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FIXED POINT PROPERTIES AND INVERSE LIMIT SPACES

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The purpose of this paper is to prove that if $(X_\lambda, \pi_{\lambda\mu}, \Lambda)$ is an inverse system of compact Hausdorff spaces such that each X_λ has the fixed point property for the continuous multi-valued functions and each projection map is surjective, then the inverse limit space also has the fixed point property for the continuous multi-valued functions.

A topological space X is said to have the *f.p.p.* (fixed point property) if for every continuous (single-valued) function $f: X \rightarrow X$ there exists some x in X such that $x = f(x)$. Hamilton [3] has proved that the chainable metric continua have the *f.p.p.* A topological space X is said to have the *F.p.p.* (fixed point property for multi-valued functions) if every continuous (see Definition 1) multi-valued function $F: X \rightarrow X$ has a fixed point; that is, there exists some point x in X such that $x \in F(x)$. If a space has the F.p.p. then it has the f.p.p., but the converse need not be true [12]. Mardešić [8] has exhibited an inverse sequence, (X_m, π_{mn}), of polyhedra, X_m, such that all X_m have the f.p.p. and all bonding maps π_{mn} are surjective, but the inverse limit space, $\varprojlim (X_m, \pi_{mn})$, fails to have the f.p.p. This answered an open question raised by Mioduszewski and Rochowski [9 and 10], in the negative. Thus, our result stated in the first paragraph serves as an interesting counter-theorem to the result of Mardešić [*op. cit.*]. As a corollary, we obtain Ward's generalization [13] of the Hamilton theorem [*op. cit.*] that every metric chainable continuum has the F.p.p. In effect, our result is stronger than that of ward, since it includes some of the nonmetrizable chainable continua as well.

1. Preliminaries. *In all that follows, all spaces are assumed to be Hausdorff spaces.* A *multifunction*, $F: X \rightarrow Y$, from a space X to a space Y is a point-to-set correspondence such that, for each $x \in X$, $F(x)$ is a subset of Y. For any $y \in Y$, we write $F^{-1}(y)$ for the set $\{x \in X \mid y \in F(x)\}$. If $A \subset X$ and $B \subset Y$, then $F(A) = \cup \{F(x) \mid x \in A\}$ and $F^{-1}(B) = \cup \{F^{-1}(y) \mid y \in B\}$.

DEFINITION 1. A multifunction, $F: X \rightarrow Y$, is said to be *continuous* if and only if (i) $F(x)$ is closed for each x in X, (ii) $F^{-1}(B)$ is closed for each closed set B in Y, and (iii) $F^{-1}(V)$ is open for each open set V in Y.

Our definition of continuity here is weaker than that of Berge [1,

p. 109], but these two definitions coincide when the range space Y is compact.

A proof of the following lemma may be found in Berge [1, Th. 3, p. 110].

LEMMA 1. *If $f: X \to Y$ is a continuous multifunction and if A is a compact subset of X such that $F(a)$ is compact for each $a \in A$, then $F(A)$ is compact.*

DEFINITION 2. The triple, $(X_\lambda, \pi_{\lambda\mu}, \Lambda)$, is an *inverse system of spaces* if and only if:

(i) Λ is a directed set directed by $<$,

(ii) for each $\lambda \in \Lambda$, X_λ is a (Hausdorff) space,

(iii) if $\lambda > \mu$, $\pi_{\lambda\mu}$ is a continuous function of X_λ to X_μ,

(iv) if $\lambda > \mu$ and $\mu > \nu$, then $\pi_{\lambda\nu} = \pi_{\mu\nu}\pi_{\lambda\mu}$.

Each function $\pi_{\lambda\mu}$ is called a *bonding map*. If λ is in Λ, let S_λ be the subset of the Cartesian product $P\{X_\lambda \mid \lambda \in \Lambda\}$ defined by

$$S_\lambda = \{x \mid \text{if } \lambda > \mu \text{ then } \pi_{\lambda\mu}x(\lambda) = x(\mu)\}\ ,$$

where $x(\lambda)$ denotes the λ-th coordinate of x.

DEFINITION 3. The *inverse limit space*, X_∞, of the inverse system of spaces $(X_\lambda, \pi_{\lambda\mu}, \Lambda)$ is defined to be

$$\bigcap \{S_\lambda \mid \lambda \in \Lambda\}$$

endowed with the relative topology inherited from the product topology for $P\{X_\lambda \mid \lambda \in \Lambda\}$. In notation, we shall write X_∞ and $\varprojlim (X_\lambda, \pi_{\lambda\mu}, \Lambda)$ interchangeably for the inverse limit space defined above.

