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SOME DUAL SERIES EQUATIONS INVOLVING LAGUERRE POLYNOMIALS

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In this paper an exact solution is found for the dual series equations

(1) $\sum_{n=0}^{\infty} C_n \Gamma(\alpha+\beta+n) L_n(\alpha\,;x) = f(x)\,, \qquad 0 \leqq x < d\,,$

(2) $\sum_{n=0}^{\infty} C_n \Gamma(\alpha+1+n) L_n(\alpha\,;x) = g(x)\,, \qquad d < x < \infty\,,$

where $\alpha+\beta>0, 0<\beta<1, L_n(\alpha\,;x) = L_n^{\alpha}(x)$ **is the Laguerre polynomial and** $f(x)$ **and** $g(x)$ **are known functions.**

In a recent paper Srivastava [3] has solved the equations

$$(3) \qquad \sum_{n=0}^{\infty} \{A_n/\Gamma(\alpha+1+n)\}\, L_n(\alpha\,;x) = f(x)\,, \qquad 0 \leqq x < d\,,$$

$$(4) \quad \sum_{n=0}^{\infty} \{A_n/\Gamma(\alpha+1/2+n)\} L_n(\alpha\,;x) = g(x)\,, \quad d < x < \infty, \alpha > -1/2\,,$$

by considering separately the equations when (a) $g(x) \equiv 0$, (b) $f(x) \equiv 0$, and reducing the problem in each case to that of solving an Abel integral equation. Srivastava's equations are a special case of (1) and (2) with $\beta = 1/2$ and $A_n = \Gamma(\alpha+1+n)\Gamma(\alpha+1/2+n)C_n$.

The solution presented in this paper employs a multiplying factor technique which is more direct than the method given in [3] and is similar to that used by Noble [2] to solve some dual series equations involving Jacobi polynomials.

2. In the course of the analysis we shall use the following results.

From [1, p. 293(5), p. 405(20)] it is readily shown that

$$(5) \quad \int_0^y x^{\alpha}(y-x)^{\beta-1} L_n(\alpha\,;x)dx = \frac{\Gamma(\beta)\Gamma(\alpha+1+n)}{\Gamma(\alpha+\beta+1+n)}\, y^{\alpha+\beta} L_n(\alpha+\beta\,;y)\,,$$

where $-1 < \alpha, \beta > 0$, and

$$(6) \quad \int_y^{\infty} (x-y)^{-\beta} e^{-x} L_n(\alpha\,;x)dx = \Gamma(1-\beta) e^{-y} L_n(\alpha+\beta-1\,;y)\,,$$

where $1 > \beta, \alpha+\beta > 0$.

The orthogonality relation for the Laguerre polynomials is

$$(7) \qquad \int_0^{\infty} x^{\alpha} e^{-x} L_n(\alpha\,;x) L_m(\alpha\,;x)dx = \frac{\Gamma(\alpha+1+n)}{\Gamma(n+1)}\, \delta_{mn}, \alpha > -1\,,$$

where δ_{mn} is the Kronecker delta.

3. **Solution of the problem.** Multiplying equation (1) by $x^{\alpha}(y-x)^{\beta-1}$, equation (2) by $(x-y)^{-\beta}e^{-x}$ and integrating with respect to x over $(0, y)$ and (y, ∞) respectively we find on using the results (5) and (6)

$$(8)\quad \sum_{n=0}^{\infty} C_n \frac{\Gamma(\alpha+1+n)}{(\alpha+\beta+n)} L_n(\alpha+\beta\,;y) = \frac{y^{-\alpha-\beta}}{\Gamma(\beta)}\int_0^y x^{\alpha}(y-x)^{\beta-1}f(x)dx\,,$$

where $0 < y < d$, $\alpha > -1$, $\beta > 0$, and

$$(9)\quad \begin{aligned}&\sum_{n=0}^{\infty} C_n\,\Gamma(\alpha+1+n)L_n(\alpha+\beta-1\,;y)\\ &\quad = \frac{e^y}{\Gamma(1-\beta)}\int_y^{\infty}(x-y)^{-\beta}e^{-x}g(x)dx\,,\end{aligned}$$

for $d < y < \infty$, $1 > \beta$, $\alpha+\beta > 0$.

