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# ON A CLASS OF CONVOLUTION TRANSFORMS

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**In this paper the convolution transform**

$$(1.1)\qquad f(x)=\int_{-\infty}^{\infty} G(x-t)\varphi(t)dt \equiv (G*\varphi)(x)$$

**whose kernel $G(t)$ is the Fourier transform of $[E(iy)]^{-1}$ where $E(s)$ is defined by**

$$(1.2)\qquad E(s) = e^{bs}\prod_{k=1}^{\infty}(1-s/a_k)\exp(sRe\,a_k^{-1})\,,$$

$$Reb = b \text{ and } \sum |a_k|^{-2} < \infty$$

**will be studied. An inversion theory similar to that achieved when $a_k$ of (1.2) are real will be obtained. The results will show that under certain rather weak conditions, an infinite subsequence $a_{k(i)}$ of $a_k$ can satisfy**

$$\min\{|\arg a_{k(i)}|, |\arg - a_{k(i)}|\} \geqq \frac{\pi}{4}\,.$$

**Classes of transforms will be introduced that allow the occurrence of $\min\{|\arg a_k|, |\arg - a_k|\} \geqq \pi/4$ for all $k$.**

We hope this will partly answer a problem set by Dauns and Widder [1] in Remark 1, page 441.

The inversion operator $P_m(D)$ is defined by

$$(1.3)\qquad P_m(D) = \exp((b-b_m)D)\prod_{k=1}^{m}\left(1-\frac{D}{a_k}\right)\exp\left(\left(Re\frac{1}{a_k}\right)D\right)$$

where $D \equiv d/dx$, $\exp(kD)f(x) = f(x+k)$ and $\lim_{m\to\infty} b_m = 0$.

The inversion formula will be

$$(1.4)\qquad \lim_{i\to\infty} P_{m(i)}(D)f(x) = \varphi(x)\,.$$

This inversion formula was achieved under general conditions on $\varphi(x)$ in the case $a_k$ were real by I. I. Hirschman and D. V. Widder in a series of papers and in their book, "The convolution transform" [7]. Hirschman and Widder [6] also found a slightly changed version of (1.4) when $\sum_{r=1}^{\infty}(\mathrm{Im}\,a_k/\mathrm{Re}\,a_k)^2 < \infty$. A. O. Garder [5] showed that if $a_{2k-1} = \bar{a}_{2k}$ then arg $a_{2k}$ can tend to 0 or $\pi$ slower than is required in [6]. Dauns and Widder [1] showed that if $a_{2k-1} = -a_{2k}$, $0 \leqq \mathrm{Re}\,a_{2k-1} \in \uparrow$ and $|\arg a_{2k-1}| < (\pi/4) - \eta$, where $\eta$ is independent of $k$, then (1.4) can be achieved.

It will be noted that in [1] and [5] the $a_k$'s were in a special order. The order of the $a_k$'s, though having no influence on $E(s)$, may be quite important when treating (1.4) as discussed with some examples in [2] and [4].

We shall define class $A(2)$ (that will depend also on the order of the $a_k$'s). The sequence $\{a_k\}$ belongs to class $A(2)$ if $\operatorname{Re} a_k \neq 0$,

$$\sum_{k=1}^{\infty} (\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2/(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) < \infty , \tag{1.5}$$

$$(1-\theta)(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) + 4 \operatorname{Im} a_{2k-1}^{-1} \operatorname{Im} a_{2k}^{-1} > 0 \tag{1.6}$$

for $k > k_0$ for some $\theta, 0 < \theta < 1$ where $\theta$ is independent of $k$, and

$$\frac{(\operatorname{Im}\{\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})} a_{2k-1}^{-1} a_{2k}^{-1}\})^2 \, |a_{2k-1} a_{2k}|^2}{|a_{2k-1}|^2 + |a_{2k}|^{-2} + 4 \operatorname{Im} a_{2k-1}^{-1} \operatorname{Im} a_{2k}^{-1}} < 1 - \eta \tag{1.7}$$

for $k \geqq k_1$ for some $\eta, 0 < \eta < 1$ where $\eta$ is independent of $k$.

