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ISOMETRIC MULTIPLIERS

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Let G be a locally compact group with right Haar measure. A left multiplier on $L^p(G)$ is a bounded operator which commutes with all the operators induced by left translations. The main theorem of this paper states that every isometric left multiplier on $L^p(G)$ for $1 \leqq p < \infty, p \neq 2$, is a scalar multiple of an operator induced by a right translation.

Wendel proved this for $p = 1$ and used it to show that if $L^1(G_1)$ and $L^1(G_2)$ are isomorphic as Banach algebras under convolution, then G_1 and G_2 are isomorphic as topological groups. In § 5 we obtain some extensions of this result to L^p. An interesting byproduct is a theorem which states that an operator which is simultaneously a contraction on L^p and unitary on L^2 (of a finite measure space) is actually an isometry on L^p.

Curiously, the proofs given below do not rely in any crucial way on the fact that the measure spaces $L^p(G)$ are defined with respect to Haar measure, and consequently the results are valid for a much larger class of measures. In § 4 this fact is used to obtain examples of operators on L^p which commute with no isometries (save scalar multiples of the identity).

An enlightening example is provided by taking G to be the group of complex numbers of modulus one. It is not difficult to show that a multiplier on $L^p(G)$ sends a function $\sum_{n=-\infty}^{\infty} a_n z^n$ into $\sum_{n=-\infty}^{\infty} c_n a_n z^n$, where $\{c_n\}$ is a fixed sequence. If the multiplier is to be an isometry, each c_n must have modulus one, and if $p = 2$, this condition is also sufficient. For $p \neq 2, \infty$, the main theorem states that the multiplier is an isometry if and only if it is a scalar multiple of an operator induced by a rotation of the circle, which means there are constants b, d of modulus one such that $c_n = d \cdot b^n$ for all n.

2. Preliminaries. Throughout, G denotes a locally compact topological group with the group operation written multiplicatively. Elements of G are indicated by $g, h, x, y, \cdots$, and Roman capitals $F, G, H, \cdots$, usually denote functions. The only L^p spaces considered are those with $1 \leqq p < \infty$, and usually p refers to a number in this range different from 2. The L^p spaces may be either real or complex, and all operators are assumed to be bounded. The characteristic function of the set Δ is called χ_Δ.

The left and right translation operators L_g and R_g are defined by $(L_gF)(x) = F(gx)$ and $(R_gF)(x) = F(xg)$. We shall also denote the left translate $L_g F$ of F by F_g.

The fact that the theorems to be presented are valid even if $L^p(G)$ is defined with respect to a measure other than Haar measure indicates that these results are more measure-theoretic than algebraic in nature. (In fact, if μ is not Haar measure, $L^p(G)$ is not even an *algebra* because convolution is not associative.) Essentially, they are consequences of the fact that there are relatively few isometries on L^p of a measure space for $p \neq 2$.

This observation is probably more interesting than the particular generalizations thus obtained, so to avoid complications we shall restrict our attention to a smaller subclass of measures on G than is strictly necessary. Specifically, we assume that the spaces $L^p(G)$ are defined with respect to a measure μ of the form $du = \rho d\nu$, where ν is right Haar measure and ρ is a positive function which is both bounded above and bounded away from zero. This hypothesis will not be stated separately in each theorem, and various properties of μ which follow from the corresponding properties of Haar measure will be used without comment.

We shall require an interesting theorem of Banach [1, Chapter 11], later refined and extended by Lamperti [5], which goes as follows. Let (X, μ) and (Y, ν) be measure spaces and M an isometry from $L^p(X, \mu)$ into $L^p(Y, \nu)$, $p \neq 2$. Then M is of the form $S_\varphi U$, where, roughly, S_φ is multiplication by a function and U is induced by a "measurable transformation." More precisely, φ is a function on Y whose restriction to any sigma-finite measurable set is measurable, and S_φ is defined by $S_\varphi(F) = \varphi \cdot F$. The (possibly unbounded) operator U is induced by a nonsingular isomorphism of the Boolean algebra of sigma-finite measurable sets in (X, μ) into the Boolean algebra of sigma-finite measurable sets in (Y, ν) (see [2], [5] for details). The pertinent facts about U are that it sends characteristic functions into characteristic functions, preserves pointwise multiplication of L^∞ functions $(U(F \cdot G) = (UF) \cdot (UG))$, and is an isometry of $L^\infty(X, \mu)$ into $L^\infty(Y, \nu)$. Usually, U is induced by a point transformation τ from Y onto X: $(UF)(y) = F(\tau y)$. (The statement of the theorem in [4] includes the hypothesis that (X, μ) and (Y, ν) be sigma-finite, but the extension to the situations to be encountered below is immediate.)