If we now multiply equation (8) by $y^{\alpha+\beta}$, differentiate with respect to y and use the formula

$$(10)\quad \frac{d}{dx}\{x^{\alpha}L_n(\alpha\,;x)\} = (n+\alpha)x^{\alpha-1}L_n(\alpha-1\,;x)\,,$$

we find

$$(11)\quad \begin{aligned}&\sum_{n=0}^{\infty} C_n\,\Gamma(\alpha+1+n)L_n(\alpha+\beta-1\,;y)\\ &\quad = \frac{y^{1-\alpha-\beta}}{\Gamma(\beta)}\frac{d}{dy}\int_0^y x^{\alpha}(y-x)^{\beta-1}f(x)dx\,,\end{aligned}$$

where $0 < y < d$, $\beta > 0$, $\alpha > -1$.

The left hand sides of equations (9) and (11) are now identical and using the orthogonality relation (7) we see that the solution of equations (1) and (2) for $\alpha+\beta > 0$, $0 < \beta < 1$, is given by

$$(12)\quad C_n = \frac{\Gamma(n+1)}{\Gamma(\alpha+1+n)\Gamma(\alpha+\beta+n)}B_n(\alpha,\beta\,;d)\,,$$

where

$$(13)\quad \begin{aligned}B_n(\alpha,\beta\,;d) &= \frac{1}{\Gamma(\beta)}\int_0^d e^{-y}L_n(\alpha+\beta-1\,;y)F(y)dy\\ &+ \frac{1}{\Gamma(1-\beta)}\int_d^{\infty} y^{\alpha+\beta-1}L_n(\alpha+\beta-1\,;y)G(y)dy\,,\end{aligned}$$

and

$$(14)\quad F(y) = \frac{d}{dy}\int_0^y x^{\alpha}(y-x)^{\beta-1}f(x)dx\,,$$

$$G(y) = \int_y^\infty (x-y)^{-\beta} e^{-x} g(x)dx \,. \tag{15}$$

To obtain the solution of Srivastava's equations (3) and (4) we write $\beta = 1/2$, $A_n = \Gamma(\alpha+1+n)\Gamma(\alpha+1/2+n)C_n$ in (12) and find that

$$\begin{aligned} A_n = \frac{\Gamma(n+1)}{\Gamma(1/2)} \Big\{ & \int_d^d e^{-y} L_n(\alpha - 1/2\,; y) F_1(y) dy \\ & + \int_d^\infty y^{\alpha-1/2} L_n(\alpha - 1/2\,; y) G_1(y) dy \Big\} \,, \end{aligned} \tag{16}$$

for $\alpha > -1/2$, and where $F_1(y)$ and $G_1(y)$ are given by equations (14) and (15) respectively with $\beta = 1/2$.

Comparing the above solution with that obtained in [3] it can be seen that they are in agreement except for the form of the function $G_1(y)$. The limits on the integrals of equations (4.7) and (4.8) in Srivastava's paper are wrong and should read (x, ∞) and (u, ∞) respectively. When these corrections have been made we find that his term corresponding to $G_1(y)$ can be written in the notation of the present paper as

$$-\frac{d}{dy}\int_y^\infty (x-y)^{-1/2} dx \int_x^\infty e^{-u} g(u) du \,. \tag{17}$$

After inverting the order of integration, carrying out the integration in x and performing the differentiation with respect to y it is found that (17) is equal to $G_1(y)$. Hence with this simplification Srivastava's solution reduces to that given by equation (16).

4. It is also possible without computing the coefficients C_n to find the values of series (1) and (2) in the regions where their values are not specified. We define (1) to have the value $h(x)$, $d < x < \infty$, and (2) to have the value $k(x)$, $0 \leqq x < d$.