A transform belongs to $A(2)$ if there is an order under which $\{a_n\} \in A(2)$. Class $A(2)$ includes the transforms of [1], [5] and [6].

LEMMA 1.1. *$\sum_{k=1}^{\infty} (\operatorname{Im} a_k/\operatorname{Re} a_k)^2 < \infty$ implies $\{a_k\} \in A(2)$ (and the order does not matter).*

*Proof.* $\sum_{k=1}^{\infty} (\operatorname{Im} a_k/\operatorname{Re} a_k)^2 < \infty$ implies $\sum_{k=1}^{\infty} (\operatorname{Im} a_k/|a_k|)^2 < \infty$ which implies $\sum_{k=1}^{\infty} (\operatorname{Im} a_k^{-1})^2/|a_k|^{-2} < \infty$ which implies (1.5). To prove that $a_k$ satisfies (1.6) and (1.7) is not difficult.

REMARK. The inversion operator introduced by Hirschman and Widder [6] was slightly different from (1.4) but since

$$\sum_{k=1}^{\infty} \{(\operatorname{Re} a_k)^{-1} - \operatorname{Re} a_k^{-1}\} = \sum_{k=1}^{\infty} \frac{(\operatorname{Im} a_k)^2}{|a_k|^2 \operatorname{Re} a_k} < \infty ,$$

the difference is a change in $b$ and $b_m$ without changing $\lim_{m\to\infty} b_m = 0$.

LEMMA 1.2. *Let $a_{2k-1} = -a_{2k}$, let $\operatorname{Re} a_{2k} > 0$ and $|\arg a_{2k}| < (\pi/4) - \eta_1$ for $k > k_2$, where $\eta_1$ satisfies $0 < \eta_1 < \pi/4$ and $\eta_1$ is independent of $k$, then $\{a_k\} \in$ class $A(2)$.*

*Proof.* It is easy to see that the sum in (1.5) is equal to zero and the right side of (1.7) is equal to zero. $|\arg a_{2k}| < (\pi/4) - \eta_1$ implies (1.6), with $\theta = 1 - 2(\operatorname{Sin}((\pi/4) - \eta_1))^2$, for $k > k_2$.

This shows that the transforms treated in [1] are included in class $A(2)$.

LEMMA 1.3. *Let $a_{2k-1} = \overline{a_{2k}}$ and let $\min\{|\arg a_{2k}|, |\arg - a_{2k}|\} < (\pi/4) - \eta_2$ for $k \geqq k_2$ where $\eta_2$, $0 < \eta_2 < \pi/4$, is independent of $k$, then $\{a_k\} \in A(2)$.*

*Proof.* It is easy to see that the sum in (1.5) and the right side of (1.7) are equal to zero. One can show that $\min\{|\arg a_{2k}|, |\arg - a_{2k}|\} < (\pi/4) - \eta_2$ implies (1.6) with $\theta = 1 - 2(\operatorname{Sin}((\pi/4) - \eta_2))^2$ for $k \geqq k_2$.

Lemma 1.3 shows that the transforms treated by A. O. Garder [5] belong to class $A(2)$. Some cases which do not belong to class $A(2)$ will be treated, among them will be the case when $a_{2k-1} = -a_{2k}$ and $\min\{|\arg a_{2k}|, |\arg - a_{2k}|\} = \pi/4$ (see Remark 2, [1], p. 442) where estimates different from those achieved for class $A(2)$ will be obtained.

For the definition of $G(t)$

$$G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} [E(s)]^{-1} e^{st} dt \tag{1.8}$$

we have to assume that the integral on the right converges.

For the convergence of (1.8) we shall have to estimate $E(iy)$ and to these estimates the various classes correspond.

