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SUM AND PRODUCT OF COMMUTING SPECTRAL OPERATORS

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Let E be a separated, quasi-complete and barreled locally convex space. Let T_1 and T_2 be two commuting, continuous spectral operators on E. The conditions under which $T_1 + T_2$ and T_1T_2 are spectral operators are obtained. Further, let X be a locally compact and σ-compact space. Let μ be a positive Radon measure on X. Let $\Omega^p(X, \mu)(1 \leqq p < \infty)$ be the linear space of all complex valued functions defined on X, whose p^{th} powers are locally integrable with respect to the measure μ. This space is given a certain topology under which it becomes a complete metrisable locally convex space. The sum and product of two commuting scalar operators on $\Omega^p(X, \mu)(2 \leqq p < \infty)$ are scalar operators and the sum and the product of two commuting spectral operators are spectral operators provided that the spectrum of each operator is compact.

In this paper we prove that under certain conditions the sum and the product of two commuting spectral operators on a locally convex space are again spectral operators. We also obtain expressions for the spectral measures of such sum and product. The technique used is, in principle, similar to one employed by Pedersen [9]. Also, if X is a locally compact and σ-compact space and μ is a positive Radon measure on X, we consider the space $\Omega^p(X, \mu)(1 \leqq p < \infty)$ which consists of the complex valued functions, f, defined on X such that for each compact set K in X, $f\zeta_K(\zeta_K$ is the characteristic function of $K)$ belongs to $L^p(K, \mu)$. This is a linear space and on this space, we define a locally convex topology by a family $\{p_K: K \text{ compact in } X\}$ of semi-norms given by

$$p_K(f) = \left(\int_K |f|^p d\mu\right)^{1/p} .$$

Dieudonné [1] obtained some of the properties of Ω^1. By using his methods, we prove that the space $\Omega^p(\equiv \Omega^p(X, \mu))(1 < p < \infty)$ is a complete metrisable space and is also weakly sequentially complete. We also obtain the dual of Ω^p. By using some inequalities obtained by McCarthy [7], we show that the sum and the product of two commuting scalar operators on $\Omega^p(2 \leqq p < \infty)$ are again scalar opetators and the sum and the product of two commuting spectral operators are spectral operators provided that the spectrum of each operator is compact.

In the original version of this paper Theorem 2.5 (and hence also Theorem 3.1) was proved under the additional assumption that E is bornological. The author wishes to thank the referee for pointing out that the results, in fact, are valid without this additional assumption.

1. Preliminaries.

In this section we give some basic definitions and collect some known results. Most of these results are taken from [4] and [5].

Throughout this work C will denote the field of complex numbers and $\hat{C}$ the one point compactification of C. By $\mathscr{B}$ we denote the class of Borel subsets of C and by E a complex locally convex space which we shall always assume to be quasi-complete and barreled. By E' we denote the dual of E and by $\mathscr{L}(E)$ the space of all continuous linear maps of E into itself. By I we denote the identity map of E onto itself. We shall always assume that $\mathscr{L}(E)$ is provided with the topology τ_b of uniform convergence on the bounded subsets of E.

We give the following definition due to Waelbroeck [10].

DEFINITION 1.1. $\lambda \in \hat{C}$ is said to belong to the *resolved set* $\rho(T)$ of $T \in \mathscr{L}(E)$ if and only if there is a neighborhood V_λ of λ in $\hat{C}$ such that there is a function $\mu \to R_\mu$ on $V_\lambda \cap C$ to $\mathscr{L}(E)$ satisfying, for each $\mu \in V_\lambda \cap C$ the conditions

(i) $R_\mu(\mu I - T) = (\mu I - T)R_\mu = I$;

(ii) $\{R_\mu : \mu \in V_\lambda \cap C\}$ is bounded in $\mathscr{L}(E)$.

The *spectrum*, $\text{sp}(T)$, of T is defined by $\text{sp}(T) = \hat{C} \sim \rho(T)$. If $\infty \notin \text{sp}(T)$ then T is called a *regular element* of $\mathscr{L}(E)$. The set of all regular elements of $\mathscr{L}(E)$ is denoted by $\mathscr{L}_r(E)$.

DEFINITION 1.2. A function $\{P(\sigma) : \sigma \in \mathscr{B}\}$ of commuting projection valued operators defined on $\mathscr{B}$ with values in $\mathscr{L}(E)$ is called a *spectral measure* on E if

(a) for each $x \in E$, $P(\cdot)x$ is countably additive in E;

(b) $P(C) = I$;

(c) $\{P(\sigma) : \sigma \in \mathscr{B}\}$ is an equicontinuous part of $\mathscr{L}(E)$.

Under the assumption that E is barreled, condition (c) may be deduced from condition (a).

It follows from (a) that for each $x \in E$ and $x' \in E'$, $\langle P(\cdot)x, x' \rangle$ is a (countably additive) complex-valued measure.

LEMMA 1.3. (a) $P(\phi) = 0$; (ϕ *is the null set*)
(b) $P(\sigma \cap \delta) = P(\sigma)P(\delta)$, $\sigma, \delta \in \mathscr{B}$.

Proof. This is proved in [8, Lemma 2.15].

An operator $T \in \mathscr{L}(E)$ is said to *commute* with a spectral measure $P(\cdot)$ if $P(\sigma)T = TP(\sigma)$ for all $\sigma \in \mathscr{B}$.

DEFINITION 1.4. An operator $T \in \mathscr{L}(E)$ is called a *spectral operator* if there exists a spectral measure $P(\cdot)$ on E such that

(a) T commutes with $P(\cdot)$;

(b) $\operatorname{sp}(T \mid P(\sigma)E) \subset \bar{\sigma}, \sigma \in \mathscr{B}$;

(c) for each $x \in E$ and $x' \in E'$, the complex measure $\langle P(\cdot)x, x'\rangle$ has compact support.

Here $T \mid P(\sigma)E$ denotes the restriction of T to the subspace $P(\sigma)E$ of E.

Such a spectral measure is unique, if it exists. Also, if $A \in \mathscr{L}(E)$ commutes with T then A commutes with the spectral measure corresponding to T.

Since we shall be interested only in spectral measures corresponding to spectral operators we may and shall assume that whenever $P(\cdot)$ is a spectral measure on E, for each $x \in E$ and $x' \in E'$ $\langle P(\cdot)x, x'\rangle$ has compact support.

