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A NOTE ON EXTREMAL PROPERTIES CHARACTERIZING WEAKLY λ-VALENT PRINCIPAL FUNCTIONS

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On a planar bordered Riemann surface $\bar{W}$, a weakly λ-valent function is one whose every image point has at most λ anti-images. In this note, extremal properties characterizing weakly λ-valent principal functions are developed. The functionals extremized are, in a rather natural way, analogous to those of the univalent cases. However, the class of competing functions consists not only of weakly λ-valent analytic functions on $\bar{W}$, but of all analytic functions which are λ-valent near an interior point $\zeta \in W$ and near the isolated border γ of $\bar{W}$, and are of arbitrary finite valence elsewhere. Such competing classes contain the λ-th powers of competing univalent functions, as would be expected. That these classes contain functions of arbitrary finite valence perhaps would not be anticipated.

An interpretation is given for that situation in which the competing classes consist of those analytic functions which are λ-valent near two isolated border components.

Since the slit mapping F_0 of [5] maximizes among univalent functions of class A, the functional $2\pi \log r(F) - \int_\beta \log |F(z)| \, d \arg F(z)$, it follows that F_0^λ, interpreted as the λ-th power of F_0, maximizes the functional $\Psi(F) = 2\pi\lambda \log r(F) - \int_\beta \log |F(z)| \, d \arg F(z)$ among all λ-th powers of functions in A.

It would seem rather a natural question to ask whether the weakly λ-valent canonical maps $F_0^\lambda(z)$ and $F_1^\lambda(z)$ of [3] extremize such functionals $\Psi(F)$ and $\Phi(F)$, not only in a class of λ-th powers of univalent maps, but also in a class of weakly λ-valent mappings, whose behavior near the border and near a preselected interior point ζ, is analogous to that of functions in A. The answer to this question is essentially in the affirmative, and indeed, for a surprising reason. Namely, the functions of class A, in which Ψ and Φ are extreme, are required to be λ-valent only near γ, the border of $\bar{W}$ and near ζ, an interior point of $\bar{W} - \{\gamma\}$. The valence of such functions is *arbitrary* elsewhere on $\bar{W}$, and certainly this class contains the weakly λ-valent functions.

Similar results will hold if the bordered surface has two border components as, for instance, in [2]; namely, the class of functions over which one may extremize is essentially that class of functions which are λ-valent near the border components. And the valence of the competing functions, elsewhere on the planar bordered Riemann surface, is arbitrary.

However, the functional extremized by the λ-th principal function $F_1^\lambda(z)$ is slightly different than anticipated. It does, however, reduce to the anticipated functional when special cases such as those of [5] or [2] are considered.

2. Notation and constructions.

In order that the notation be consistent with that of [3], we consider, on the one hand $\bar{W}$, a planar bordered Riemann surface with an isolated compact border component γ, and a point ζ, belonging to the interior of $\bar{W} - \{\gamma\}$. On the other hand, we consider a planar bordered surface $\bar{V}$ with two compact isolated border components γ and δ. In the development, we shall deal with the surface $\bar{W}$, but shall be able to make interpretations in terms of the surface $\bar{V}$.

As usual, $\bar{W}$ is exhausted by a sequence of approximating bordered surfaces $\{W_n\}$, each compact, and of finite connectivity. Furthermore, for each n, W_n has ζ as an interior point, γ for one border component, and $\bar{\beta}_n$ as the union of the remaining border components $\beta_1, \cdots, \beta_{k(n)}$. We shall be concerned with the following class of functions defined on W_n.

DEFINITION. The class $H_n(\lambda)$ is the set of functions $p(z)$ harmonic on $W_n - \{\zeta\}$ such that (i) $p(z) = \text{const.} = c(p)$ for $z \in \gamma$ and $\int_\gamma dp^* = 2\pi\lambda$, (ii) the function $h(z) = p(z) - \lambda \log|z - \zeta|$ has a harmonic continuation to ζ, with $h(\zeta) = 0$.