We write p_λ: $P\{X_\lambda \mid \lambda \in \Lambda\} \to X_\lambda$ for the λ-th projection of $P\{X_\lambda \mid \lambda \in \Lambda\}$, i.e., $p_\lambda(x) = x(\lambda)$ for all x in $P\{X_\lambda \mid \lambda \in \Lambda\}$; the restriction $p_\lambda \mid X_\infty$ will be denoted by π_λ which will be called a *projection map*. It is readily seen from the definition that an element x of $P\{X_\lambda \mid \lambda \in \Lambda\}$ is in X_∞ if and only if $\pi_{\lambda\mu}p_\lambda(x) = p_\mu(x)$ whenever $\lambda > \mu$. A more detailed account of inverse limit spaces may be found in Lefschetz [6], Capel [2] and Mardešić [7].

The following known results (see, e.g., [2], [6]) will be used.

LEMMA 2. (i) *The collection $\{\pi_\lambda^{-1}(U_\lambda) \mid \lambda \in \Lambda$ and U_λ is an open subset of $X_\lambda\}$ forms a basis for the topology of X_∞.*

(ii) *The inverse limit space, X_∞, is Hausdorff; if $\lambda \in \Lambda$, S_λ is a closed subset of $P\{X_\lambda \mid \lambda \in \Lambda\}$ so that X_∞ is closed in $P\{X_\lambda \mid \lambda \in \Lambda\}$.*

(iii) *If X_λ is compact for each λ in Λ, then X_∞ is compact; if, in addition, each X_λ is nonvoid, then X_∞ is nonvoid.*

(iv) *If X_λ is a continuum for each $\lambda \in \Lambda$, then the inverse limit* *space is a continuum.*

LEMMA 3. *If A is a compact subset of X_∞ and if $\pi'_{\lambda\mu} = \pi_{\lambda\mu} \mid \pi_\lambda(A)$, then $(\pi_\lambda(A), \pi'_{\lambda\mu}, \Lambda)$ is an inverse system of spaces such that $A = \varprojlim (\pi_\lambda(A), \pi'_{\lambda\mu}, \Lambda)$, and each bonding map $\pi'_{\lambda\mu}$ is surjective.*

2. Main results. In the sequel, since we are only interested in compact spaces, *each projection map π_λ will be assumed to be surjective*; for if otherwise, by virtue of Lemma 3, each X_λ may be replaced by $\pi_\lambda(X_\infty)$ without disturbing the resulting inverse limit space. We are now ready to state our main result.

MAIN THEOREM. *Let $(X_\lambda, \pi_{\lambda\mu}, \Lambda)$ be an inverse system of compact spaces such that each X_λ has the* F.p.p., *then the inverse limit space X_∞ also has the* F.p.p.

We divide the proof of this theorem into the following steps. In Lemmas 4, 5 and 6, X_∞ will be the inverse limit space of the inverse system $(X_\lambda, \pi_{\lambda\mu}, \Lambda)$ of *compact* spaces.

LEMMA 4. *If $F: X_\infty \to X_\infty$ is a continuous multifunction, define $F_\lambda: X_\lambda \to X_\lambda$ by $F_\lambda = \pi_\lambda F \pi_\lambda^{-1}$ for each λ, then F_λ is a continuous multifunction.*

Proof. (i) By Lemma 1, $F(\pi^{-1}(t))$ is compact in X_λ for each t in X_λ, and consequently each $F_\lambda(t)$ is closed in X_λ.

(ii) If C_λ is a closed subset of X_λ, then $F_\lambda^{-1}(C_\lambda)$ is closed. For, the set $F^{-1}\pi_\lambda^{-1}(C_\lambda)$ is closed in X_∞ and hence compact; therefore $\pi_\lambda F^{-1}\pi_\lambda^{-1}(C_\lambda) = F_\lambda^{-1}(C_\lambda)$ is compact and hence closed.

(iii) Since each π_λ is also an open map, as a dual of (ii) above, $F_\lambda^{-1}(U_\lambda)$ is open for each open set U_λ in X_λ.

Thus, by (i), (ii) and (iii) above, $F_\lambda: X_\lambda \to X_\lambda$ is continous.

LEMMA 5. *$F: X_\infty \to X_\infty$ be a continuous multifunction, let $F_\lambda: X_\lambda \to X_\lambda$ be defined as in Lemma 4. Then, for each x in X_∞,*

(i) *$(F_\lambda\pi_\lambda(x), \pi_{\lambda\mu}, \Lambda)$*[1] *and $(\pi_\lambda F(x), \pi_{\lambda\mu}, \Lambda)$ are inverse systems of compact spaces,*

(ii) $\varprojlim (F_\lambda\pi_\lambda(x), \pi_{\lambda\mu}, \Lambda) = \varprojlim (\pi_\lambda F(x), \pi_{\lambda\mu}, \Lambda)$,

(iii) $F(x) = \varprojlim (F_\lambda\pi_\lambda(x), \pi_{\lambda\mu}, \Lambda)$.