It is now easy to describe why every isometric multiplier is a scalar multiple of a right translation. For simplicity, assume that $\varphi(x)$ is never 0 and U is induced by a point transformation τ. Forget for the moment that measurable functions and transformations are only defined modulo sets of measure 0. Then the relation $S_{\varphi_g} L_g U = L_g S_\varphi U = S_\varphi U L_g$ suggests that $\varphi(gx)F(\tau(gx)) = \varphi(x)F(g \cdot \tau x)$ for all x, g in G and F in L^p. Consideration of this for characteristic functions

F suggests that $\varphi(gx) = \varphi(x)$ for all x, g (hence φ is constant), and τ commutes with left translations (hence τ is right translation by $\tau(e)$, where e is the group identity). To make this rigorous, we shall transform "almost everywhere" considerations into pointwise ones via a standard result in the theory of commutative Banach algebras. This approach was suggested by Alessandro Figa-Talamanca.

3. Isometric multipliers.

THEOREM 1. *Every left multiplier (not necessarily isometric) on $L^p(G, \mu)$, $1 \leqq p < \infty$, $p \neq 2$, of the form $S_\varphi U$ is a scalar multiple of a right translation R_g. In particular, every isometric left multiplier is a scalar multiple of a right translation.*

Before proving this, we state a few simple lemmas. Lemmas 1 and 2 merely insure that measure-theoretic pathology cannot arise in the cases under consideration, and Lemma 3 is unnecessary if $S_\varphi U$ is assumed to map L^p *onto* L^p. Thus the casual reader may profitably skip directly to the proof of Theorem 1.

LEMMA 1. *Let φ be a function on G such that for each sigma-finite set E,*

(1) *The restriction $\varphi \mid E$ of φ to E is measurable.*

(2) *$\varphi \mid E = \varphi_g \mid E$ almost everywhere for each $g \in G$.*

Then the restriction of φ to any sigma-finite set is constant almost everywhere, and the operator S_φ is a scalar multiple of the identity.

Proof. For $M > 0$, let $\varphi_M(X) = \varphi(x)$ or 0 according as $|\varphi(x)| \leqq M$ or $|\varphi(x)| > M$. Then φ_M also satisfies (1) and (2). Let $\{V_\alpha\}$ be a basis of compact neighborhoods of the identity in G, and let I_α be the characteristic function of V_α divided by $\mu(V_\alpha)$. Then, as is well-known,

$$\lim_\alpha (\varphi_M * I_\alpha)(x) = \lim_\alpha \int \varphi_M(xy^{-1}) I_\alpha(y) d\mu(y)$$

exists for each x, and if Ψ_M denotes the limit function, Ψ_M agrees with φ_M almost everywhere on each sigma-finite set. But Ψ_M is clearly identically constant, since $\Psi_M(x) = \Psi_M(gx)$ for *all* $x, g \in G$, and hence φ_M and φ are constant almost everywhere on each sigma-finite set.

LEMMA 2. *Let φ be a function in $L^\infty(G)$ such that given $\varepsilon > 0$, there is a neighborhood V of the identity such that for all $g \in V$, $\|\varphi - \varphi_g\|_\infty < \varepsilon$. Then φ coincides almost everywhere with some left* *uniformly continuous function* Ψ.

Proof. Again, set $\Psi(x) = \lim_\alpha (\varphi * I_\alpha)(x)$.

LEMMA 3. *Let* M *be a nonzero multiplier on* $L^p(G)$, *and let* Δ *be a set of positive measure. Then there is an* F *in* L^p *such that the intersection of* Δ *and the support of* MF *is nonzero. In particular, if* $M = S_\varphi U$, *the restriction of* φ *to any sigma-finite set is nonzero almost everywhere.*

Proof. If $F \in L^p$, then $L_g F \in L^p$, and $M(L_g F) = L_g(MF)$. The support of $L_g(MF)$ is g^{-1} times the support of MF. If the support of MF has positive measure, then there is a $g \in G$ such that g^{-1} times the support of MF intersects Δ in a set of positive measure [4, p. 260, Th. E].

Proof of Theorem 1. We have $S_\varphi(UL_g) = L_g S_\varphi U = S_{\varphi_g}(L_g U)$. If Δ is a set with $0 < \mu(\Delta) < \infty$, then $\chi_\Delta \in L^p$, and $\varphi \cdot (UL_g \chi_\Delta) = \varphi_g \cdot (L_g U \chi_\Delta)$.