By the construction of [4] already employed in [3], we find, in the class $H_n(\lambda)$, the functions $p_{0n}^\lambda(z)$ and $p_{1n}^\lambda(z)$. The first of these has its normal derivative $\partial p_{0n}^\lambda/\partial n$ equal to zero on $\bar{\beta}_n$, while the second has constant value $c_{\beta_i}(p_{1n}^\lambda)$ on each component β_i of $\bar{\beta}_n$, with

$$\int_{\beta_i} dp_{1n}^{\lambda *} = 0 .$$

Furthermore, these functions are characterized in $H_n(\lambda)$ by the following extremal properties.

3. Extremal properties for approximating functions.

PROPOSITION 1. The function $p_{0n}^\lambda(z)$ maximizes the functional $\Psi_n(p) = 2\pi\lambda c(p) - \int_{\bar{\beta}_n} pdp^*$ among all $p \in H_n(\lambda)$, and the deviation from the maximum is $D_{W_n}(p - p_{0n}^\lambda)$.

PROPOSITION 2. The function $p_{1n}^\lambda(z)$ minimizes the functional

$$\Phi_n(p) = 2\pi\lambda c(p) + \int_{\bar{\beta}_n} pdp^* - 2\int_{\bar{\beta}_n} p_{1n}^{\lambda} dp^*$$

among all $p \in H_n(\lambda)$, and the deviation from the minimum is $D_{W_n}(p - p_{1n}^{\lambda})$.

The proof of each of these propositions is similar to a proof in [5]. For instance, in Proposition 2, one establishes with the usual application of Green's formula, that

$$(1) \qquad D_{W_n}(p - p_{1n}^{\lambda}) = \int_{\gamma+\delta} pdp_{1n}^{\lambda *} - p_{1n}^{\lambda} dp^* + \int_{\bar{\beta}_n} pdp^* - 2\int_{\bar{\beta}_n} p_{1n}^{\lambda} dp^* ,$$

where δ is the oriented boundary of a disk with center at ζ. By applying condition (ii) for the family $H_n(\lambda)$, we find that

$$\int_{\delta} pdp_{1n}^{\lambda *} - p_{1n}^{\lambda} dp^* = 0 ,$$

and by applying condition (i), we may write Equation 1 as

$$(2) \qquad D_{W_n}(p - p_{1n}^{\lambda}) + 2\pi\lambda c(p_{1n}^{\lambda}) = 2\pi\lambda c(p) + \int_{\bar{\beta}_n} pdp^* - 2\int_{\bar{\beta}_n} p_{1n}^{\lambda} dp^* .$$

Proposition 2 now follows.

4. Extremal properties for arbitrary surfaces. The uniqueness of the solutions to the operator equation of [4] implies that the principal functions p_{in}^{λ} are λp_{in} $(i = 1, 2)$. The limit functions p_i^{λ} are then λ-multiples of principal functions of [4]. Each of these belongs to the following enlarged class of competing functions.

DEFINITION. The class $H(\lambda)$ is the set of functions $p(z)$ harmonic on $\bar{W} - \{\zeta\}$, whose restriction to W_n, for each n, belongs to $H_n(\lambda)$. If, for functions $p \in H(\lambda)$, we define the limit functional $\Psi(p)$ as the $\lim_n \Psi_n(p)$, it is only mechanical to check the conditions of the Reduction Theorem [6] and so establish the following theorem.

THEOREM 1. *The function $p_0^{\lambda}(z)$, and only this function, maximizes the functional $\Psi(p) = 2\pi\lambda c(p) - \int_{\beta} pdp^*$ among all $p \in H(\lambda)$. The deviation of this functional from its maximum is, for each such $p(z)$, equal to $D_{\bar{W}}(p - p_0^{\lambda})$.*

Due to the presence of the last term of the functional $\Phi_n(p)$, the extremal property for the function $p_1^{\lambda}(z)$ is not as readily established, and the following sequence of lemmas is presented for this purpose.