**2. Estimates for $E_{2m}(s)$ when $\{a_k\} \in$ class $A(2)$.** In previous papers (see [1] and [6] for example) it was found useful and important to estimate $E_m(s)$ which is defined by

$$E_m(s) = e^{b_m s} \prod_{k=m+1}^{\infty} (1 - s/a_k) \exp(s \operatorname{Re} a_k^{-1}) . \tag{2.1}$$

In order to estimate $E_m(s)$ we shall estimate one term first.

LEMMA 2.1. *Let $\{a_k\} \in$ class $A(2)$ then for $k \geqq K$*

$$\begin{aligned} &|(1 - iy/a_{2k-1})(1 - iy/a_{2k})|^2 \\ &\quad \geqq (1 + \alpha y^2/|a_{2k-1}|^2)(1 + \alpha y^2/|a_{2k}|^2)(1 - \alpha^{-1}[(\operatorname{Im}(a_{2k-1}^{-1} \\ &\qquad + a_{2k}^{-1}))^2/(|a_{2k-1}|^{-2} + |a_{2k}|^{-2})]) . \end{aligned} \tag{2.2}$$

*where $0 < \alpha < 1$ and $\alpha$ is independent of $k$. ($\alpha$ does depend on $\theta$ and $\eta$ of the definition of class $A(2)$).*

*Proof.* By a simple calculation we get

$$\begin{aligned} I_k \equiv |(1 - iy/a_{2k-1})(1 - iy/a_{2k})|^2 &= 1 + 2y \operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}) \\ &+ y^2\{|a_{2k-1}|^{-2} + |a_{2k}|^{-2} + 4 \operatorname{Im} a_{2k-1}^{-1} \operatorname{Im} a_{2k}^{-1}\} \\ &+ 2y^3 \operatorname{Im}\{a_{2k-1}^{-1} a_{2k}^{-1} \overline{(a_{2k-1}^{-1} + a_{2k}^{-1})}\} + y^4 |a_{2k-1}|^{-2} |a_{k2}|^{-2} . \end{aligned}$$

We assume $K \geqq k_1$ and therefore by (1.7) we get

$$\frac{(\operatorname{Im}\{\overline{(a_{2k-1}^{-1}+a_{2k}^{-1})}(a_{2k-1}^{-1}\cdot a_{2k}^{-1})\})^2}{\left[\left(1-\frac{\eta}{2}\right)(|a_{2k-1}|^{-2}+|a_{2k}|^{-2}+4\operatorname{Im}a_{2k-1}^{-1}\operatorname{Im}a_{2k}^{-1})\right]\cdot\left[\left(1-\frac{\eta}{2}\right)|a_{2k-1}\cdot a_{2k}|^{-2}\right]}$$
$$<(1-\eta)\Big/\left(1-\eta+\frac{\eta^2}{4}\right)<1-\frac{\eta^2}{4}.$$

It is easy to see that $y^2(A+2By+Cy^2)\geqq 0$ whenever $A>0$, $C>0$ and $B^2<AC$. We substitute

$$A-\left(1=\frac{\eta}{2}\right)\{|a_{2k-1}|^{-2}+|a_{2k}|^{-2}+4\operatorname{Im}a_{2k-1}^{-1}\operatorname{Im}a_{2k}^{-1}\},$$
$$B=\operatorname{Im}\{\overline{(a_{2k-1}^{-1}+a_{2k}^{-1})}a_{2k-1}^{-1}a_{2k}^{-1}\}\quad\text{and}$$
$$C=\left(1-\frac{\eta}{2}\right)|a_{2k-1}a_{2k}|^{-2}.$$