DEFINITION 1.5. An operator $S \in \mathscr{L}(E)$ is called a *scalar operator* if there exists a spectral measure $P(\cdot)$ on E such that λ is integrable with respect to $\langle P(\cdot)x, x'\rangle$ for each $x \in E$ and $x' \in E'$ and such that

$$\langle Sx, x'\rangle = \int_C \lambda d\langle P(\lambda)x, x'\rangle, x \in E, x' \in E' .$$

Every scalar operator is a spectral operator and the spectral measure corresponding to a scalar operator is unique.

An operator $N \in \mathscr{L}(E)$ is called *quasi-nilpotent* if for each $x \in E$ and $x' \in E'$

$$\lim_{n \to \infty} |\langle N^n x, x'\rangle|^{1/n} = 0 .$$

If E is quasi-complete and barreled then $N \in \mathscr{L}(E)$ is quasi-nilpotent if and only if $\operatorname{sp}(N) = \{0\}$.

A spectral measure $P(.)$ on E is said to satisfy *condition* PC_0 if for each $x \in E$ and $x' \in E'$ there exists a compact set $\sigma(x, x')$ such that $\operatorname{Supp} \langle P(.)Qx, x'\rangle \subset \sigma(x, x')$ for each operator $Q \in \mathscr{L}(E)$ and commuting with $P(.)$.

We take the following theorem from [5]:

THEOREM 1.6. *Let E be quasi-complete and barreled. Let $T \in \mathscr{L}(E)$ be a spectral operator whose corresponding spectral measure satisfies the condition PC_0. Then T can be uniquely expressed as $T = S + N$ where S is a scalar operator having the same spectral measure as T and commuting with T, and N is a quasi-nilpotent operator commuting with S.*

Conversely, if $S \in \mathscr{L}(E)$ is a scalar operator and $N \in \mathscr{L}(E)$ is *any quasi-nilpotent operator commuting with* S *then* $T = S + N$ *is a spectral operator having the same spectral measure as* S.

The above decomposition of a spectral orerator is called the *canonical decomposition*. S is called the *scalar part* of T and N the *radical part* of T. In the sequel, whenever we consider the canonical decomposition of a spectral operator it will be tacitly assumed that the corresponding spectral family satisfies the condition PC_0. It may be *remarked that if* $\mathrm{sp}(T)$ is compact then $\bigcup_{\substack{x \in E \\ x' \in E'}} \mathrm{Supp}\langle P(.)x, x'\rangle$ is compact and $P(.)$ automatically satisfies the condition PC_0.

We give the following generalization of Orlicz-Pettis theorem which is proved in [5, Lemma 1.1.1].

LEMMA 1.7. *A set function* m *defined on* $\mathscr{B}$ *with values in* E *is countably additive in* E *if and only if it is weakly countably additive.*

We now prove a few lemmas which will be of use in the following sections.

LEMMA 1.8. *Let* $P(.)$ *be any spectral measure on* E. *Then for each bounded set* $B \subset E$ *and each equicontinuous part* $A \subset E'$ *there exists a constant* K *such that* $|\langle P(\sigma)x, x'\rangle| \leqq K$ *for all* $x \in B$, $x' \in A$ *and for all* $\sigma \in \mathscr{B}$.

Proof. Since $\{P(\sigma): \sigma \in \mathscr{B}\}$ is an equicontinuous part of $\mathscr{L}(E)$ and B is bounded in E, $\{P(\sigma)x: \sigma \in \mathscr{B}, x \in B\}$ is bounded in E. Since A is equicontinuous in E' it follows that $\{\langle P(\sigma)x, x'\rangle: \sigma \in \mathscr{B}, x \in B, x' \in A\}$ is bounded in C. This establishes the lemma.

LEMMA 1.9. *Let* E *be barreled and let* Γ *be any equicontinuous part of* $\mathscr{L}(E)$. *Let* A *be any equicontinuous part of* E'. *Then the set* $\{T'x': x' \in A, T \in \Gamma\}$ *is an equicontinuous part of* E'. (*Here* T' *is the adjoint of* $T \in \mathscr{L}(E)$.)

Proof. Since E is barreled, it is enough to prove that for $x \in E$, $\{\langle x, T'x'\rangle: x' \in A, T \in \Gamma\}$ is bounded in C. Since Γ is equicontinuous in $\mathscr{L}(E)$, $\{Tx: T \in \Gamma\}$ is bounded in E and hence $\{\langle Tx, x'\rangle: T \in \Gamma, x' \in A\}$ is bounded in C. This proves the lemma.

LEMMA 1.10. *Let* $P(.)$ *be a spectral measure on a barreled space* E. *Then* $\sum_j \mu_j P(\delta_j)$ *is an equicontinuous part of* $\mathscr{L}(E)$ *if* j *varies over any finite index set and* $|\mu_j| \leqq 1$ *for all* j. (δ_j *are mutually* *disjoint sets in* $\mathscr{B}$.)

Proof. Since E is barreled, it is enough to prove that for each $x \in E$ and each equicontinuous part $A \subset E'$ $\sup_j |\langle \sum \mu_j P(\delta_j)x, x' \rangle|$ is bounded in C. Now

$$\left|\left\langle \sum_j \mu_j P(\delta_j)x, x' \right\rangle\right| \leqq \sup_j |\mu_j| \sum |\langle P(\delta_j)x, x'\rangle| \leqq \sup_j |\mu_j| 4K \leqq 4K ,$$

where $K = \sup_{\substack{\delta \in \mathscr{B} \\ x' \in A}} |\langle P(\delta)x, x'\rangle| < \infty$. Since K is independent of x' and depends only on x and A, the lemma is proved.

LEMMA 1.11. *Let* $x \in E$ *and* $x' \in E'$ *be fixed. Let* $P(.)$ *be a spectral measure on* E *and let* σ *be the compact support of* $\langle P(.)x, x'\rangle$. *Then for any finite disjoint subdivision* $(\sigma_i)_{i=1}^N$ *of* σ *into Borel subsets of* C, *such that* $\operatorname{diam}(\sigma_i) < \varepsilon$, $i = 1, 2, \cdots, N$,

$$\left|\int \lambda d\langle P(\lambda)x, x'\rangle - \sum_{i=1}^N \lambda_i \langle P(\sigma_i)x, x'\rangle\right| \leqq 8\varepsilon K$$

where $K = \sup |\langle P(\delta)x, x'\rangle|$ *and* $\lambda_i \in \sigma_i$.

The lemma can be easily established by following the steps in the proof of [2, Th. 7].

2. The product measure of two spectral measures. In this section we shall define the product measure of two commuting spectral measures.