LEMMA 1. *For the family of functions $\{p_{1n}^{\lambda}(z)\}$, whose limit is $p_1^{\lambda}(z)$, we have the relation $\lim_n D_{W_n}(p_{1n}^{\lambda} - p_1^{\lambda}) = 0$.*

Proof. We have already observed that $p_{ln}^{\lambda}(z) = \lambda p_{ln}(z)$ for each n and also that $p_l^{\lambda}(z) = \lambda p_1(z)$. Hence, the quadratic character of the Dirichlet integral furnishes us with $D_{W_n}(p_1^{\lambda} - p_{1n}^{\lambda}) = \lambda^2 D_{W_n}(p_1 - p_{1n})$. Furthermore, due to the univalence of $F_1(z)$, it follows that $\int_{\beta_i} dp_1^* = 0$ for each contour of the set $\bar{\beta}_n$. Certainly, then $p_1(z) \in \{p\}$ of [5, No. 5] and according to Lemma 2 of that reference, $D_{W_n}(p_1 - p_{1n})$ is merely the deviation $2\pi c(p_1) + \int_{\bar{\beta}_n} p_1 dp_1^* - 2\pi c(p_{1n})$, and this goes to zero with increasing n. Hence $D_{W_n}(p_1^{\lambda} - p_{1n}^{\lambda}) = \lambda^2 D_{W_n}(p_1 - p_{1n})$ goes to zero as well, and Lemma 1 is proved.

LEMMA 2. *If $p \in H(\lambda)$, and if for some disk Δ containing ζ, $D_{\bar{W}-\Delta}(p) < \infty$, then* $\lim_n D_{W_n-\Delta}(p_{1n}^{\lambda} - p_1^{\lambda}, p) = 0$.

Proof. According to the Cauchy-Schwarz inequality, we have

$$(D_{W_n-\Delta}(p_{1n}^{\lambda} - p_1^{\lambda}, p))^2 \leqq D_{W_n-\Delta}(p_{1n}^{\lambda} - p_1^{\lambda}) D_{W_n-\Delta}(p) ,$$

and Lemma 2 now follows directly from Lemma 1.

LEMMA 3. *If $D_{\bar{W}-\Delta}(p) < \infty$, then $\lim_n D_{W_n-\Delta}(p_{1n}^{\lambda}, p)$ exists for each $p \in H(\lambda)$, and is equal to $D_{\bar{W}-\Delta}(p_1^{\lambda}, p)$.*

Proof. By the linearity of the mixed Dirichlet integral, we know that $D_{W_n-\Delta}(p_{1n}^{\lambda}, p) - D_{W_n-\Delta}(p_1^{\lambda}, p) = D_{W_n-\Delta}(p_{1n}^{\lambda} - p_1^{\lambda}, p)$. Hence our result will follow from Lemma 2 if $\lim_k D_{W_k-\Delta}(p_1^{\lambda}, p)$ exists. Upon using the Cauchy-Schwarz inequality again, we find

$$(D_{W_q-W_k}(p_1^{\lambda}, p))^2 \leqq D_{W_q-W_k}(p_1^{\lambda}) D_{W_q-W_k}(p) .$$

Since each of these can be made small, it follows that the sequence $\{D_{W_n-\Delta}(p_1^{\lambda}, p)\}$ converges.

For later use, we now draw the following corollary.

COROLLARY. *If $D_{\bar{W}-\Delta}(p) < \infty$, then*

$$\lim_n D_{W_n}(p - p_{1n}^{\lambda}) = D_{\bar{W}}(p - p_1^{\lambda}) .$$

Proof. Certainly it suffices to prove our relation with W_n and $\bar{W}$ replaced respectively by $W_n - \Delta$ and $\bar{W} - \Delta$. Due to the linearity of the Dirichlet integral, we have

$$D_{W_n-\Delta}(p - p_{1n}^{\lambda}) = D_{W_n-\Delta}(p) - 2D_{W_n-\Delta}(p, p_{1n}^{\lambda}) + D_{W_n-\Delta}(p_{1n}^{\lambda}) . \tag{3}$$

In the limit, the first term becomes $D_{\bar{W}-\Delta}(p)$, and according to Lemma 3, the second term approaches $-2D_{\bar{W}-\Delta}(p, p_1^{\lambda})$. As for the third term, we need only observe that

$$\sqrt{D_{W_n-\Delta}(p_{1n}^\lambda - p_1^\lambda)} \geqq |\sqrt{D_{W_n-\Delta}(p_{1n}^\lambda)} - \sqrt{D_{W_n-\Delta}(p_1^\lambda)}| .$$

Hence it follows from Lemma 1 that the third term on the right in Equation 3 approaches $D_{\bar{W}-\Delta}(p_1^\lambda)$, and the corollary is proved.