Because both $(UL_g)\chi_\Delta$ and $(L_g U)\chi_\Delta$ are characteristic functions and φ is nonzero a.e. (Lemma 3), they are characteristic functions of the same set, say Δ'. Thus for each g in G, $\varphi = \varphi_g$ almost everywhere on each set of the form Δ' with $\chi_{\Delta'} = (UL_g)\chi_\Delta = U(\chi_{g^{-1}\cdot\Delta})$, $0 < \mu(\Delta) < \infty$. The class of sets Δ with $0 \leqq \mu(\Delta) < \infty$ is mapped onto itself by left translation, so $\varphi = \varphi_g$ almost everywhere on each set of the form Δ' with $\chi_{\Delta'} = U\chi_\Delta$, $0 < \mu(\Delta) < \infty$. Given $g \in G$, if Λ is a measurable set such that $\varphi(x) \neq \varphi(gx)$ for x in Λ, then Λ is disjoint from all sets of the form Δ' above, and hence Λ is disjoint from the support of every UF and $S_\varphi UF$ with $F \in L^p$. Lemma 3 implies that Λ has measure 0, and Lemma 1 shows that S_φ is a scalar multiple of the identity.

Let F be a continuous function with compact support Δ. Then F is left uniformly continuous, and the relation $\| UF - (UF)_g \|_\infty = \| UF - U(F_g) \|_\infty = \| F - F_g \|_\infty$ together with Lemma 2 show that UF coincides almost everywhere with a unique left uniformly continuous function which we shall call $\hat{U}F$. Further, $\hat{U}F$ has compact support because $F \cdot F_g = 0$ for all g not in the compact set $\Delta \cdot \Delta^{-1}$ and thus $(\hat{U}F) \cdot (\hat{U}F)_g = \hat{U}(F \cdot F_g) = 0$ for all $g \notin \Delta \cdot \Delta^{-1}$. (The support of $\hat{U}F$ is contained in $\Delta \cdot \Delta^{-1} \cdot x$, where x is any point in the support of $\hat{U}F$.)

The Banach algebra $C_0(G)$ consisting of all continuous functions on G vanishing at infinity (with the supremum norm) is generated by the set of continuous functions with compact support, and the preceding remarks show that $\hat{U}$ is an isometric isomorphism of $C_0(G)$ into itself which commutes with translations. It is known that each homomorphism of $C_0(G)$ into the complex numbers is of the form Ψ_g, where $\Psi_g(F) = F(g)$ [6, p. 123]. Therefore if e is the group identity, the homomorphism $\Psi_e \circ \hat{U}$ is Ψ_h for some $h \in G$. For each $F \in C_0(G)$, $F(h) = \Psi_h(F) = (\Psi_e \circ \hat{U})(F) = (\hat{U}F)(e)$. And, for any $g \in G$,

$$(\hat{U}F)(g) = (L_g \hat{U}F)(e) = (\hat{U}L_g F)(e) = (L_g F)(h) = F(gh) .$$

Therefore, $\hat{U} = R_h$ and also $U = R_h$ because $C_0(G)$ is dense in $L^p(G)$.

4. A class of operators which commute with no isometries. Theorem 1 states that every isometry on $L^p(G, \mu), p \neq 2$, which commutes with all left translations is a scalar multiple of a right translation. Of course, if μ is not right Haar measure, not all right translations, will be isometries. If μ is a measure such that no right translation R_g with $g \neq e$ is an isometry, then no isometries except scalar multiples of the identity commute with all left translations. Thus if μ is of this type and L_g is a left translation whose powers are dense in the weak operator topology in the set of all left translations, L_g commutes with no nontrivial isometry.

It is easy to construct such situations. For instance, let G be the group of complex numbers of modulus one with a measure μ defined by $d\mu = \varphi d\nu$, where ν is Lebesque measure and $\varphi(z) = 1$ or 2 according as z is on the upper or lower half circle. Clearly, no nontrivial translation is an isometry on $L^p(G, \mu)$. If c is not a root of unity, the powers of the operator generated by the translation $z \rightarrow c \cdot z$ are easily shown to be dense in the group of translation operators, and hence this operator commutes with no nontrivial isometry on $L^p(G, \mu), p \neq 2$.

