PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 3, 1968

SUBDIRECT DECOMPOSITIONS OF LATTICES OF WIDTH TWO

OSCAR TIVIS NELSON, JR.

The class of nontrivial distributive lattices is the class of subdirect products of two-element chains. Lattices of width one are distributive and hence are subdirect products of two element chains. Below it is shown that lattices of width two are subdirect products of two element chains and nonmodular lattices of order five (N_5). (width = greatest number of pairwise incomparable elements.)

The statement follows from several lemmas. Throughout we shall assume that a, b are arbitrary noncomparable elements of a lattice L of width two.

LEMMA 1. *$x\cdot(a+b)+y\cdot(a+b)=(x+y)\cdot(a+b)$ and*

$$(x+a\cdot b)\cdot(y+a\cdot b)=x\cdot y+a\cdot b$$

for any $x, y\in L$.

Proof. In any lattice

$$x\cdot(a+b)+y\cdot(a+b)\leqq(x+y)\cdot(a+b)\ . \tag{1}$$

Trivially, if x and y are related, the identity holds. Thus, assume that x and y are unrelated. There are three possibilities:

(i) Suppose $x\leqq a$ and $y\leqq b$. Then

$$x\cdot(a+b)+y\cdot(a+b)=x+y=(x+y)\cdot(a+b)\ .$$

(ii) In case $a\leqq x$ and $b\leqq y, a+b\leqq x+y$. If $a+b\leqq x$ or y, it is easy to verify that the identity holds. If $a+b\nleqq x$ or y, then x or $y\leqq a+b$. Suppose $x\leqq a+b$. Then

$$(x+y)\cdot(a+b)=a+b\leqq x+y\cdot(a+b)=x\cdot(a+b)+y\cdot(a+b)\ .$$

This relation and (1) yield the equality.

(iii) Now suppose $a\leqq x$ and $y\leqq b$. $b\leqq x$ implies that x and y are comparable while $x\leqq b$ implies that a and b are comparable. Thus, x and b are unrelated. Since L is of width two, $a+y$ is related to either x or b, $a+y\leqq x$ and $a+y\leqq b$ imply that $y\leqq x$ and $a\leqq b$ respectively. Thus, either $x\leqq a+y$ or $b\leqq a+y$. In case $x\leqq a+y, x\leqq a+y\leqq a+b$ and $y\leqq b\leqq a+b$. Hence

$$x\cdot(a+b)+y\cdot(a+b)=x+y=(x+y)(a+b)\ .$$

In case $b \leqq a + y, y \leqq b \leqq a + y \leqq a + b$. Thus,

$$(x + y)\cdot(a + b) \leqq a + b \leqq a + y \leqq x\cdot(a + b) + y$$
$$= x\cdot(a + b) + y\cdot(a + b)$$

and the identity holds in all cases. A dual argument yields the other identity.

By Lemma 1, if s and t are unrelated elements of a lattice of width two, the mappings $x \to x\cdot(s + t)$ and $x \to x + s\cdot t$ determine congruence relations θ_{s+t} and $\psi_{s\cdot t}$.

LEMMA 2. $\theta_{a+b} \cap \psi_{a\cdot b} = 0$.

Proof. If $x \equiv y(\theta_{a+b} \cap \psi_{a\cdot b})$, $x\cdot(a + b) = y\cdot(a + b)$ and $x + a\cdot b = y + a\cdot b$. x and y are each related to either a or b. Thus $x \leqq a + b$ or $a\cdot b \leqq x$. Similarly, $y \leqq a + b$ or $a\cdot b \leqq y$. If

$$x, y \leqq a + b, x = x\cdot(a + b) = y\cdot(a + b) = y\ .$$

If $a\cdot b \leqq x, y$; $x = x + a\cdot b = y + a\cdot b = y$. Finally, if $x \leqq a + b$ and $a\cdot b \leqq y$, $a\cdot b \leqq y\cdot(a + b) = x\cdot(a + b) = x$, i.e., $a\cdot b \leqq x, y$ again. Thus $x = y$ in every case, and $\theta_{a+b} \cap \psi_{a\cdot b} = 0$.

