PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 3, 1968

A NOTE ON LEFT MULTIPLICATION OF SEMIGROUP GENERATORS

KARL GUSTAFSON

It is shown in this note that if A is the infinitesimal generator of a strongly continuous semigroup of contraction operators in any Banach space X, then so is BA for a broad class of bounded operators B; the only requirement on B is that it transforms "in the right direction".

In the recent paper [1] the following interesting result was obtained.

THEOREM 1 (*Dorroh*). *Let X be the Banach space of bounded functions on a set S under the supremum norm, let A be the infinitesimal generator of a contraction semigroup in X, and let B be the operator given by multiplication by p, $pX \subseteqq X$, where p is a positive function defined on S, bounded above, and bounded below above zero. Then BA is also the infinitesimal generator of a contraction semigroup in X.*

This leads naturally to the general question of preservation of the generator property under left multiplication; the purpose of this note is to present Theorem 2 below, which shows that for any Banach space, a large class of operators B are acceptable. In the following, the word "generator" will always mean generator of contraction semigroup.

In this note we will consider only left multiplication by everywhere defined bounded operators B. It is easily seen (e.g., [2, Corollary 3]) that A generates a contraction semigroup if and only if cA does, $c > 0$. Also by [4, Th. 2.1], if A is bounded, BA is a generator if and only if BA is dissipative; in this case clearly right multiplication also yields a generator. See [4, 5] for dissipativeness; we use dissipativeness in the sense [4], and recall that if BA is a generator, then BA is dissipative in all semi-inner products on X.

THEOREM 2. *Let X be any Banach space, A the infinitesimal generator of a contraction semigroup in X, and B a bounded operator in X such that $\|\varepsilon B - I\| < 1$ for some $\varepsilon > 0$. Then BA generates a contraction semigroup in X if and only if BA is dissipative, (i.e., $\operatorname{Re}[BAx, x] \leqq 0$, all $x \in D(A)$, $[u, v]$ a semi-inner product (see [4])).*

Proof. We note that $R(B) = X$ when $\|\varepsilon B - I\| < 1$ for some $\varepsilon > 0$; to show that BA is a generator it suffices to show that εBA is a generator for some positive ε. From the relation $\|\varepsilon B - I\| < 1 \leqq \|(I - \varepsilon BA)^{-1}\|^{-1}$ we have by [2, Lemma 1] that:

$$\beta(I - \varepsilon BA) = \beta((I - \varepsilon BA) + (\varepsilon B - I)) \equiv \beta(\varepsilon B(I - A)) = \beta(\varepsilon B) = 0\ ,$$

where $\beta(T) = \dim X/\mathrm{Cl}\,(R(T))$ is the deficiency index of an operator T. A closed implies εBA closed (and therefore $I - \varepsilon BA$ closed), since $\varepsilon BA = A + (\varepsilon B - I)A$ and $\|\varepsilon B - I\| < 1$; BA dissipative implies that $I - \varepsilon BA$ possesses a continuous inverse, so that we therefore have $R(I - \varepsilon BA)$ closed, and thus BA the generator of a contraction semigroup. This result also follows quickly from [2, Theorem 2].

In the above we made use of basic index theory as may be found in [3] and the well-known characterizations of generators as may be found in [3, 4, 5], for example. The index theory notation here is a convenience only; the arguement can be presented without it.

COROLLARY 3. *Theorem* 1 *stated above.*

Proof. As shown in [1], pA is dissipative with respect to the semi-inner product used there, and clearly $0 < m \leqq p(s) \leqq M$ implies that $|\varepsilon p - 1| < 1 - \varepsilon m$ for small enough ε.

COROLLARY 4. *Let B be of the form $cI + C, \|C\| < c, CA$ dissipative. Then BA is a generator if A is.*

Proof. Clearly $c^{-1}B$ satisfies the conditions of Theorem 2; note $\|\varepsilon B - I\| < 1$ for some $\varepsilon > 0$ if and only if B is of the form $cI + C$, $\|C\| < c$.

Remarks. The condition BA dissipative in Theorem 2, necessary for BA to be a generator, requires (in general) that B be in a "positive" rather than a dissipative direction. For example, if A, B, and BA are self-adjoint operators on a Hilbert space, then A is a generator if and only if A is negative, and then BA is a generator if B is positive.

The condition $\|\varepsilon B - I\| < 1$ in Theorem 2 is easily seen to be equivalent to the condition: B strongly accretive, i.e., $\exists m = m(B)$ such that $\mathrm{Re}\,[Bx, x] \geqq m > 0$ for $\|x\| = 1$, where $[u, v]$ is the semi-inner product being used (see [4]). It is a sharp condition since equality $\|\varepsilon B - I\| = 1$ cannot be permitted in general, as seen from the example $B = 0$, A unbounded, for then BA is not closed.

The effect of Theorem 2 is that, after the application of index theory therein, one sees that the essential question concerning when BA is a generator is the question of when BA is dissipative. Three situations which can then occur are: (i) as in [1], for special operators B, one can find a semi-inner product for which BA is dissipative; (ii) A commutes with B (see [3]), for which one can easily obtain results such as A self-adjoint, dissipative, and B accretive imply BA dissipative; (iii) general (noncommuting) A and B. For case (iii) one can obtain the following interesting result (proof given in forthcoming paper by the author, Math. Zeitschrift). Let $-A$ and B be strongly accretive operators on a Banach space. If

$$\min_{\varepsilon} \| \varepsilon B - I \| \leqq m(-A) \cdot \| A \|^{-1} ,$$

then BA is dissipative. In particular, let A and B be self-adjoint operator: then $(\| B \| - m(B)) \cdot (\| B \| + m(B))^{-1} \leqq m(-A) \cdot \| A \|^{-1}$ is sufficient. Moreover these conditions can be sharpened by introducing the concept of the cosine of an operator. For certain operators the condition for BA to be dissipative can then be written as $\sin B \leqq \cos A$.

The author appreciates useful expository suggestions from the referee. Extensions of these results to unbounded right and left multiplication will appear in a forthcoming paper by the author and G. Lumer.

References

1. J. R. Dorroh, *Contraction semigroups in a function space*, Pacific J. Math. **19**, (1966), 35-38.

2. K. Gustafson, *A perturbation lemma*, Bull. Amer. Math. Soc. **72** (1966), 334-338.

3. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966.

4. G. Lumer and R. S. Phillips, *Dissipative operators in a Banach space*, Pacific J. Math. **11** (1961), 679-698.

5. E. Nelson, *Feynman integrals and the Schrödinger equation*, J. Mathematical Phys. **5** (1964), 332-343.

Received April 3, 1967, and in revised form July 11, 1967. This work was partially supported by N.S.F. G.P. 7041X.

UNIVERSITY OF MINNESOTA
AND
INSTITUT BATTELLE, GENEVA