For each $p(z)$ with $D_{\bar{W}-\Delta}(p) < \infty$, we have now established, according to Lemma 3, a functional $\Phi(p)$. For the existence of

$$\lim_n D_{W_n-\Delta}(p_{1n}^\lambda, p)$$

determines, and is determined by the existence of $\lim_n \int_{\bar{\beta}_n} p_{1n}^\lambda dp^*$.

DEFINITION. For each $p \in H(\lambda)$, $\Phi(p)$ is understood as $\lim_n \Phi_n(p)$, where $\Phi_n(p)$ is defined in Proposition 2.

Certainly such is defined if $D_{\bar{W}-\Delta}(p)$ is finite. If, on the other hand, $D_{\bar{W}-\Delta}(p)$ is infinite, then so is $\lim_n D_{W_n}(p - p_{1n}^\lambda)$. It then follows from Equation 2 that $\Phi(p)$ is infinite as well.

We are now able to state Theorem 2, which seems to extend the results of [5] and [2] in that it applies to a wider class of harmonic functions. The same will be true for corresponding theorems concerning λ-th principal analytic functions which are rather naturally associated with our principal harmonic functions.

THEOREM 2. *The function $p_1^\lambda(z)$, and only this function, minimizes the functional $\Phi(p) = 2\pi\lambda c(p) + \int_\beta p dp^* - 2\int_\beta p_1^\lambda dp^*$ among all $p \in H(\lambda)$. The minimum value $\Phi(p_1^\lambda)$ is $\lim_n \Phi_n(p_{1n}^\lambda)$, and the deviation of this functional from this value is $D_{\bar{W}}(p - p_1^\lambda)$.*

Proof. Since each $p(z)$ of $H(\lambda)$ is, when properly restricted, automatically in $H_n(\lambda)$, it follows from Proposition 2 that

$$(4) \qquad \Phi_n(p) - \Phi_n(p_{1n}^\lambda) = D_{W_n}(p - p_{1n}^\lambda) .$$

Hence it follows from Lemma 1 that $\Phi(p_1^\lambda) = \lim_n \Phi_n(p_{1n}^\lambda)$. Furthermore, it follows from Proposition 2 that $\Phi_n(p_{1n}^\lambda) \leqq \Phi_n(p)$ for each $p \in H(\lambda)$, and with the equality just established we have $\Phi(p_1^\lambda) \leqq \Phi(p)$. Hence $p_1^\lambda(z)$ minimizes $\Phi(p)$ among all $p \in H(\lambda)$.

If $D_{\bar{W}-\Delta}(p)$ is infinite, then so are $D_{\bar{W}}(p - p_1^\lambda)$ and

$$\Phi(p) = \lim_n \Phi_n(p) = \lim_n D_{W_n}(p - p_{1n}^\lambda) + 2\pi\lambda c(p_1^\lambda) .$$

The deviation formula is understood in the sense that $\Phi(p) - \Phi(p_1^\lambda)$ and $D_{\bar{W}}(p - p_1^\lambda)$ are both infinite. Suppose now that $D_{\bar{W}-\Delta}(p)$ is finite. The deviation formula will follow, according to the corollary to

Lemma 3, from taking limits in Equation 4.