Let Δ denote the algebra generated by the sets of the form $\sigma \times \delta$ where $\sigma \in \mathscr{B}$ and $\delta \in \mathscr{B}$. Let Δ^* be the σ-algebra generated by Δ. Each $\alpha \in \Delta$ may be expressed as

$$(*) \qquad \alpha = \bigcup_{i=1}^n (\sigma_i \times \delta_i)$$

where $\sigma_i \in \mathscr{B}$ and $\delta_i \in \mathscr{B} (i = 1, 2, \cdots, n)$ and

$$(\sigma_i \times \delta_i) \cap (\sigma_j \times \delta_j) = \phi,\ i \neq j .$$

For two commuting spectral measures $P(.)$ and $Q(.)$ on E, we define a set function R_0 with values in $\mathscr{L}(E)$ by

$$R_0(\alpha) = \sum_{i=1}^n P(\sigma_i)Q(\delta_i) ;$$

where $\alpha \in \Delta$ is represented in the form (*). We remark that this definition is independent of the representation of α.

By making use of representation (*) we can easily prove the following lemma.

LEMMA 2.1. *For* $\alpha \in \Delta$ *and* $\beta \in \Delta$,
(a) $R_0(\alpha \cap \beta) = R_0(\alpha)R_0(\beta)$;
(b) $R_0(\alpha \cup \beta) = R_0(\alpha) + R_0(\beta)$, *if* $\alpha \cap \beta = \phi$;
(c) $R_0(C \times C) = I$.

COROLLARY 2.2. *$R_0(.)$ is a finitely additive set function on Δ to $\mathscr{L}(E)$. The set $\{R_0(\alpha)\colon \alpha \in \Delta\}$ consists of commuting projections.*

REMARK. For given $x \in E$ and $x' \in E'$, since $\langle P(.)x, x'\rangle$ and $\langle Q(.)x, x'\rangle$ have compact support, it follows that $\langle R_0(.)x, x'\rangle$ has a compact support in $C \times C$.

Let X be a topological space and let A be an algebra of sets in X. Let $S(.)$ be a set function defined on A with values in $\mathscr{L}(E)$. We shall say that $S(.)$ is *regular* on A if for each $x \in E$, for each equicontinuous part $A \subset E'$, for each $\sigma \in A$ and for each $\varepsilon > 0$ there exist a closed set $Z \subset \sigma$ and an open set $U \supset \sigma$ such that if $\delta \subset U \sim Z$ and $\delta \in A$ then

$$\sup_{x' \in A} |\langle S(\delta)x, x'\rangle| < \varepsilon .$$

PROPOSITION 2.3. If $P(.)$ is a spectral measure on E, then $P(.)$ is regular on $\mathscr{B}$.

Proof. Let $x \in E$ and A an equicontinuous part in E' be fixed. We first show that the set $S = \{\langle P(.)x, x'\rangle\colon x' \in A\}$ of complex measures is weakly sequentially compact as a subset of ca $(C, \mathscr{B})$, the space of all bounded complex measures on $\mathscr{B}$. By Lemma 1.8 there exists a constant K such that $|\langle P(\sigma)x, x'\rangle| \leqq K$ for all $x' \in A$ and for all $\sigma \in \mathscr{B}$ so that S is uniformly bounded. Let (σ_n) be a decreasing sequence of sets in $\mathscr{B}$ such that $\sigma_n \downarrow \phi$. Let A^0 be the polar of A (with respect to the duality $\langle E, E'\rangle$) so that A^0 is a 0-neighborhood in E. Let p be the gauge function of A^0 (and hence a continuous seminorm on E). Since $P(.)x$ is countably additive in E, $\lim_{n\to\infty} P(\sigma_n)x = 0$ in E. Therefore, for given $\varepsilon > 0$, there exists a positive integer N such that $p(P(\sigma_n)x) < \varepsilon$ for all $n \geqq N$. This in turn implies that $\langle P(\sigma_n)x, x'\rangle \to 0$ uniformly on A. Hence S is weakly sequentially compact [3, Th. IV, 9.1]. By [3, Th. IV, 9.2], there exists a positive (regular) measure $\lambda \in$ ca $(C, \mathscr{B})$ such that $\lim_{\lambda(\sigma)\to 0} \langle P(\sigma)x, x'\rangle = 0$ uniformly on A.

Let $\varepsilon > 0$ be given. There exists a $\delta > 0$ such that if $\gamma \in \mathscr{B}$ and $\lambda(\gamma) < \delta$ then $|\langle P(\gamma)x, x'\rangle| < \varepsilon/2$ for all $x' \in A$. Since λ is regular, for a given $\sigma \in \mathscr{B}$ there exists a closed set $Z \subset \sigma$ and an open set $U \supset \sigma$ such that $\lambda(\beta) < \delta$ whenever $\beta \subset U \sim Z$ and $\beta \in \mathscr{B}$. This shows that $|\langle P(\beta)x, x'\rangle| < \varepsilon/2$ for all $x' \in A$ so that $\sup_{x' \in A} |\langle P(\beta)x, x'\rangle| < \varepsilon$. This proves the regularity of $P(.)$ on $\mathscr{B}$.

PROPOSITION 2.4. Let $P(.)$ and $Q(.)$ be two commuting spectral measures on E such that the Boolean algebra generated by $P(.)$ and $Q(.)$ is an equicontinuous part of $\mathscr{L}(E)$. Then $R_0(.)$ is regular and countably additive on Δ.

Proof. By virtue of representation (*) it is sufficient to prove the result for a set $\sigma \times \delta \in \Delta$ where $\sigma, \delta \in \mathscr{B}$.

Let $x \in E$ and A an equicontinuous part of E' be fixed. Let $Z_1 \subset \sigma \subset U_1$ and $Z_2 \subset \delta \subset U_2$ where Z_1 and Z_2 are closed sets and U_1 and U_2 open sets in C to be specified afterwards. Let α be any set in Δ such that $\alpha \subset (U_1 \times U_2) \sim (Z_1 \times Z_2)$. We have, (Lemma 2.1 and Corollary 2.2,)

$$\begin{aligned}
\sup_{x' \in A} |\langle R_0(\alpha)x, x'\rangle| &= \sup_{x' \in A} |\langle R_0((U_1 \times U_2) \sim (Z_1 \times Z_2))R_0(\alpha)x, x'\rangle| \\
&= \sup_{x' \in A} |\langle R_0((U_1 \sim Z_1) \times U_2 \cup Z_1 \times (U_2 \sim Z_2))x, R_0(\alpha)'x'\rangle| \\
&\leqq \sup_{x' \in A} |\langle R_0((U_1 \sim Z_1) \times U_2)x, R_0(\alpha)'x'\rangle| \\
&\quad + \sup_{x' \in A} |\langle R_0(Z_1 \times (U_2 \sim Z_2))x, R_0(\alpha)'x'\rangle| \\
&= \sup_{x' \in A} |\langle Q(U_2)P(U_1 \sim Z_1)x, R_0(\alpha)'x'\rangle| \\
&\quad + \sup_{x' \in A} |\langle P(Z_1)Q(U_2 \sim Z_2)x, R_0(\alpha)'x'\rangle| .
\end{aligned}$$