We use (1.6), (1.7) and the above calculation to show that, for $k>\max(k_0,k_1)$, $A>0, C>0$ and $B^2>AC$. By omitting $y^2(A+2By+Cy^2)$ from the right side of the equation defining $I_k$ we obtain

$$I_k\geqq 1+2y\operatorname{Im}(a_{2k-1}^{-1}+a_{2k}^{-1})+\frac{\eta\theta}{2}y^2(|a_{2k-1}|^{-2}+|a_{2k}|^{-2})+\frac{\eta y^4}{2}|a_{2k-1}a_{2k}|^{-2}\tag{2.3}$$

by minimum consideration

$$1+2y\operatorname{Im}(a_{2k-1}^{-1}+a_{2k}^{-1})+\frac{\eta\theta}{4}y^2(|a_{2k-1}|^{-2}+|a_{2k}|^{-2})\geqq 1-\frac{\frac{4}{\eta\theta}(\operatorname{Im}(a_{2k-1}^{-1}+a_{2k}^{-1}))^2}{(|a_{2k-1}|^{-2}+|a_{2k}|^{-2})}\tag{2.4}$$

the last term tends to 1 for large $k$ because of (1.5). Using (2.3), (2.4) and letting the coefficients of $y^2$ and $y^4$ be smaller, we obtain (2.2) with $\alpha=\eta\theta/4$.

LEMMA 2.2. *Suppose $\{a_k\}\in$ class $A(2)$. Then for $k>K$ there exist $A$ and $B$, $0<A<B<1$ independent of $k$ (but they depend on $\eta$ and $\theta$) so that for any $r$, $r<\min(|a_{2k-1}|, |a_{2k}|)$, we shall have:*

(a) *For $|\sigma|\leqq Ar$ and $|y|\leqq Br$*

$$H_k(\sigma)\equiv|(1-(\sigma+iy)/a_{2k-1})(1-(\sigma+iy)/a_{2k})|^2\exp(2\sigma\operatorname{Re}(a_{2k-1}^{-1}+a_{2k}^{-1}))$$
$$\geqq 1-2\alpha^{-1}\frac{\operatorname{Im}(a_{2k-1}^{-1}+a_{2k}^{-1})}{|a_{2k-1}|^{-2}+|a_{2k}|^{-2}}-\frac{r^2}{4}(|a_{2k-1}|^{-2}+|a_{2k}|^{-2})$$
$$-4\sigma^2(\operatorname{Re}(a_{2k-1}^{-1}+a_{2k}^{-1}))^2.$$

(b) *For $|\sigma|\leqq Ar$ and $|y|\geqq Br$*

$$H_k(\sigma) \geqq \Big(1 + \frac{\alpha}{4} y^2 \,|\, a_{2k-1} |^{-2}\Big)\Big(1 + \frac{\alpha}{4} y^2 \,|\, a_{2k} |^{-2}\Big)$$
$$\times \Big(1 - \frac{2}{\alpha} \frac{(\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2}{|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}}\Big),$$

*where $\alpha$ is that of Lemma* 2.1.

*Proof.* By a simple calculation

$$\Big|\Big(1 - \frac{\sigma + iy}{a_{2k-1}}\Big)\Big(1 - \frac{\sigma + iy}{a_{2k}}\Big)\Big|^2 = \Big|\Big(1 - \frac{iy}{a_{2k-1}}\Big)\Big(1 - \frac{iy}{a_{2k}}\Big)\Big|^2$$
$$- 2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}) + [\sigma^2(|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2} + 4 \operatorname{Re} a_{2k-1}^{-1} \operatorname{Re} a_{2k}^{-1})$$
$$+ \sigma^4 \,|\, a_{2k-1}a_{2k} |^{-2} + 2\sigma^2 y^2 \,|\, a_{2k-1}a_{2k} |^{-2} - 4\sigma y \operatorname{Im}(a_{2k-1}^{-1}a_{2k}^{-1})$$
$$- 2(\sigma^2 + y^2)\sigma \operatorname{Re}\{\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})a_{2k-1}^{-1}a_{2k}^{-1}}\}$$
$$+ 2\sigma^2 y \operatorname{Im}\{\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})a_{2k-1}^{-1}a_{2k}^{-1}}\}] \equiv I_k - 2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}) + J_k \,.$$