LEMMA 3. *If* $\theta_{a+b} = 0, a + b = 1$; *and if* $\psi_{a\cdot b} = 0, a\cdot b = 0$.

Proof. By definition $x\cdot(a + b) \equiv x(\theta_{a+b})$. Thus $\theta_{a+b} = 0$ implies that

$$x\cdot(a+ b) = x$$

for all x, and consequently that $a + b = 1$. Similarly, $\psi_{a\cdot b} = 0$ implies that $a\cdot b = 0$.

LEMMA 4. *If L is subdirectly irreducible,* $\theta_{a+b} = 0$, *and* $a\cdot b \neq 0$, *then there exists* $p \in L$ *such that p and* $a\cdot b$ *are noncomparable.*

Proof. *If* $\theta_{a+b} = 0, a\cdot b \neq 0$, and there exists no p as above, than it is easy to verify that $\theta_{a\cdot b} \cap \psi_{a\cdot b} = 0$. (Note that $x \equiv y\ (\theta_{a\cdot b})$ if and only if $x = y$ or $a\cdot b \leqq x, y$, and that $x \equiv y(\psi_{a\cdot b})$ if and only if $x = y$ or $x, y \leqq a\cdot b$). Since $a\cdot b \neq 0$, neither $\theta_{a\cdot b}$ nor $\psi_{a\cdot b} = 0$. Thus L is reducible. This contradiction implies that p must exist.

If L and p are as in Lemma 4, p must be related to a or b, but $a \leqq p$ or $b \leqq p$ implies that p and $a\cdot b$ are comparable. Thus we can assume that p is less than one of a, b; assume $p < a$.

LEMMA 5. *If L and p are as in Lemma* 4, $p + a\cdot b$ *and b are* *noncomparable.*

Proof. Clearly $a \cdot b \leqq b \cdot (p + a \cdot b)$. Since $p < a, p + a \cdot b \leqq a$, and hence $b \cdot (p + a \cdot b) \leqq a \cdot b$. Thus $b \cdot (p + a \cdot b) = a \cdot b$. Since a and b are noncomparable, $a \cdot b \neq b$; and since $a \cdot b, p$ are noncomparable, $p + a \cdot b \neq a \cdot b$. Thus b and $p + a \cdot b$ are noncomparable.

LEMMA 6. *If L and p are as in Lemma 4,*

$$L = \{x \mid x \leqq p + a \cdot b\} \cup \{x \mid a \cdot b \leqq x\} .$$

Proof. Trivially, if z is related to $a \cdot b, z$ is in one of the sets. Thus suppose that $z, a \cdot b$ are unrelated. Since $a \cdot b, p$ are noncomparable and L is of width two, z must be related to p. If $z \leqq p$, $z \leqq p + a \cdot b$. If z is also related to $p + a \cdot b, z$ is in one of the sets. Thus, suppose that $p \leqq z$ and that z and $p + a \cdot b$ are unrelated. By Lemma 5, $p + a \cdot b$ and b are unrelated. Thus, z must be related to b. If $b \leqq z, a \cdot b \leqq z$; and if $z \leqq b, p \leqq z < b(p + a \cdot b \leqq b)$. But both conclusions are impossible. Thus L is the union of the two sets.

LEMMA 7. *If L and p are as in Lemma 4, $\theta_{p+a \cdot b} \cap \psi_{a \cdot b} = 0$.*

Proof. If $x \equiv y(\theta_{p+a \cdot b} \cap \psi_{a \cdot b}), x \cdot (p + a \cdot b) = y \cdot (p + a \cdot b)$ and $x + a \cdot b = y + a \cdot b$. If

$$x, y \leqq p + a \cdot b, x = x \cdot (p + a \cdot b) = y \cdot (p + a \cdot b) = y;$$

and if $a \cdot b \leqq x, y, x = x + a \cdot b = y + a \cdot b = y$. Thus suppose

$$x \leqq p + a \cdot b$$

and $a \cdot b \leqq y$ (By Lemma 6, we can assume that this is the only remaining possibility.) Then $x = x \cdot (p + a \cdot b) = y \cdot (p + a \cdot b) \leqq y$, i.e., $x \leqq y$. Also, $x + a \cdot b = y + a \cdot b = y$. Since

$$x \leqq p + a \cdot b, y = x + a \cdot b \leqq p + a \cdot b .$$

Thus, $x \leqq y \leqq p + a \cdot b$, and $x = x \cdot (p + a \cdot b) = y \cdot (p + a \cdot b) = y$.