Now, by hypothesis $\{R_0(\alpha): \alpha \in \Delta\}$ is an equicontinuous part of $\mathscr{L}(E)$. Hence, by Lemma 1.9, $\{R_0(\alpha)'x': x' \in A\}$ and hence $\{Q(U_2)'R_0(\alpha)'x': x' \in A\}$ and $\{P(Z_1)'R_0(\alpha)'x': x' \in A\}$ are equicontinuous subsets of E'. Since $P(.)$ and $Q(.)$ are regular measures, for given $\varepsilon > 0$ there exist open sets $U \supset \sigma$ and $U' \supset \delta$ and closed sets $Z \subset \sigma$ and $Z' \subset \delta$ such that

$$\sup_{x' \in A} |\langle P(\beta)x, Q(U_2)'R_0(\alpha)'x'\rangle| < \varepsilon/2, \beta \subset U \sim Z, \beta \in \mathscr{B} ;$$

and

$$\sup_{x' \in A} |\langle Q(\beta)x, P(Z_1)'R_0(\alpha)'x\rangle| < \varepsilon/2, \beta \in U' \sim Z', \beta \in \mathscr{B} .$$

By taking $U_1 = U$, $U_2 = U'$; $Z_1 = Z$ and $Z_2 = Z'$ we have, if $\alpha \subset (U \times U') \sim (Z \times Z')$ then

$$\sup_{x' \in A} |\langle R_0(\alpha)x, x'\rangle| < \varepsilon ,$$

which proves the regularity of $R_0(.)$ on Δ.

To show that for each $x \in E$, $R_0(.)x$ is countably additive on Δ, by Lemma 1.7, it is enough to prove that for each $x' \in E'$, $\langle R_0(.)x, x'\rangle$ is countably additive. This may be proved as in [9, Lemma 3]. (Refer, also, [3, Th. III, 5.13].)

THEOREM 2.5. *Let E be weakly (sequentially) complete and barreled space. Let $P(.)$ and $Q(.)$ be two commuting spectral measures on E such that the Boolean algebra generated by $P(.)$ and $Q(.)$ is an equicontinuous part of $\mathscr{L}(E)$. Then there exists a unique set function $R(.)$ defined on Δ^* such that*

(a) *$R(.)$ is an extension of $R_0(.)$;*

(b) *for each $x \in E$, $R(.)x$ is countably additive;*

(c) *$\{R(\delta): \delta \in \Delta^*\}$ is an equicontinuous part of $\mathscr{L}(E)$.*

Proof. For each $x \in E$ and $x' \in E'$, by the Hahn extension theorem, there is a unique countably additive complex measure $u(., x, x')$ defined on Δ^* such that

$$u(\delta, x, x') = \langle R_0(\delta)x, x'\rangle, \delta \in \Delta ,$$

and such that

$$(**) \qquad \sup_{\delta \in \Delta^*} | u(\delta, x, x') | = \sup_{\delta \in \Delta} |\langle R_0(\delta)x, x'\rangle | .$$

From the uniqueness of $u(\cdot, x, x')$ it follows that u is linear in x and x'. Since $\{R_0(\delta): \delta \in \Delta\}$ is an equicontinuous part of $\mathscr{L}(E)$, (**) shows that the set $\{u(\delta, x, x'): \delta \in \Delta^*\}$ is bounded in C. Since, E is barreled, it follows that the set $\{u(\delta, \cdot, x'): \delta \in \Delta^*\}$ is an equicontinuous part of $\mathscr{L}(E, C)$. In particular, the mapping $x \to u(\delta, x, x')$ is a continuous linear form on E. We denote it by $R'(\delta)x'$ so that $R'(\delta)x'$ is an element of E'. The mapping $x' \to R'(\delta)x'$ maps E' into E'. For each $x \in E$ and $x' \in E'$, the scalar valued set function $u(\cdot, x, x') = \langle x, R'(\cdot)x'\rangle$ is an extension of $\langle R_0(\cdot)x, x'\rangle$ so that $R'(\cdot)$ is an extension of $R_0'(\cdot)$.

Let M be the class of all $\sigma \in \Delta^*$ for which $R'(\sigma)$ is the adjoint of an operator, say $R(\sigma)$. We shall show that M is a monotone class containing Δ.

If $\sigma \in \Delta$ then $\langle x, R'(\sigma)x'\rangle = \langle R_0(\sigma)x, x'\rangle$ so that $R'(\sigma)$ is the adjoint of $R_0(\sigma)$ and, hence, $\sigma \in M$. Next, let (σ_n) be a monotone sequence in M and let $\sigma = \lim \sigma_n$. We claim that $\sigma \in M$. For,

$$\begin{aligned}\langle x, R'(\sigma)x'\rangle &= \lim_n \langle x, R'(\sigma_n)x'\rangle \\ &= \lim_n \langle R(\sigma_n)x, x'\rangle .\end{aligned}$$

Since, E is weakly complete, for each $x \in E$, there is an element $S(\sigma)x$ in E such that

$$\langle S(\sigma)x, x'\rangle = \lim_n \langle R(\sigma_n)x, x'\rangle .$$

The mapping $x \longrightarrow S(\sigma)x$ is clearly linear. By making use of the fact that E is barreled, it is easy to show that it is also continuous. It is, therefore, an element of $\mathscr{L}(E)$ and the adjoint of $R'(\sigma)$. Thus $\sigma \in M$. Since $\varDelta^*$ is the σ-algebra generated by $\varDelta$, we have proved (a) and also that $R(\cdot)x$ is weakly countably additive. Part (b) now follows from Lemma 1.7.

To prove (c), we need only to remark that from (**) it follows that for fixed x and x'

$$\{\langle R(\sigma)x, x'\rangle : \sigma \in \varDelta^*\}$$

is bounded in C; so that $\{R(\sigma)x : \sigma \in \varDelta^*\}$ is weakly bounded and hence bounded in E. The result now follows from the fact that E is barreled.