For the estimation of $J_k$ we shall recall that

$$|\, (a_{2k-1}^{-1} + a_{2k}^{-1})a_{2k-1}^{-1}a_{2k}^{-1} \,| \leqq 2(|\, a_{2k-1} |^{-3} + |\, a_{2k} |^{-3}) \tag{2.5}$$

and

$$\begin{aligned} |\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2} + 4 \operatorname{Re} a_{2k-1}^{-1} \operatorname{Re} a_{2k}^{-1} &\geqq -2 \,|\operatorname{Re} a_{2k-1}^{-1} \operatorname{Re} a_{2k}^{-1} | \\ &\geqq -(|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}) \,. \end{aligned} \tag{2.6}$$

To prove (a) assume $|\sigma| \leqq Ar$, $|y| \leqq Br$. Using (2.5) and (2.6) and dropping positive terms we obtain for $A < B$

$$J_k \geqq (-A^2 - |\, 2AB)r^2(|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2})$$
$$+ (-4(A^2 + B^2)A - 4A^2B)r^3(|\, a_{2k-1} |^{-3} + |\, a_{2k} |^{-3})$$
$$\geqq (-3B^2 - 12B^3)r^2(|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}) \,.$$

Choosing $A < B$ and (for instance) $B = 3^{-2}$ and using Lemma 2.1 with $y = 0$ we obtain

$$H_k(\sigma) \geqq \Big(1 - \frac{\alpha^{-1}(\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2}{|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}} - \frac{1}{9} r^2(|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2})$$
$$- 2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1})\Big) \exp(2\sigma \operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}))$$
$$\geqq 1 - 2\alpha^{-1}(\operatorname{Im}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2[|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}]^{-1}$$
$$- \frac{1}{4} r^2(|\, a_{2k-1} |^{-2} + |\, a_{2k} |^{-2}) - 4\sigma^2(\operatorname{Re}(a_{2k-1}^{-1} + a_{2k}^{-1}))^2 \,.$$

(The coefficients in the above estimation are not the best but they are convenient). To prove (b) (for which we are free to choose $A, A < B$) we recall that for $A \leqq \beta B$, $0 < \beta < 1$ and $|\sigma| < Ar$ we have

$$| 2\sigma^2 \operatorname{Re} a_{2k-1}^{-1} \operatorname{Re} a_{2k}^{-1}) | \leqq \beta^2 B^2 r^2(| a_{2k-1} |^{-2} + | a_{2k} |^{-2}) ,$$

$$| 2\sigma^2 y \operatorname{Im} \{\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})} a_{2k-1}^{-1} \cdot a_{2k}^{-1}\} |$$

$$\leqq \sigma^2 y^2 | a_{2k-1} a_{2k} |^{-2} + 2\sigma^2(| a_{2k-1} |^{-2} + | a_{2k} |^{-2}) ,$$

$$| 2y^2 \sigma \operatorname{Re} \{\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})} a_{2k-1}^{-1} a_{2k}^{-1}\} |$$

$$\leqq \beta y^4 | a_{2k-1} a_{2k} |^{-2} + 2\beta y^2(| a_{2k-1} |^{-2} + | a_{2k} |^{-2})$$

and

$$| 2\sigma^3 \operatorname{Re} \{\overline{(a_{2k-1}^{-1} + a_{2k}^{-1})} a_{2k-1}^{-1} a_{2k}^{-1}\} |$$

$$\leqq \sigma^4 | a_{2k-1} a_{2k} |^{-2} + 2\sigma^2(| a_{2k-1} |^{-2} + | a_{2k} |^{-2}) .$$