5. Isomorphisms of convolution algebras.

THEOREM 2. *Let G_1 and G_2 be locally compact groups with respective measures μ_1, μ_2 as described in* § 2. *Let T be an isometry of $L^p(G_1, \mu_1)$ onto $L^p(G_2, \mu_2), 1 \leqq p < \infty, p \neq 2$, such that $T(F*G) = TF*TG$ whenever $F*G \in L^p(G_1)$, and $T^{-1}(F*G) = (T^{-1}F)*(T^{-1}G)$ whenever $F*G \in L^p(G_2)$. Then there is a bicontinuous isomorphism τ of G_2 onto G_1. Further, if μ_1 and μ_2 are right Haar measures, there is a character λ on G_2 and a positive constant c such that $(TF)(g) = c\lambda(g)F(\tau g)$ for all $g \in G_2$.*

This theorem was proved for $p = 1$ and Haar measures μ_1, μ_2 by Wendel [7]. A later paper [9] gave a simpler proof and extended the theorem to the case in which T is only assumed to be norm-decreasing. The solution of the isometric multiplier problem for $p \geqq 1$

(Theorem 1) enables us to easily adapt Wendel's later proof to establish Theorem 2. Only a sketch of the proof will be given here, and the reader may consult [9] for details.

Sketch of proof of Theorem 2. Let ν_1 and ν_2 be right Haar measures for G_1 and G_2 respectively, and suppose $d\mu_1 = \rho_1 d\nu_1$, $d\mu_2 = \rho_2 d\nu_2$. Easy computations show that for any $F, G \in L^p(G_1, \mu_1)$, $L_g(F*G) = (L_g F)*G$ and $R_g(F*G) = F*(SR_g G)$, where $S(F) = (R_g \rho_1/\rho_1)\cdot F$. Further for any $g \in G_1$ and $F, G \in L^p(G_2, \mu_2)$,

$$(TR_g T^{-1})(F*G) = F*(TSR_g T^{-1}G) .$$

Thus, it is apparent that $TR_g T^{-1}$ is a left multiplier on $L^p(G_2, \mu_2)$. If $T = S_\varphi U$ as described in § 1, $TR_g T^{-1} = S_\Psi(UR_g U^{-1})$, where $\Psi = \varphi\cdot(UR_g U^{-1}(\varphi^{-1}))$. Now $UR_g U^{-1}$ is an operator induced by a Boolean set map, so by Theorem 1, $TR_g T^{-1}$ is a scalar multiple of the operator induced by a right translation on $L^p(G_2)$. Define a map τ from G_2 onto G_1 and a function λ on G_2 by $TR_{\tau g} T^{-1} = \lambda(g) R_g$. The proof that τ is a bicontinuous isomorphism from G_2 onto G_1 and the rest is now identical to that in [9].

Wendel established Theorem 2 under the weaker hypothesis that $\| T \| \leqq 1$ by first proving that any convolution-preserving contraction of $L^1(G_1)$ onto $L^1(G_2)$ is automatically an isometry. The author does not know if this is true in general for L^p, $p \neq 2$, but a more modest result can be obtained quite simply. First we make the following observation, which is perhaps of interest in its own right. The L^p norm of a function F is denoted by $\| F \|_p$.

THEOREM 3. *Let (X, μ) and (Y, ν) be measurable spaces with $\mu(X) = \mu(Y) < \infty$, and let $1 \leqq p < q \leqq \infty$. Suppose T is an isometry of $L^p(X, \mu)$ into $L^p(Y, \nu)$ such that for each F in $L^q(X, \mu)$, $\| TF \|_q \leqq \| F \|_q$. Then T is an isometry of $L^r(X, \mu)$ into $L^r(Y, \nu)$ for all r, $1 \leqq r \leqq \infty$. In fact, T is of the form $S_\varphi U$ described in § 2, with U induced by a measure-preserving transformation and $|\varphi| = 1$.*

Proof. We assume the measure spaces are normalized so that $\mu(X) = \nu(Y) = 1$. A simple application of Holder's inequality shows that for all $F \in L^p$, $\| F \|_p \leqq \| F \|_q$, and equality occurs if and only if F has constant modulus one. For,

$$\| F \|_p^p = \int | F |^p \leqq \left(\int (| F |^p)^{q/p}\right)^{p/q} \cdot \quad \left(\int 1^{q/q-p}\right)^{q-p/q} = \| F \|_q^p .$$

If F has modulus one, $\| F \|_p = \| F \|_q$, and by hypothesis

$$\| TF \|_q \leqq \| F \|_q = \| F \|_p = \| TF \|_p .$$

Hence $\| TF \|_q = \| TF \|_p$ and TF has constant modulus one. If Δ is any set, and $|c| = 1$, $|\chi_\Delta + c\chi_{X-\Delta}| = 1$ a.e. and $|T\chi_\Delta + cT\chi_{X-\Delta}| = 1$ a.e. This can happen for all $|c| = 1$ only if $T\chi_\Delta$ and $T\chi_{X-\Delta}$ have disjoint supports.