THEOREM 2.6. *The family $\{R(\sigma): \sigma \in \varDelta\}$ obtained in Theorem 2.5 is a family of commuting projection valued operators in $\mathscr{L}(E)$ which satisfies the condition,*

$$R(\sigma \cap \delta) = R(\sigma)R(\delta), \sigma, \delta \in \varDelta^* .$$

This may be proved as [8, Th. 2.16].

3. Sum and product of two commuting spectral operators. In this section we shall prove that under certain conditions sum and the product of two commuting spectral operators are again spetral operators.

REMARKS. Let T_1 and T_2 be two commuting spectral operators on E. Let $T_1 = S_1 + N_1$ and $T_2 = S_2 + N_2$ be their cannonical decompositions. Since T_1 and T_2 commute, it follows that $N_1 + N_2$ commutes with $S_1 + S_2$ and also with $T_1 + T_2$. Moreover, $N_1 + N_2$ is quasi-nilpotent. Hence, $T_1 + T_2$ is a spectral operator if and only if $S_1 + S_2$ is a scalar operator.

The same, however, cannot be said about the product T_1T_2. Maede [6] has given an example to show that the product of an operator T on a locally convex space with a quasi-nilpotent operator N on the same space need not be quasi-nilpotent. But if the spectrum of T is compact then TN and NT are quasi-nilpotent, since the quasi-nilpotent operators form a two sided ideal in $\mathscr{L}_r(E)$. Thus, if T_1 and T_2 have compact spectra then T_1T_2 is a spectral operator if and only if S_1S_2 is a scalar operator on E.

We now state and prove one of the main theorems of this paper.

THEOREM 3.1. *Let E be a quasi-complete, barreled and weakly sequentially complete locally convex space. Let T_1 and T_2 be two commuting spectral operators on E with corresponding spectral measures $P(.)$ and $Q(.)$. Let the Boolean algebra generated by $P(.)$ and $Q(.)$ form an equicontinuous part of $\mathscr{L}(E)$. Then* (a) *if $P(.)$ and $Q(.)$ satisfy condition PC_0 then $T_1 + T_2$ is a spectral operator whose spectral measure $G(.)$ is given by*

$$G(\alpha) = R\{(\lambda, \mu): \lambda + \mu \in \alpha\}, \alpha \in \mathscr{B} ;$$

(b) *if T_1 and T_2 have compact spectra then T_1T_2 is a spectral operator whose spectral measure $H(.)$ is given by*

$$H(\alpha) = R\{(\lambda, \mu): \lambda\mu \in \alpha\}, \alpha \in \mathscr{B} .$$

Proof: (a) Let S_1 and S_2 be the scalar parts of T_1 and T_2, respectively. By the remarks preceding the statement of the theorem, it is enough to show that $S_1 + S_2$ is a scalar operator.

We have, $G(\alpha) = R(\Psi^{-1}(\alpha))$ where Ψ is the measurable map $(\lambda, \mu) \to \lambda + \mu$ of $C \times C$ into C. It follows from Therems 2.5 and 2.6, that $G(.)$ is a spectral measure and hence is regular on $\mathscr{B}$. Let $x \in E$ and $x' \in E'$ be fixed and let $\varepsilon > 0$ be given. Let Ω and τ be the compact supports of $\langle P(.)x, x'\rangle$ and $\langle Q(.)x, x'\rangle$ and σ the compact support of $\langle G(.)x, x'\rangle$. Let $K = 8 \sup_{\alpha \in \mathscr{B}} \{|\langle P(\alpha)x, x'\rangle|, |\langle Q(\alpha)x, x'\rangle|, |\langle G(\alpha)x, x'\rangle|\} < \infty$. Let $(\alpha_i)_{i=1}^{N_1}, (\beta_j)_{j=1}^{N_2}, (\gamma_k)_{k=1}^{N_3}$ be any finite disjoint subdivisions of σ, Ω and τ, respectively, of norm less than ε. By Lemma 1.11,

$$\left|\int \lambda d\langle G(.)x, x'\rangle - \sum \langle \lambda_i G(\alpha_i)x, x'\rangle\right| < K\varepsilon , \tag{1}$$

$$\left|\langle S_1x, x'\rangle - \sum_j \langle \mu_j P(\beta_j)x, x'\rangle\right| < K\varepsilon , \tag{2}$$

$$|\langle S_2x, x'\rangle - \sum \langle \nu_k Q(\gamma_k)x, x'\rangle| < K\varepsilon , \tag{3}$$

where $\lambda_i \in \alpha_i$, $\mu_j \in \beta_j$ and $\nu_k \in \gamma_k$. It follows from Lemmas 1.10 and 1.9 that the set $\{\sum_j \mu_j P(\beta_j)'x'\} = A$ (say) is an equicontinuous part of E', for all the partitions of Ω. Hence, by the regularity of $G(.)$ there exist closed sets Γ_i such that if $\chi_i \subset \alpha_i \sim \Gamma_i$ $(i = 1, 2, \cdots, N_1)$ then

$$\sup_{y' \in A} |\langle (G(\alpha_i) - G(\chi_i))x, y'\rangle| < \varepsilon/N_1 . \tag{4}$$

Similarly, if we write B for the equicontinuous part $\{\sum_k \mu Q(\gamma_k)'x\}$ of E' then there exist closed sets Γ_i' such that if $\chi_i \subset \alpha_i \sim \Gamma_i'$ $(i = 1, 2, \cdots, N_1)$ then

$$\sup_{y' \in B} |\langle (G(\alpha_i) - G(\chi_i))x, y' \rangle| < \varepsilon / N_1 . \tag{5}$$

Let $Z_i = \Gamma_i \cup \Gamma'_i$ for $i = 1, 2, \cdots, N_1$. We have

$$|\langle G(\alpha_i) - G(Z_i))x, x' \rangle| < \varepsilon / N_1 , \tag{6}$$

$$\left| \sum_i \langle \lambda_i (G(\alpha_i) - G(Z_i))x, x' \rangle \right| < \varepsilon M , \tag{7}$$

where $M = \sup_{\lambda \in \sigma} |\lambda|$. Also

$$\begin{aligned} &\left| \left\langle \left(\sum_i \mu_j P(\beta_j) \right) \left(\sum_i (G(\alpha_i) - G(Z_i)) \right) x, x' \right\rangle \right| \\ &= \left| \left\langle \sum_i (G(\alpha_i) - G(Z_i))x, \sum_j \mu_j P(\beta_j)' x' \right\rangle \right| \\ &\leqq \varepsilon, \text{ by } (4) . \end{aligned} \tag{8}$$