LEMMA 8. *If L is a subdirectly irreducible lattice of width two and a, b are noncomparable elements of $L, a + b = 1$ and $a \cdot b = 0$.*

Proof. By Lemma 2, $\theta_{a+b} \cap \psi_{a \cdot b} = 0$. Since L is irreducible, $\theta_{a+b} = 0$ or $\psi_{a \cdot b} = 0$. Suppose $\theta_{a+b} = 0$. Then $a + b = 1$ by Lemma 3. If $a \cdot b \neq 0, \psi_{a \cdot b} \neq 0$ by Lemma 3. Also, by Lemma 4, there is an element p of L which is noncomparable to $a \cdot b$. For this p,

$$\theta_{p+a\cdot b} \cap \psi_{a\cdot b} = 0$$

by Lemma 7. Hence $\theta_{p+a\cdot b} = 0$. But this is impossible since it implies that $1 = p + a \cdot b \leqq a$ or b. Hence $a \cdot b = 0$. If $\psi_{a\cdot b} = 0$, a dual argument completes the proof.

(Note that Lemma 8 implies that a subdirectly irreducible lattice of width two has a zero and a one.)

Let L be a subdirectly irreducible lattice of width two. If there were an element z of $L - \{0, 1\}$ which was comparable to each element of L, $\theta_z \cap \psi_z = 0$ with $\theta_z \neq 0$ and $\psi_z \neq 0$. Thus, since L is irreducible, it must be the union of the pairwise disjoint sets $\{0, 1\}$, C_1, C_2 where C_1, C_2 are chains such that the sum of elements from different chains is 1 and the product, 0. If each chain has at least two elements, then one can define two congruence relations R_1, R_2 as follows:

$x \equiv y(R_i)$ if and only if $x = y$ or $x, y \in C_i$ $(i = 1, 2)$. Clearly, $R_1 \cap R_2 = 0$, but $R_1, R_2 \neq 0$ since each chain contains at least two elements. Thus, one chain must contain exactly one element. If both chains consist of a single element, L is a direct product of two-element chains, and hence is reducible. Thus, L consists of $\{0, 1\}$, C_1, C_2 where C_1 contains only one element and C_2 contains at least two elements. Suppose C_2 contains at least three elements $p < q < r$. Define relations S_1, S_2 on L by

$$x \equiv y\,(S_1) \text{ if and only if } x = y \text{ or } 0 < x, y \leqq q,$$
$$x \equiv y\,(S_2) \text{ if and only if } x = y \text{ or } q \leqq x, y < 1\,.$$

It is easy to show that these are congruence relations. Clearly $S_1 \cap S_2 = 0$. Thus $S_1 = 0$ or $S_2 = 0$. But $p \equiv q(S_1)$ and $q \equiv r(S_2)$, a contradiction. Thus C_2 consists of exactly two elements, and $L \cong N_5$. Hence

THEOREM. *Every lattice of width two is a subdirect product of two-element chains and* N_5.

COROLLARY. *The only subdirectly irreducible lattice of width two is* N_5.

For each $n \geqq 3$, one can exhibit a lattice to show that it is false that all lattices of width n are subdirect products of lattices from some class of finite lattices. For a fixed n, it would be of interest to find a lattice property P such that if L were of width n and had property P, that L would be a subdirect product of finite lattices.

References

1. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloquium Publications, Vol. 25, 1948 (rev. ed.).
2. ———, *Subdirect unions in universal algebras*, Bull. Amer. Math. Soc. **50** (1944) 764-768.

Received January 16, 1967.

EMORY UNIVERSITY