Let e be the function constantly one, and let $U = S_{T(e)}^{-1} T$. The new operator U satisfies the hypotheses because $|T(e)| = 1$. Now $U\chi_\Delta + U\chi_{X-\Delta} = Ue = e$, and $U\chi_\Delta$ and $U\chi_{X-\Delta}$ have disjoint supports, so $U\chi_\Delta$ is a characteristic function. Hence if $F = \sum c_i \chi_{E_i}$ is a simple function with E_i pairwise disjoint, then for all $r \geqq 1$,

$$\| UF \|_r^r = \int | UF |^r d\nu = \sum | c_i |^r \int | U\chi_{E_i} |^r d\nu$$

$$= \sum | c_i |^r \int | U\chi_{E_i} |^p d\nu = \sum | c_i |^r \mu(E_i) = \| F \|_r^r .$$

Thus U is an isometry on all the spaces $L^r(X, \mu)$.

The last statement of the theorem follows from a result of Lamperti [4] which states that an operator which is an isometry on L^r for two distinct values of r must be of the form given above. This may also be deduced from the observation that the set map τ defined by $U\chi_\Delta = \chi_{\tau(\Delta)}$ is Boolean.

Lamperti's theorem holds even if $\mu(X) = \mu(Y) = \infty$, while Theorem 3 does not. Theorem 3 may therefore be regarded as a partial generalization of Lamperti's result. Robert Strichartz has pointed out that the hypothesis $\mu(X) = \mu(Y)$ in Theorem 3 is essential. For, take $X = [0, 1]$, $Y = [0, 2]$, and μ, ν Lebesque measures. Let $(Tf)(x) = (1/2)f((1/2)x)$. Then T is an isometry on $L^1(X, \mu)$, but $\| T \|_p = 2^{1-p/p}$.

COROLLARY. *Let* $\mu(X) = \nu(Y) < \infty$ *and* $1 \leqq p, q \leqq \infty, p \neq q$. *Suppose* T *is an isometry of* $L^p(X, \mu)$ *onto* $L^p(Y, \nu)$ *such that for all* F *in* $L^q(X, \mu)$, $\| TF \|_q \leqq \| F \|_q$. *Then* T *is an isometry of each space* $L^r(X, \mu)$ *onto* $L^r(Y, \nu), 1 \leqq r \leqq \infty$.

Proof. For $p > q$ this is Theorem 3. For $p > q$, apply Theorem 3 to T^*, which is an isometry on $L^{p'}$ and a contraction on $L^{q'}$, where $L^{p'}$ and $L^{q'}$ are the conjugate spaces of L^p and L^q respectively (so $p' < q'$).

THEOREM 4. *If* G_1 *and* G_2 *are compact Abelian groups, and* $L^p(G_1), L^p(G_2)$ *are defined with respect to Haar measures, then Theorem 2 is valid when the hypothesis that* T *be an isometry is replaced by the hypothesis that* $\| T \| \leqq 1$.

Proof. We show that a convolution-preserving contraction of

$L^p(G_1)$ onto $L^p(G_2)$, $p \neq 2, \infty$, is automatically an isometry.

It is well known that any convolution-preserving operator must send characters onto characters. (For a quick proof, note that γ is a character if and only if $\gamma * \gamma = \gamma$ and $\gamma * F$ is a scalar multiple of γ for every $F \in L^p$.) Since the characters on a group form an orthonormal basis for L^2 of the group, T is an isometry from $L^2(G_1)$ onto $L^2(G_2)$, and the corollary applies.

REMARKS 1. The analogues of Theorems 1 and 2 for L^2 are false. The falsity of Theorem 1 in this context is apparent from the example given in § 1. And, Gaudry [3] has shown that there is a convolution-preserving isometry from L^2 of the unit circle onto L^2 of the torus $\{(z, w) \mid |z| = |w| = 1\}$, but these groups are certainly not topologically isomorphic.

2. Since this paper was submitted, [7] has appeared in which Theorems 1 and 2 are proved in slightly less generality.

3. The analogue of Theorem 2 for compact groups (with Haar measures) and $p = \infty$ may be found in [3] and [7].

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