Similarly,

$$\left| \left\langle \left(\sum_k \nu_k Q(\gamma_k) \right) \left(\sum_i (G(\alpha_i) - G(Z_i)) \right) x, x' \right\rangle \right| < \varepsilon . \tag{9}$$

For $n = 0, 1, 2, \cdots$, and for each pair of integers p and q, we denote by $\beta_n(p, q)$ the square consisting of all z such that $2^{-n}p < \operatorname{Re} z \leqq 2^{-n}(p+1)$ and

$$2^{-n}q < \operatorname{Im} z \leqq 2^{-n}(q+1) .$$

For any closed set Z in C we have

$$\bigcup_{p,q} (\beta_n(p, q) \times (Z - \beta_n(p, q)) \downarrow \{(\lambda, \mu) \colon \lambda + \mu \in Z\} \quad \text{as } n \to \infty .$$

Therefore,

$$G(Z)x = \lim_{n \to \infty} \sum_{p,q} P(\beta_n(p, q)) Q(Z - \beta_n(p, q))x ,$$

so that for each equicontinuous part D of E' there exists a positive integer n_D such that

$$\sup_{y' \in D} \left| \left\langle \left(G(Z_i) - \sum_{p,q} P(\beta_n(p, q)) Q(Z_i - \beta_n(p, q)) \right) x, y' \right\rangle \right| < \varepsilon / N_1 \quad \text{for } n \geqq n_D . \tag{10}$$

Since the diameter of $\beta_n(p, q) \downarrow 0$ as $n \to \infty$, we can take n sufficiently large so that diam $\beta_n(p, q) < \varepsilon$. Also, we shall assume that $n \geqq \max(n_x, n_A, n_B)$. If n is so chosen, we shall write $\beta(p, q)$ instead of $\beta_n(p, q)$. For simplicity of notations we shall write $y_{i,p,q}$ for $Z_i - \beta(p, q)$. We have from (10),

$$\left|\left\langle\left(\sum_i \lambda_i\left(G(Z_i) - \sum_{p,q} P(\beta(p, q))Q(y_{i,p,q})\right)x, x'\right\rangle\right| < \varepsilon M . \tag{11}$$

$$\begin{aligned} &\left|\left\langle\left(\sum_j \mu_j P(\beta_j)\right)\left(\sum_{i}\sum_{p,q} (P(\beta(p, q))Q(y_{i,p,q}) - G(Z_i))\right)x, x'\right\rangle\right| \\ &= \left|\left\langle \sum_{i,p,q} (P(\beta(p, q))Q(y_{i,p,q}) - G(Z_i))x, \sum_j \mu_j P(\beta_j)'x'\right\rangle\right| \\ &< \varepsilon . \end{aligned} \tag{12}$$

Similarly,

$$\left|\left\langle\left(\sum_k \nu_k Q(\gamma_k)\right)\left(\sum_{i,p,q} (P(\beta(p, q))Q(y_{i,p,q}) - G(Z_i))\right)x, x'\right\rangle\right| < \varepsilon . \tag{13}$$

Next, let $\lambda_{p,q} \in \beta(p, q)$ be arbitrary. We have

$$\begin{aligned} &\left|\left\langle\left(\sum_{i,p,q} (\lambda_i - \lambda_{p,q})P(\beta(p, q))Q(y_{i,p,q})\right.\right.\right. \\ &\qquad \left.\left.\left. - \left(\sum_k \nu_k Q(\gamma_k)\right)\left(\sum_{i,p,q} P(\beta(p, q))Q(y_{i,p,q})\right)\right)x, x'\right\rangle\right| \\ &\leqq \sum_{i,p,q,k} |\lambda_i - \lambda_{p,q} - \nu_k| |\langle P(\beta(p, q))Q(\gamma_k \cap y_{i,p,q})x, x'\rangle| \\ &\leqq 4\varepsilon \operatorname{var} \langle R(.)x, x'\rangle \\ &\leqq 16\varepsilon K_0, \text{ where } K_0 = \sup_{\delta \in \Delta^*} |\langle R(\delta)x, x'\rangle . \end{aligned} \tag{14}$$

Also,

$$\begin{aligned} &\left|\left\langle\left(\sum_{i,p,q} \lambda_{p,q} P(\beta(p, q))Q(y_{i,p,q})\right.\right.\right. \\ &\qquad \left.\left.\left. - \left(\sum_j \mu_j P(\beta_j)\right)\left(\sum_{i,p,q} P(\beta(p, q)Q(y_{i,p,q}))\right)\right)x, x'\right\rangle\right| \\ &< 8\varepsilon K_0 . \end{aligned} \tag{15}$$

By the successive application of triangle inequality, it follows that

$$\left|\int \lambda d\langle G(\lambda)x, x'\rangle - \langle S_1 x, x'\rangle - \langle S_2 x, x'\rangle\right| < \Lambda\varepsilon ,$$

where Λ is independent of ε, so that

$$\langle (S_1 + S_2)x, x'\rangle = \int \lambda d\langle G(\lambda)x, x'\rangle .$$

Hence, $S_1 + S_2$ is a scalar operator with the corresponding spectral measure $G(.)$. This proves (a).

(b) may be proved similarly. We note that if a closed set Z in C does not contain the origin then

$$\bigcup_{p,q} \beta_n(p, q) \times \frac{Z}{\beta_n(p, q)} \downarrow \{(\lambda, \mu): \lambda\mu \in Z\} \quad \text{as } n \to \infty .$$

so that

$$H(Z)x = \lim_{n\to\infty} \sum_{p,q} P(\beta_n(p, q))Q\left(\frac{Z}{\beta_n(p, q)}\right)x .$$

If the support of $\langle H(.)x, x'\rangle$ contains the origin then in the estimation of $\int \lambda d\langle H(.)x, x'\rangle$ by the sums of the form $\sum_i \langle \lambda_i H(\alpha_i)x, x'\rangle$ we take $\lambda_0 = 0$ where it is assumed that α_0, and none of the other α's, contains the origin.

REMARK. If the above theorem is to be proved for the scalar operators instead of the spectral operators then we do not need the condition PC_0 for $P(.)$ and $Q(.)$ and also the spectra of S_1 and S_2 need not be compact.

4. The space $\Omega^p(1 \leqq p < \infty)$. Let X be a locally compact and σ-compact space so that $X = \bigcup_{n=1}^{\infty} K_n$ where K_n are compact subsets of X. Let μ be a positive Radon measure on X. All the integrations over X will be assumed to be with respect to the measure μ. Also, we shall write X for the measure space (X, μ) and we shall identify the functions which are equal almost everywhere in X.

