both J and S/J are semilattices of order less than n, we have by induction that RJ and $R_0(S/J)$ are direct sums of copies of R, and hence so is RS.

Similarly $R_0S \cong R_0J \oplus R_0(S/J)$ and induction gives R_0S as a sum of copies of R.

As a partial converse we have

THEOREM 4.2. *Let S be a semigroup of order n, and let R be an integral domain such that no prime $p \leqq n$ is a unit in R. If RS is the direct sum of copies of R, then S is a semilattice.*

Proof. Let $RS \cong R \oplus \cdots \oplus R$, and let K be the quotient field of R. Then $KS \cong K \oplus \cdots \oplus K$, so that KS is semisimple. Hence by [2, Cor. 5.15] S is a semisimple commutative semigroup. Thus S has a principal series $\phi < S_1 < S_2 < \cdots < S_k = S$ such that the kernel $S_1 = G_1$ is a group and S_i/S_{i-1} is a group with zero $G_i \cup 0$ for $i = 2, \cdots, k$. Thus $RS \cong RG_1 \oplus \cdots \oplus RG_k$. By [3] each RG_i is indecomposable; but by hypothesis each is the direct sum of copies of R. Thus each G_i is trivial, so that S is a semilattice.

Using Theorem 4.2 and the results of § 3, we have

PROPOSITION 4.3. Let S be a semilattice, let T be a commutative semigroup of the same order, and let K be a field of characteristic 0. Then $KS \cong KT$ if and only if T is the disjoint union of groups $G_1 \cup \cdots \cup G_k$ such that if G_i has exponent m_i, then K contains the m_i-th roots of unity.

Using Theorem 4.2 and the fact that for a band S, KS is semisimple if and only if S is commutative [2, p. 169], we see

PROPOSITION 4.4. Let S be a band, and let G be a group of the same order n. Let K be a field whose characteristic does not divide n. Then $KS \cong KG$ if and only if S and G are commutative and F contains the m-th roots of unity, where m is the exponent of G.

Let $R = GF(2)$. Using the fact that $RS \cong R \oplus \cdots \oplus R$ for a finite semilattice S, we may derive the following well known result:

Every semilattice S of order n can be embedded in the lattice 2^n (subsets of the set $\{1, 2, \cdots, n\}$. In fact, S can be considered as linearly independent subset of 2^n, where 2^n is viewed as $R \times \cdots \times$

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