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POINTLIKE SUBSETS OF A MANIFOLD

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Morton Brown introduced the concept of a cellular subset of S^n. As a consequence of the generalized Schoenflies Theorem it is easy to show that a subset of S^n is pointlike if and only if it is cellular. In this paper the obvious generalization of the definitions of pointlike and cellular sets are made and thier relationship in a manifold is considered. It is easy to show that a cellular subset of a manifold is pointlike. While it is not true that a pointlike subset of a manifold is cellular, it is shown that a pointlike subset of a compact n-manifold lies in a contractible n-manifold with $(n-1)$-sphere boundary. As a consequence of this it is shown that K is a pointlike subset of a compact n-manifold $(n \neq 4)$ if and only if K is cellular. The case $n = 4$ is still unsolved.

DEFINITIONS. An *n-manifold* is a connected separable locally Euclidean metric space. A connected separable metric space in which every point has a neighborhood whose closure is an n-cell is an *n-manifold with boundary.* Note that a manifold is a manifold with boundary boundary but not conversely. A compact connected subset K of an n-manifold M is *pointlike* if $M \sim K$ is homeomorphic with $M \sim \{p\}$ where $p \in M$. A subset K of an n-manifold M is *cellular* if there is a sequence of n-cells $C_1, C_2, \cdots$ such that $C_{i+1} \subset \text{Int } C_i$ and $K = \cap C_i$. An $(n-1)$-sphere S^{n-1} that separates an n-manifold M into components A and B is *collared on the side containing* A if there is an embedding $h: S^{n-1} X[0, 1] \rightarrow \bar{A}$ such that $h(x, 0) = x$. An $(n-1)$-sphere S^{n-1} in an n-manifold M is *bicollared* if there is an embedding $h: S^{n-1}X[0, 1] \rightarrow M$ such that $h(x, 1/2) = x$. A *pseudo-sphere* is a compact manifold that is a homotopy sphere. A compact contractible n-manifold with boundary is called a pseudo-cell. The Poincare Conjecture—known to be true for $n \neq 3, 4$ [7]—says that a pseudo-sphere is a sphere.

PRELIMINARY THEOREMS. The following theorem follows from the corresponding theorem for E^n which is proved by the same methods as used in [4].

THEOREM 1. *A cellular subset of a manifold is pointlike.*

One might think that a pointlike subset of a manifold is cellular. That this is not the case is shown by the following example.

EXAMPLE 1. Let M be E^3 minus the integers on the positive x-axis, and minus 1-spheres of radius 1/4 centered at the negative integers on the x-axis. The 1-sphere of radius 1/4 and center at 0 is pointlike but not cellular. A similar construction using linked 1-spheres gives an example of a pointlike subset of a manifold containing a loop that is homotopically nontrivial in the manifold. A cellular subset of a manifold is not necessarily contractible, for example the crumpled cube bounded by the Alexander Horned sphere is not simply connected even though it is cellular.

LEMMA 2. *Let K be a pointlike subset of a compact manifold M with boundary. Let $h': M \sim K \rightarrow M \sim \{p\}$ be a homeomorphism. Then h' can be extended to a continuous map $h: M \rightarrow M$ such that $h^{-1}(p) = K$.*

Proof. Define h by

$$h(x) = \begin{cases} h'(x) & \text{for} \quad x \in M \sim K\,, \\ p & \text{for} \quad x \in K\,. \end{cases}$$

Let U be an open neighborhood of p. Then $\sim U$ is compact; hence, $h^{-1}(\sim U)$ is compact so $M \sim h^{-1}(\sim U)$ is open. Clearly this set contains K. Thus h is continuous.

LEMMA 3. *If K is a pointlike subset of a compact n-manifold M with boundary and K lies in an open n-cell, then K is cellular.*

Proof. We shall show that if U is a neighborhood of K then there is an n-cell C such that $K \subset \operatorname{Int} C \subset U$. Using this a simple inductive argument completes the proof. Let $h: M \rightarrow M$ be the continuous map given by the previous lemma. Then $h(U)$ is a neighborhood of p. Let C' be an n-cell with bicollared boundary in $h(U)$ containing p in its interior. Then $h^{-1}(C') = C$ is a cell by the Generalized Schoenflies theorem.

By obvious modifications of the proof in [8], the Jordan-Brouwer Theorem can be shown to hold in a pseudo-n-sphere. Let K be the closure of one of the complementary domains of S^{n-1}. If an n-cell is sewn to K the result is another pseudo-sphere. Applications of the Van Kampen Theorem, the Mayer-Vietoris Sequence and the Hurewicz Isomorphism show that K is $(n-2)$-connected. Theorem 6.6.5 and Theorem 6.2.20 of [8] show that K is contractible.

LEMMA 4 (*Pseudo Schoenflies Lemma*). *A bicollared $(n-1)$-sphere S^{n-1} in a pseudo-sphere M^n is the common boundary of two psudo-cells.*

MAIN RESULT.