Choosing $\beta$ so that $5\beta^2 + 4\beta < \alpha/4$ and $K$ so that

$$\alpha^{-1}(\operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2/(| a_{2k-1} |^{-2} + | a_{2k} |^{-2}) \leqq \frac{1}{4}$$

for $k \geqq K$ we obtain by the above estimations

$$H_k(\sigma) \leqq \left(1 + \frac{\alpha}{2} y^2 | a_{2k-1} |^{-2}\right)\left(1 + \frac{\alpha}{2} | a_{2k} |^{-2}\right)\left(1 - \frac{1}{\alpha}(\operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2 ,\right.$$

$$\left. \times [| a_{2k-1} |^{-2} + | a_{2k} |^{-2}]^{-1} - 2\sigma \operatorname{Re} (a_{2k-1}^{-1} + a_{2k}^{-1})\right) \cdot \exp (2\sigma \operatorname{Re} (a_{2k-1}^{-1} + a_{2k}^{-1}))$$

$$\leqq \left(1 + \frac{3\alpha}{8} y^2 | a_{2k-1} |^{-2}\right)\left(1 + \frac{3\alpha}{8} y^2 | a_{2k} |^{-2}\right)\left(1 - \frac{2}{\alpha}(\operatorname{Im} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2 ,\right.$$

$$\left. \times [| a_{2k-1} |^{-2} + | a_{2k} |^{-2}]^{-1}\right) \cdot (1 - 4\sigma^2(\operatorname{Re} (a_{2k-1}^{-1} + a_{2k}^{-1}))^2) .$$

Since $\beta < \alpha \cdot 4^{-2}$, $\beta^2 < \alpha \cdot 4^{-4}$ we obtain (b) easily.

Define $S_m$ and $S_m^{(l)}$ (see [7], [2] and [4]) by

$$S_m = \sum_{k=m+1}^{\infty} | a_k |^{-2} \tag{2.7}$$

$$S_m^{(l)} = S_m - \max_{k(1) < \cdots < k(l)} \sum_{i=1}^{l} | a_{k(i)} |^{-2} . \tag{2.8}$$

Define also $r_m$ by

$$r_m = \min_{k > m} | a_k | \tag{2.9}$$

One can easily see that $S_m^{(0)} = S_m$ and $S_m^{(1)} = S_m - r_m^{-2}$.

THEOREM 2.3. *Let $\{a_k\} \in$ class $A(2)$, then for $m \geqq K$, $| \sigma | \leqq AS_{2m}^{-1/2}$, and $b_{2m} = 0$ we have*

$$| E_{2m}(\sigma + iy) | \geqq \sqrt{2}/2 . \tag{2.10}$$

*(A being that of Lemma 2.2.)*

*Proof.* To prove (2.10) we use Lemma 2.2(a) whose conditions are satisfied since $S_{2m} > r_{2m}^{-2}$, $S_{2m}^{-1/2} < r_{2m} = \min_{k>2m} |a_k|$. We also recall that for $A_n > 0$ and $\sum_{n=m+1}^{\infty} A_n < 1/2$ we have

$$\prod_{n=m+1}^{\infty} (1 - A_n) \geqq 1 - \sum_{n=m+1}^{\infty} A_n \geqq \frac{1}{2} .$$

Remembering that for large $m$

$$2\alpha^{-1} \sum_{k=m+1}^{\infty} (\mathrm{Im}\,(a_{2k-1}^{-1} + a_{2k}^{-1}))^2/(|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) < \frac{1}{8}$$

and

$$\begin{aligned} 4\sigma^2 \sum_{k=m+1}^{\infty} (\mathrm{Re}\,(a_{2k-1}^{-1} + a_{2k}^{-1}))^2 &\leqq 8A^2 S_{2m} \sum_{k=m+1}^{\infty} (|a_{2k-1}|^{-2} + |a_{2k}|^{-2}) \\ &\leqq 8A^2 < 8^{-1} \end{aligned}$$

and using Lemma 2.2(a) we conclude the proof of (2.10) in the case where $|\sigma| \leqq S_{2m}^{-1/2}$ and $|y| \leqq BS_{2m}^{-1/2}$. Using Lemma 2.2(b), (2.10) in the case where $|\sigma| \leqq AS_{2m}^{-1/2}$, $|y| \geqq BS_{2m}^{-1/2}$ follows by an argumentation similar to that used in the first part. Then:

THEOREM 2.4. *Let* $\{a_k\} \in A(2)$, $b_{2m} = 0$, *then for* $m \geqq k$, $|\sigma| \leqq AS_{2m}^{-1/2}$ *and* $|y| \geqq BS_{2m}^{-1/2}$ *we have*

$$\begin{aligned} |E_{2m}(\sigma + iy)| &\geqq \frac{3}{4}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} y^{2n} \cdot \left(\frac{\alpha}{4}\right)^n \cdot \prod_{l=0}^{n-1} S_{2m}^{(l)}\right)^{1/2} \\ &\geqq \frac{3}{4}\left(1 + \frac{1}{n!} y^{2n} \left(\frac{\alpha}{4}\right)^n \prod_{l=0}^{n-1} S_{2m}^{(l)}\right)^{1/2} . \end{aligned} \tag{2.11}$$

*Proof.* Using (1.5) we can choose, by the method in the proof of Theorem 2.3, $m$ so that

$$\sum_{k=m+1}^{\infty} \left(1 - \frac{2}{\alpha}[(\mathrm{Im}\,(a_{2k-1}^{-1} + a_{2k}^{-1}))^2/(|a_{2k-1}|^{-2} + |a_{2k}|^{-2})]\right) \geqq \frac{9}{16} \tag{2.12}$$

(9/16 can be replaced of course by any $1 - \varepsilon$).

$$\sum_{n=1}^{\infty} \frac{1}{n!} y^{2n} \left(\frac{\alpha}{2}\right)^n \prod_{l=0}^{n-1} S_{2m}^{(l)}$$

converges for all $y$ since $S_{2m} = S_{2m}^{(0)} > S_{2m}^{(1)} > \cdots > S_{2m}^{(l)}$. By Lemma 2.2 and (2.12) we have

$$|E_{2m}(\sigma + iy)| \geqq \frac{3}{4}\left(1 + \sum_{n=1}^{\infty} y^{2n} \left(\frac{\alpha}{4}\right)^n \sum_{\substack{k(1)>2m \\ k(i)<k(i+1)}} |a_{k(1)} \cdots a_{k(n)}|^{-2}\right)^{1/2} .$$

But we have

$$I(n, m) \equiv \sum_{\substack{2m<k(1) \\ k(i)<k(i+1)}} |a_{k(1)} \cdots a_{k(n)}|^{-2} = \frac{1}{n!} \sum_{\substack{k(i)>2m \\ k(i)\neq k(j), j\neq i}} |a_{k(1)} \cdots a_{k(n)}|^{-2}$$
$$\geqq \frac{1}{n!} \sum_{\substack{k(i)>2m \\ k(i)\neq k(j), j\neq i}} \left(S_{2m} - \sum_{r=1}^{n-1} |a_{k(r)}|^{-2}\right) |a_{k(1)} \cdots a_{k(n-1)}|^{-2}$$
$$\geqq \frac{1}{n!} S_{2m}^{(n-1)} \sum_{\substack{k(i)>2m \\ k(i)\neq k(j), j\neq i}} |a_{k(1)} \cdots a_{k(n-1)}|^{-2} .$$

Since

$$\sum_{k(i)>2m} |a_{k(i)}|^{-2} = S_{2m} = S_{2m}^{(0)} ,$$

by induction $I(n, m) \geqq 1/n!. \prod_{l=0}^{n-1} S_{2m}^{(l)}$, which concludes the proof of the theorem.

THEOREM 2.5. *Let* $\{a_k\} \in A(2)$, $b_{2m} = 0$, *and* $\sigma$ *satisfies* $\operatorname{Re} a_k \neq \sigma$ *for all* $k > n$, *then for* $p, n = 0, 1, 2, \cdots$ there exist $k_1(p, \sigma, n)$ *and* $k_2(p, \sigma, n)$ *so that*

$$|E_{2n}(\sigma + i\tau)|^2 \geqq k_1(p, \sigma, n) + k_2(p, \sigma, n)\tau^{2p} . \tag{2.13}$$

*Proof.* Since $S_{2m} = o(1)m \to \infty$ we can choose $m$ so that $AS_{2m}^{-1/2} \geqq \sigma$ (for $A$ of Theorems 2.3 and 2.4). Combining Theorems 2.3, 2.4 and the fact that $|\prod_{k=2n+1}^{2m} (1 - \sigma + i\tau/a_k))e^{\sigma \operatorname{Re} a_k^{-1}}| \geqq \delta$ whenever $\operatorname{Re} a_k \neq \sigma$, we obtain (2.13).