A complex-valued measurable function f defined on X is said to have compact support if there exists a compact set K in X such that f vanishes in the complement, $\sim K$, of K in X.

Let f be a complex-valued measurable function defined on X. Let for each compact set K in X, $f\zeta_K \in L^p(K)(1 \leqq p < \infty)$. The class of all such functions form a linear space which we shall denote by Ω^p. For each compact set K in X, we define a semi-norm p_K on Ω^p by

$$P_K(f) = \left(\int_X |f\zeta_K|^P\right)^{1/P}, f \in \Omega^P .$$

The family of the semi-norms $\{p_K: K \text{ compact in } X\}$ defines a separated, locally convex topology τ on Ω^p. We shall write Ω^p for the separated, locally convex space (Ω^p, τ).

LEMMA 4.1. *Ω^p is a complete metrisable space.*

Proof. In fact, the topology τ can be generated by a countable family of increasing semi-norms P_{K_n}, where $K_n \subset K_{n+1}$ and $X = \bigcup_{n+1}^{\infty} K_n$. Hence Ω^p is metrisable.

Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in Ω^p. For each compact set K

in X, $(f_n\zeta_K)$ is a Cauchy sequence in $L^p(K)$; and since $L^p(K)$ is complete, there is a function $f_K \in L^p(K)$ such that $f_n\zeta_K \to f_K$ in $L^p(K)$ as $n \to \infty$. For two compact sets K_1 and K_2 in X, f_{K_1} and f_{K_2} coincide on $K_1 \cap K_2$. Since X is σ-compact, there exists a function $f \in \Omega^p$ whose restriction to any compact K is f_K. The given sequence converges to f so that Ω^p is complete.

LEMMA 4.2. *Let $1 < p < \infty$ and let $1/p + 1/q = 1$. Then if F is any continuous linear functional on Ω^p there exists a $g \in \Omega^q$ such that g has compact support in X and such that*

$$F(f) = \int_X gf \quad \textit{for all } f \in \Omega^p .$$

Proof. The subspace $\Omega^p(K)$ of Ω^p consisting of functions which are zero outside a compact set K in X, can be identified with $L^p(K)$. By the Riesz representation theorem, the restriction F_K of F to this subspace is of the form

$$F_K(f) = \int_K g_K f ,$$

where $g_K \in L^q(K)$. If we define g_K to be zero outside K, then $g_K \in \Omega^q$. If K_1 and K_2 are two compact sets, then g_{K_1} and g_{K_2} agree on $K_1 \cap K_2$. Since X is σ-compact, there exists a function $g \in \Omega^q$ such that for each compact K, the restriction of g to K is equal to g_K; and since F is continuous there exists a compact set K_0 such that $|F(f)| \leqq 1$ if $\int_{K_0} |f|^p \leqq 1$. This implies that g vanishes outside K_0 so that g has compact support.

For any $f \in \Omega^p$, we have $f = f_1 + f_2$ where $f_1 = f\zeta_{K_0}$ and $f_2 = f\zeta_{\sim K_0}$. Since f_2 vanishes on K_0, $|F(f_2)| \leqq 1$ and the same is true for the function λf_2 where λ is any scalar. Hence $F(f_2) = 0$ and we have

$$F(f) = F(f_1) = \int_{K_0} g_{K_0} f = \int_X gf$$

and the lemma is proved.

THEOREM 4.3. *For $1 < p < \infty$, the dual of Ω^p is the set of all $g \in \Omega^q$ having compact support. The duality is given by*

$$\langle f, g \rangle = \int_X fg .$$

Proof. Let $g \in \Omega^q$ have compact support and let g vanish outside a compact set K. We show that the mapping $F\colon f \to \int_X gf$ is a continuous linear functional on Ω^p. We write $M = \left(\int_K |g|^q\right)^{1/q} < \infty$.

Since, g vanishes on $\sim K$, we have $\int_{\sim K} gf = 0$. Also $gf \in L^1(K)$. Hence, $\int_X gf = \int_K gf < \infty$. Clearly F is linear. To show that F is continuous, let (f_n) be a sequence in Ω^p such that $f_n \to f \in \Omega^p$. Let $\varepsilon > 0$ be given. For each continuous semi-norm p on Ω^p, there exists a positive integer N such that $p(f_n - f) < \varepsilon/M$ for all $n \geqq N$. In particular, there exists an integer N_0 such that

$$\left(\int |f_n - f|^p\right)^{1/p} < \varepsilon/M , \qquad \text{for } n \geqq N_0 .$$

Now, if we apply Hölder's inequality, we obtain

$$\begin{aligned} \left|\int_X g(f_n - f)\right| &\leqq \int_X |g(f_n - f)| \\ &= \int_K |g(f_n - f)| \\ &\leqq \left(\int |g|^q\right)^{1/q}\left(\int |f_n - f|^p\right)^{1/p} \\ &< \varepsilon, \quad \text{for } n \geqq N_0 . \end{aligned}$$

Thus, we have proved that $f \to \int_X gf$ is a continuous and linear functional on Ω^p. The theorem now follows from Lemma 4.2.

PROPOSITION 4.4. For $1 < p < \infty$. The space Ω^p is weakly (sequentially) complete.

This may be proved like Lemma 4.1 by making use of the fact that for each compact $K \subset X$, $L^p(K)$ is weakly sequentially complete.

For $1 \leqq p < \infty$ we define, after McCarthy [7],

$$\operatorname*{average}_{|c_j|=1}\left|\sum_{j=1}^n c_j \mu_j\right|^p = (2\pi)^{-n}\int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \left|\sum_{j=1}^n e^{i\theta_j}\mu_j\right|_p .$$

LEMMA 4.5. *For any complex numbers λ_{jk} and for $2 \leqq p < \infty$, we have*

$$\begin{aligned} \left(\sum_{j,k} |\lambda_{jk}|^2\right)^{p/2} &\leqq \operatorname*{average}_{|c_j|=1, |d_k|=1} \left|\sum_{j,k} c_j d_k \lambda_{jk}\right|^p \\ &\leqq \Gamma\left(\frac{p}{2} + 2\right)^2 \left(\sum_{j,k} |\lambda_{j,k}|^2\right)^{p/2} . \end{aligned}$$

This is proved in [7, Propositions 1 and 2].

For the proof of the next theorem we shall need the following well-known result on locally convex spaces.