Proof. (i) It is obvious that each $F_\lambda\pi_\lambda(x)$ is compact. To show that $(F_\lambda\pi_\lambda(x), \pi_{\lambda\mu}, \Lambda)$ forms an inverse system, it suffices to show $\pi_{\lambda\mu}F_\lambda\pi_\lambda(x) \subset F_\mu\pi_\mu(x)$ whenever $\lambda > \mu$. To this end we first observe

[1] For simplicity in symbolism, henceforth if $A \subset \varprojlim (X_\lambda, \pi_{\lambda\mu}, \Lambda)$, then $(\pi_\lambda(A), \pi_{\lambda\mu}, \Lambda)$ will mean $(\pi_\lambda(A), \pi_{\lambda\mu} \mid \pi_\lambda(A), \Lambda)$.

$$\pi_\lambda(x) \in (\pi_{\lambda\mu}^{-1}\pi_{\lambda\mu})\pi_\lambda(x) = \pi_{\lambda\mu}^{-1}\pi_\mu(x) ,$$

since $\pi_{\lambda\mu}\pi_\lambda = \pi_\mu$. From this, with some computations,

$$\pi_{\lambda\mu}F_\lambda\pi_\lambda(x) \subset F_\mu\pi_\mu(x)$$

follows.

The fact that $(\pi_\lambda F(x), \pi_{\lambda\mu}, \Lambda)$ forms an inverse system follows from Lemma 3.

(ii) For each $\lambda \in \Lambda$ and any $x \in X_\infty$, we have

$$\pi_\lambda F(x) \subset \pi_\lambda F\pi_\lambda^{-1}\pi_\lambda(x) = (\pi_\lambda F\pi_\lambda^{-1})\pi_\lambda(x) = F_\lambda\pi_\lambda(x) ,$$

and thus,

$$\varprojlim (\pi_\lambda F(x), \pi_{\lambda\mu}, \Lambda) \subset \varprojlim (F_\lambda\pi_\lambda(x), \pi_{\lambda\mu}, \Lambda) .$$

To prove the other inclusion, we show

$$X_\infty - \varprojlim (\pi_\lambda F(x), \pi_{\lambda\mu}, \Lambda) \subset X_\infty - \varprojlim (F_\lambda\pi_\lambda(x), \pi_{\lambda\mu}, \Lambda) .$$

Let y be in $X_\infty - \varprojlim (\pi_\lambda F(x), \pi_{\lambda\mu}, \Lambda)$, then by Lemma 3 there exists a $\mu \in \Lambda$ such that $\pi_\mu(y) \notin \pi_\mu F(x)$. Let U_μ and V_μ be two disjoint open sets in X_μ such that

$$\pi_\mu(y) \in U_\mu \text{ and } \pi_\mu F(x) \subset V_\mu$$

so that

$$F(x) \subset \pi_\mu^{-1}(V_\mu) .$$

If follows then from Lemma 2(i) and the continuity of F that there exists a $\delta \in \Lambda$ and an open set U_δ in X_δ such that $x \in \pi_\delta^{-1}(U_\delta)$, and

$$(*) \qquad F(\pi_\delta^{-1}(U_\delta)) \subset \pi_\mu^{-1}(V_\mu) .$$

Since Λ is directed, there is a $\lambda_0 \in \Lambda$ such that $\lambda_0 > \mu$ and $\lambda_0 > \delta$, we shall use this λ_0 throughout the proof of lemma. If we denote $U_{\lambda_0} = \pi_{\lambda_0\delta}^{-1}(U_\delta)$ and using the equality $\pi_\delta^{-1} = \pi_{\lambda_0}^{-1}\pi_{\lambda_0\delta}^{-1}$, then (*) may be rewritten as

$$F(\pi_{\lambda_0}^{-1}(U_{\lambda_0})) \subset \pi_\mu^{-1}(V_\mu) ,$$

and hence

$$F_{\lambda_0}(U_{\lambda_0}) = \pi_{\lambda_0}F\pi_{\lambda_0}^{-1}(U_{\lambda_0}) \subset \pi_{\lambda_0}\pi_\mu^{-1}(V_\mu) = \pi_{\lambda_0}(\pi_{\lambda_0\mu}\pi_\lambda)^{-1}(V_\mu) = \pi_{\lambda_0\mu}^{-1}(V_\mu) .$$

In particular,

$$F_{\lambda_0}\pi_{\lambda_0}(x) \subset \pi_{\lambda_0\mu}^{-1}(V_\mu) .$$

Similarly, one obtains $\pi_{\lambda_0}(y) \in \pi_{\lambda_0\mu}^{-1}(U_\mu)$.
Since $\pi_{\lambda_0\mu}^{-1}(V_\mu)$ and $\pi_{\lambda_0\mu}^{-1}(U_\mu)$ are disjoint, $\pi_{\lambda_0}(y) \in F_{\lambda_0}\pi_{\lambda_0}(x)$. From this we

conclude $y \notin \varprojlim (F_\lambda \pi_\lambda(x), \pi_{\lambda\mu}, \Lambda)$, as desired.