The Theorems 1 and 2 are valid for bordered Riemann surfaces $\bar{V}$ having two isolated border components γ and δ in the sense of [2]. On such surfaces, the appropriate class $H(\lambda)$ of harmonic functions $p(z)$ consists of those functions satisfying (i) $p(z) = \text{const.} = c_\gamma(p)$ for each $z \in \gamma$ with flux on γ equal to $2\pi\lambda$, (ii) $p(z) = \text{const.} = c_\delta(p)$ with flux on δ equal to $-2\pi\lambda$, and (iii) $p(\zeta) = 0$ for some ζ, an interior point of $\bar{V}$. The harmonic $p_1^\lambda(z)$ belongs to $H(\lambda)$ and is a limit of functions $\{p_{1n}^\lambda(z)\}$, each with zero flux on β_i and constant there; while $p_0^\lambda(z)$ is a limit of $\{p_{0n}^\lambda(z)\}$, each with normal derivative equal to zero on $\bar{\beta}_n$. Theorems 1 and 2 are now valid verbatim, if only we interpret $c(p)$ as $c_\gamma(p) - c_\delta(p)$.

5. Extremal properties of weakly λ-valent principal functions.

We consider the λ-th principal functions $F_0^\lambda(z) = \exp(p_0^\lambda(z) + ip_0^\lambda(z)^*)$ and $F_1^\lambda(z) = \exp(p_1^\lambda(z) + ip_1^\lambda(z)^*)$ already introduced in [3]. A class of analytic functions on $\bar{W}$ naturally associated with the harmonic class $H(\lambda)$ would seem to be the following.

DEFINITION. The class $A(\lambda)$ is the set of functions $F(z)$ analytic on $\bar{W}$ satisfying (i) for $z \in \gamma$, $|F(z)| = \text{const.} = r(F)$ and

$$\int_\gamma d(\arg F(z)) = 2\pi\lambda\ ,$$

(ii) $F(z)$ has a λ-th order zero at $z = \zeta$, where $\lim_{z\to\zeta} F(z)/(z-\zeta)^\lambda = 1$.

Each of the λ-th principal functions of [3] belongs to the class $A(\lambda)$, and furthermore, for each function $F(z)$ of the class $A(\lambda)$, the harmonic $\log|F(z)|$ belongs to the class $H(\lambda)$ of No. 4. These remarks, along with Theorems 1 and 2, establish the following two theorems.

THEOREM 3. *The λ-th principal function $F_0^\lambda(z)$, and only this function maximizes the functional*

$$2\pi\lambda \log r(F) - \int_\beta \log|F(z)|\, d(\arg F(z))$$

among all functions of class $A(\lambda)$. The deviation, for each $F(z) \in A(\lambda)$, from the maximum is $D_{\bar{W}}(\log|F(z)/F_0^\lambda(z)|)$.

THEOREM 4. *The λ-th principal function $F_1^\lambda(z)$, and only this function minimizes the functional*

$$2\pi\lambda \log r(F) + \int_\beta \log|F(z)|\, d(\arg F(z)) - 2\int_\beta \log|F_1^\lambda(z)|\, d(\arg F(z))$$

among all functions of class $A(\lambda)$. The deviation, for each $F(z) \in A(\lambda)$, *from the minimum is* $D_{\overline{W}}(\log|F(z)/F_1^\lambda(z)|)$.

It is not difficult to establish that the class $A(\lambda)$ is properly larger than the class of λ-th powers of univalent functions. For one need only consider the harmonic $p(z) = p_1^\lambda(z) + t_{\beta'}(z) - t_{\beta''}(z)$. Here, β' and β'' are components of the ideal boundary β, $t_{\beta'}(z)$ and $t_{\beta''}(z)$ are the respective capacity functions for these boundary components. That is, the class $\{t\}$ [7, p. 141] is taken as those functions for which $\int_{\beta'} dt^* = 2\pi$ and $\int_\sigma dt^* = 0$ for each cycle σ not separating the point ζ from β'. Then $t_{\beta'}(z)$ is that harmonic function which minimizes $\int_\beta t dt^*$ in this class. The function $t_{\beta''}(z)$ is defined in an analogous manner. The analytic $F(z) = \exp(p(z) + ip(z)^*)$ belongs to the class $A(\lambda)$ and is according to the appendix of [3], a λ-th power of no univalent function.

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