THEOREM 5. *If K is a pointlike subset of a compact manifold M^n and $\dot{K}$ is an $(n-1)$-sphere collared on the side containing K, then K is a pseudo-cell.*

Proof. Assume $n \geqq 3$. Denote by L the set $((M^n \sim K) \cup$ collar of $\dot{K})$. Then L and K are closed and their union is M^n while their intersection is simply connected. By the Van Kampen Theorem $\pi_1(M^n) = \pi_1(L) * \pi_1(K)$, where $*$ denotes the free product. Borsuk [2] has shown that every compact manifold is dominated by a polyhedron, that is there is a finite polyhedron P and continuous maps $f: P \rightarrow M^n$ and $g: M^n \rightarrow P$ such that $f \circ g$ is a homotopic to 1_{M^n}. It follows that $\pi_1(M^n)$ is a finitely presented group. Since K is pointlike, $\pi_1(M^n \sim K) = \pi_1(L) = \pi_1(M^n \sim \{p\}) = \pi_1(M^n)$. We have $\pi_1(M^n) = \pi_1(K) * \pi_1(L) = \pi_1(K) * \pi_1(M^n)$. By Grusko's theorem [6], $\pi_1(K)$ is trivial.

To show that $\pi_q(K)$ is trivial for $q \leqq n$ we show that $H_q(K)$ is trivial for $q \leqq n-2$, then we use duality to get $H_g(K) = 0$ for $q \leqq n$. Since K and L form an excisive couple we may apply the Mayer-Vietoris Sequence to get

$$H_q(K \cap L) \rightarrow H_q(K) \oplus H_q(L) \rightarrow H_q(K \cup L) \rightarrow H_{q-1}(K \cap L)\ , \qquad 1 \leqq q \leqq n-2\ .$$

Since $K \cap L$ is an n-annulus this sequence becomes

$$0 \rightarrow H_q(K) \oplus H_q(L) \rightarrow H_q(K \cup L) \rightarrow 0\ ,$$

which implies that $H_q(K) \oplus H_p(L) \approx H_q(K \cup L)$. Since K is pointlike, $H_q(K \cup L) \approx H_q(L)$. Since there is a dominating polyhedron for M^n, $H_q(M^n)$ is a finitely generated group. It follows that $H_q(K)$ is trivial. By the Hurewicz Isomorphism Theorem, $\pi_q(K) = 0$ for $1 \leqq q \leqq n-2$. Let S be the compact manifold obtained by sewing a cell to the boundary of K. Then by duality, S is a homotopy sphere. By Lemma 4, K is contractible.

If $n = 2$ then K can be shown to shown to be a 2-cell by the classification theorem for compact 2-manifolds with contours for boundary.

COROLLARY 6. *Let K be a pointlike subset of a compact manifold M, then K lies in a pseudo-cell with sphere boundary.*

Proof: Let $h: M \rightarrow M$ be the continuous map given by Lemma 2. Let C' be a cell containing p and having a bicollared boundary. Then C' is pointlike so $h^{-1}(C') = C$ is a pointlike subset of M with bicollared sphere boundary. The previous theorem shows that C is a pseudo-cell.

COROLLARY 7. *In a compact manifold in which every pseudo-cell with sphere boundary is a cell, a pointlike subset is cellular.*

LEMMA 8. If K is a pointlike subset of a compact manifold M, then there are infinitely many disjoint homeomorphic copies of K in M.

Proof. Let $p \in M \sim K$ and let $h: M \sim K \rightarrow M \sim \{p\}$ be a homeomorphism. Let $h^{-1}(K) = K_1 \subset M \sim K$. Let g_1 be a homeomorphism of M onto itself such that $g_1(p) = p_1 \notin K \cup K_1$ and $g_1 = 1$ on K. Let $h_1 = g_1 \circ h$. Then $h_1^{-1}(K_1) = K_2$ is homeomorphic with k and

$$h_1^{-1}(K_1) \cap (K_1 \cap K) = \emptyset .$$

Continuing in this fashion we get $K, K_1, K_2, \cdots$.

The complement of two disjoint pointlike subsets of a manifold M need not be homeomorphic with the complement of two points in M; for example two linked 1-spheres in the 3-manifold of Example 1.

THEOREM 9. *A pointlike subset of a compact n-manifold ($n \neq 4$) is cellular.*

Proof. By Corollary 6, the pointlike set lies in a pseudo-cell P with sphere boundary. Sew a cell to P along their boundaries to get a homotopy sphere S^n. Since the Poincare Conjecture has been proved [7] for $n \geqq 5$, S^n must be a sphere. The generalized Schoenflies Theorem [3] shows that P is a cell. An application of Lemma 3 completes the proof when $n \geqq 5$. If K is a pointlike subset of a compact manifold M, then there are countably many disjoint homeomorphic copies of K in M. Thus if K is a pointlike subset of M that is not cellular, then M must contain countably many disjoint pseudo-cells that are not cells. If $n = 3$, M is triangulable so an application of Bing's Side Approximation Theorem [1] allows us to assume that each pseudo-cell has a polyhedral sphere boundary. Kneser [5] has shown that such a decomposition can contain only finitely many such sets that are not cells.

We note that we have a generalization of the Generalized Schoenflies theorem: If S^{n-1} is a bicollared $(n-1)$-sphere that separates a compact n-manifold M and one of the components of $M - S^{n-1}$ is pointlike, then that component is a pseudo-cell.

One should observe that the proof the Theorem 5 shows: If K is a pointlike subset of an n-manifold M, $\pi_m(M)$ is finitely generated for $1 \leqq m \leqq n$, and $\dot{K}$ is an $(n-1)$-sphere collared on the side containing K, then K is a pseudo-cell.

Using arguments like those used in the proof of Theorem 5, one can show that a compact n-manifold ($n \neq 4$) can be written as the connected sum of at most finitely many nontrivial summands.

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Question. If we drill countably many disjoint cells out of S^4 and sew in pseudo-cells, is the resulting space ever a manifold?

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