(a) *Calculation of* $h(x)$. Substituting for C_n from equation (12) into (1) and interchanging the order of integration and summation we find

$$\begin{aligned} h(x) = & \frac{1}{\Gamma(\beta)} \int_0^d e^{-y} F(y) S_1(x, y) dy \\ & + \frac{1}{\Gamma(1-\beta)} \int_d^\infty y^{\alpha+\beta-1} G(y) S_1(x, y) dy \,, \qquad d < x < \infty \,, \end{aligned} \tag{18}$$

where

$$S_1(x, y) = \sum_{n=0}^\infty \frac{\Gamma(n+1)}{\Gamma(\alpha+1+n)} L_n(\alpha\,; x) L_n(\alpha+\beta-1\,; y) \,. \tag{19}$$

Using the results (6) and (7) it is easily shown that

$$S_1(x, y) = \frac{e^y x^{-\alpha}(x-y)^{-\beta}}{\Gamma(1-\beta)} H(x-y) , \tag{20}$$

where $H(x)$ is the Heaviside unit function.

From equations (18) and (20) we see that $h(x)$ is given by

$$\begin{aligned} \Gamma(1-\beta)x^\alpha h(x) = {} & \frac{1}{\Gamma(\beta)}\int_0^d (x-y)^{-\beta}F(y)dy \\ & + \frac{1}{\Gamma(1-\beta)}\int_d^x e^y y^{\alpha+\beta-1}(x-y)^{-\beta} G(y)dy , \end{aligned} \tag{21}$$

for $d < x < \infty$, where $F(y)$ and $G(y)$ are given by equations (14) and (15).

(b) *Calculation of* $k(x)$. Using the differentiation formula

$$e^{-x} L_n(\alpha ; x) = -\frac{d}{dx}\{e^{-x} L_n(\alpha-1 ; x)\} , \tag{22}$$

we may write equation (2) as

$$\begin{aligned} & \frac{d}{dx} e^{-x} \sum_{n=0}^{\infty} C_n \Gamma(\alpha+1+n) L_n(\alpha-1 ; x) \\ & = -e^{-x}k(x) , \qquad 0 \leqq x < d . \end{aligned} \tag{23}$$

Substituting for C_n and interchanging the order of integration and summation we find

$$\begin{aligned} e^{-x} k(x) = {} & -\frac{d}{dx} e^{-x}\Big\{\frac{1}{\Gamma(\beta)}\int_0^d e^{-y} F(y)S_2(x, y)dy \\ & + \frac{1}{\Gamma(1-\beta)}\int_d^\infty y^{\alpha+\beta-1} G(y)S_2(x, y)dy\Big\} , \end{aligned} \tag{24}$$

for $0 \leqq x < d$, and

$$\begin{aligned} S_2(x, y) & = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(\alpha+\beta+n)} L_n(\alpha-1 ; x)L_n(\alpha+\beta-1 ; y) \\ & = \frac{1}{\Gamma(\beta)} e^x (y-x)^{\beta-1} y^{1-\alpha-\beta} H(y-x) , \end{aligned} \tag{25}$$

where the series has been summed using the results (6) and (7).

Substituting for $S_2(x, y)$ in (24) we see that $k(x)$ is given by

$$\begin{aligned} \Gamma(\beta)e^{-x} k(x) = {} & -\frac{d}{dx}\Big\{\frac{1}{\Gamma(\beta)}\int_x^d e^{-y}(y-x)^{\beta-1} y^{1-\alpha-\beta} F(y)dy \\ & + \frac{1}{\Gamma(1-\beta)}\int_d^\infty (y-x)^{\beta-1} G(y)dy\Big\} , \end{aligned} \tag{26}$$

when $0 \leqq x < d$.

It is perhaps interesting to note that the expressions for the functions $k(x)$ and $h(x)$ do not involve Laguerre polynomials.

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