LEMMA 4.6. *If A is any equicontinuous subset of $\mathscr{L}(E)$, then for each continuous semi-norm p on E, there exists a continuous semi-norm q on E such that $\sup_{T\in A} p(Tx) \leqq q(x)$, $x \in E$.*

THEOREM 4.7. *Let $P(.)$ and $Q(.)$ be two commuting spectral measures on $\Omega^p(2 \leqq p < \infty)$. Then the Boolean algebra generated by $P(.)$ and $Q(.)$ is an equicontinuous part of $\mathscr{L}(\Omega^p)$.*

Proof. It is enough to show that $\sum_{j=1}^n \sum_{k=1}^m a_{jk}P_jQ_k$, where $|a_{jk}| = 1$ for all j and k and $\sum_j p_j = I$ and $\sum_k Q_k = I$, is an equicontinuous part of $\mathscr{L}(\Omega^p)$. Since Ω^p is barreled, it is enough to show that for each $f \in \Omega^p$ and each continuous semi-norm p on Ω^p, there exists a constant A such that $p(\sum a_{jk}P_jQ_kf) \leqq A$.

Let $c_1, c_2, \cdots, c_n$ and $d_1, d_2 \cdots, d_m$ be any complex numbers of absolute value one. By Lemma 1.10, the sets $\{\sum_j c_jP_j\}$; $\{\sum_j \bar{c}_jP_j\}$; $\{\sum_k d_kQ_k\}$ and $\{\sum_k \bar{d}_kQ_k\}$ are all equicontinuous parts of $\mathscr{L}(\Omega^p)$. Let $f \in \Omega^p$ and p_K a semi-norm on Ω^p be fixed. By direct computations we have,

$$\sum_{j,k} a_{jk}P_jQ_kf = \Big(\sum_j \bar{c}_jP_j\Big)\Big(\sum_k \bar{d}_kQ_k\Big)\Big(\sum_{j,k} a_{jk}c_jd_kP_jQ_kf\Big) .$$

Therefore,

$$(1)\qquad \begin{aligned} p_K\Big(\sum_{j,k} a_{jk}P_jQ_kf &\leqq p_{K'}\Big(\Big(\sum_k \bar{d}_kQ_k\Big)\Big)\Big(\sum_{j,k} a_{jk}c_jd_kP_jQ_kf\Big)\Big) \\ &\leqq p_{K''}(\sum a_{jk}c_jd_kP_jQ_kf) , \end{aligned}$$

where $p_{K'}$ is the semi-norm corresponding to p_K and the equicontinuous set $\{\sum_j \bar{c}_jP_j\}$; and $p_{K''}$ is the semi-norm corresponding to $p_{K'}$ and the equicontinuous set $\{\sum_k \bar{d}_kQ_k\}$ as given by Lemma 4.6. We have, then

$$\int_K \Big|\sum_{j,k} a_{jk}P_jQ_kf\Big|^p \leqq \int_{K''} \Big|\sum_{j,k} a_{jk}c_jd_kP_jQ_kf\Big|^p .$$

Now,

$$\begin{aligned} &\operatorname*{Average}_{|c_j|=1,|d_k|=1} \int_{K''} \Big|\sum_{j,k} a_{jk}c_jd_kP_jQ_kf\Big|^p . \\ &= \int_{K''} \operatorname*{Average}_{|c_j|=1,|d_k|=1} \Big|\sum_{j,k} a_{jk}c_jd_kP_jQ_kf\Big|^p . \end{aligned}$$

By Lemma 4.5, this is bounded below by

$$\int_{K''} \Big(\sum_{j,k} | a_{jk}P_jQ_kf |^2\Big)^{p/2} = \int_{K''} (\sum | P_jQ_kf |^2)^{p/2} ;$$

and above by

$$\Gamma\left(\frac{p}{2}+2\right)^2 \int_{K''} (\sum |P_j Q_k f|^2)^{p/2} .$$

Hence, from (1) we have

$$\begin{aligned}\left(p_K\left(\sum_{j\ k} a_{jk} P_j Q_k f\right)\right)^p &\leqq \Gamma\left(\frac{p}{2}+2\right)^2 \int_{K''} \left(\sum_{j,k} |P_j Q_k f|^2\right)^{p/2} \\ &\leqq \Gamma\left(\frac{p}{2}+2\right)^2 \int_{K''} \underset{|c_j|=1, |d_k|=1}{\text{Average}} \left|\sum_{j,k} c_j d_k P_j Q_k f\right|^p \\ &= \Gamma\left(\frac{p}{2}+2\right)^2 \underset{|c_j|=1, |d_k|=1}{\text{Average}} \int_{K''} \left|\left(\sum_j c_j P_j\right)\left(\sum_k d_k Q_k\right) f\right|^p .\end{aligned}$$

Therefore, if we use Lemma 4.6 once again we find that

$$\begin{aligned} p_K&\left(\sum_{j,k} a_{jk} P_j Q_k f\right) \\ &= \Gamma\left(\frac{p}{2}+2\right)^{2/p} \underset{|c_j|=1, |d_k|=1}{\text{Average}}\, p_{K''}\left(\left(\sum_j c_j P_j\right)\left(\sum_k d_k Q_k\right) f\right) \\ &\leqq \Gamma\left(\frac{p}{2}+2\right)^{2/p} p_{K'''}(f) = \Delta, \text{ say} ;\end{aligned}$$

for some continuous semi-norm $p_{K'''}$. This proves the theorem.

We have already proved that the space $\Omega^p (1 < p < \infty)$ is complete metrisable and weakly (sequentially) complete. It is also barreled. The following theorem is now an immediate consequence of Theorems 3.1 and 4.7.

THEOREM 4.8. *Let T_1 and T_2 be two commuting spectral operators on $\Omega^p (2 \leqq p < \infty)$. Let $P(.)$ and $Q(.)$ be the corresponding spectral measures. Then*

(a) *if $P(.)$ and $Q(.)$ satisfy condition PC_0 then $T_1 + T_2$ is a spectral operator whose spectral measure $G(.)$ is given by*

$$G(\alpha) = R\{(\lambda, \mu) \colon \lambda + \mu \in \alpha\}, \alpha \in \mathscr{B} ;$$

(b) *if T_1 and T_2 have compact spectra then $T_1 T_2$ is a spectral operator whose spectral measure $H(.)$ is given by*

$$H(\alpha) = R\{(\lambda, \mu) \colon \lambda\mu \in \alpha\}, \alpha \in \mathscr{B} .$$

Added in proof. The spaces $\Omega^p (1 < p < \infty)$ are reflexive. Hence, a consideration of the adjoint operators would show that the Theorems 4.7 and 4.8 are, in fact, valid for $1 < p < \infty$.

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