(iii) This follows immediately from (ii) above and Lemma 3.

LEMMA 6. *Let* $F: X_\infty \to X_\infty$ *be a continuous multifunction, let* $F_\lambda: X_\lambda \to X_\lambda$ *be defined as in Lemma 4. Let* $E_\lambda = \{e_\lambda \mid e_\lambda \in X_\lambda$ *and* $e_\lambda \in F_\lambda(e_\lambda)\}$ *then* $(E_\lambda, \pi_{\lambda\mu}, \Lambda)$ *forms an inverse system.*

Proof. It suffices to prove $\pi_{\lambda\mu}(E_\lambda) \subset E_\mu$ whenever $\lambda > \mu$, which follows in a routine way.

Proof of main theorem. Since each X_λ has the F.p.p. and, by Lemma 4, each $F_\lambda: X_\lambda \to X_\lambda$ is continuous, each E_λ is closed and nonvoid. By Lemma 6, $(E_\lambda, \pi_{\lambda\mu}, \Lambda)$ is an inverse system of compact spaces, so it has a nonvoid inverse limit space $\varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda)$. We now conclude the proof by showing that each x in $\varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda)$ is a fixed point under F; i.e., $x \in F(x)$. If x is in $\varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda)$, then $\pi_\lambda(x) \in E_\lambda$ for all $\lambda \in \Lambda$; i.e., $\pi_\lambda(x) \in F_\lambda \pi_\lambda(x)$ for all $\lambda \in \Lambda$. Consequently, by Lemmas 3 and 5, we have

$$x = \varprojlim (\pi_\lambda(x), \pi_{\lambda\mu}, \Lambda) \in \varprojlim (F_\lambda \pi_\lambda(x), \pi_{\lambda\mu}, \Lambda) = F(x) .$$

Since the main theorem fails for single-valued functions, it should be pointed out that why the above argument breaks down in the single-valued case: given any continuous multifunction $F: X_\infty \to X_\infty$, each induced F_λ is again a continuous multifunction and hence has a fixed point; this is crucial to the proof. In the single-valued case, however, it does not follow in general that F_λ is single-valued and hence F_λ may not have a fixed point.

In fact, with the assumption of the main theorem and the notation of Lemma 6 together with the notation $E = \{x \mid x \in F(x)\}$, we can make the following sharper assertion.

THEOREM. $E = \varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda)$.

Proof. From the proof of the main Theorem, we have $E \supset \varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda)$. It remains to be proved that

$$E \subset \varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda) .$$

Let x be in E, then $x \in F(x)$ and therefore, for all $\lambda \in \Lambda$,

$$\pi_\lambda(x) \in \pi_\lambda F(x) \subset \pi_\lambda F(\pi_\lambda^{-1}\pi_\lambda)(x) = F_\lambda(\pi_\lambda(x)) .$$

That is, $\pi_\lambda(x) \in E_\lambda$ for all λ; consequently, by Lemma 3, $E \subset \varprojlim (E_\lambda, \pi_{\lambda\mu}, \Lambda)$.

A *chain* $(U_1, U_2, \cdots, U_n)$ is a finite sequence of sets U_i such that

$U_i \cap U_j \neq \square$ if and only if $|i-j| \leqq 1$, where $\square$ denotes the empty set. A Hausdorff space X is said to be *chainable* if to each open cover $\mathscr{V}$ of X there is a finite open cover $\mathscr{U} = (U_1, U_2, \cdots, U_n)$ such that (i) $\mathscr{U}$ refines $\mathscr{V}$; (ii) $\mathscr{U} = (U_1, U_2, \cdots, U_n)$ forms a chain. It follows that a chainable space is a continuum. It is implicit in the paper of Isbell [5] that each metrizable chainable continuum is the inverse limit space of a sequence of (real) arcs. This together with a theorem of Strother [12] that a bounded closed interval of the real numbers has the F.p.p. implies the following result of Ward [13] as a consequence of our main theorem.

Corollary [13]. *Each chainable metric continuum has the* F.p.p.

Examples of inverse limit spaces of inverse systems of real arcs exist which are not metrizable; for instance, the long line [4, p. 55] is one such.

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