**3. Estimates for $E_{m(i)}(s)$ in special cases when $\{a_k\} \notin A(2)$.** In this section we shall estimate $E_{2m}(s)$ in case $\{a_k\}$ does not necessarily belong to $A(2)$ but $a_{2k-1} = -a_{2k}$ or $a_{2k-1} = \bar{a}_{2k}$ and some other conditions are satisfied.

First we prove some lemmas concerning the above mentioned cases.

LEMMA 3.1. *Let $a$ be a complex number* $\operatorname{Re} a \neq 0$, *then for all real $y$ and $q \geqq 1$*

$$I(a) = \left|\left(1 - \frac{iy}{a}\right)\left(1 + \frac{iy}{a}\right)\right|^2 = \left|\left(1 - \frac{iy}{a}\right)\left(1 - \frac{iy}{\bar{a}}\right)\right|^2$$
$$\geqq \begin{cases} 1 - q\left(\dfrac{\operatorname{Re} a^2}{|a|^2}\right)^2 + \left(1 - \dfrac{1}{q}\right) y^4 |a|^{-4} \\ 1 + y^4 |a|^{-4} \qquad \operatorname{Re} a^2 \geqq 0 . \end{cases} \tag{3.1}$$

*Proof.* Simple calculation yields

$$\left|\left(1 - \frac{iy}{a}\right)\left(1 - \frac{iy}{a}\right)\right|^2 = 1 - q\left(\frac{\operatorname{Re} a^2}{|a|^2}\right)^2$$
$$+ \left(\sqrt{q}\frac{\operatorname{Re} a^2}{|a|^2} + \frac{1}{\sqrt{q}} y^2 |a|^{-2}\right)^2 + \left(1 - \frac{1}{q}\right) y^4 |a|^{-4}$$

from which (3.1) is immediate.

LEMMA 3.2. *Let a be complex number,* $\operatorname{Re} a \neq 0$, *then*

$$\left|\left(1-\frac{\sigma+iy}{a}\right)\left(1+\frac{\sigma+iy}{a}\right)\right|^2$$

$$= I(a) + 2\sigma^2(|a|^2 - 2(\operatorname{Re} a)^2|a|^{-4}) + \sigma^4|a|^{-4} + 2\sigma^2y^2|a|^{-4} + 4\sigma y(\operatorname{Im} a^2)|a|^{-4}\,, \tag{3.2}$$

$$\left|\left(1-\frac{\sigma+iy}{a}\right)\left(1-\frac{\sigma+iy}{\bar a}\right)\right|^2$$

$$= I(a) - 4\sigma \operatorname{Re} a\,|a|^{-2} + \sigma^2(2|a|^{-2} + 4(\operatorname{Re} a)^2|a|^{-4}) + \sigma^4|a|^{-4} + 2\sigma^2y^2|a|^{-4} - 4(\sigma^2+y^2)\sigma|a|^{-4}\operatorname{Re} a\,, \tag{3.3}$$

*where* $I(a)$ *is defined in Lemma* 3.1.

*Proof.* The proof is a corollary of the proof of Lemma 2.2 combined with Lemma 3.1.

LEMMA 3.3. *Let* $\operatorname{Re} a \neq 0$, *then for* $K > 1$ *there exists* $A$ *and* $B$, *independent of* $a$, $0 < A < B < 1$ *such that for* $r < |\operatorname{Re} a|$ *we have*:

(a) *For* $|\sigma| \leqq Ar$ *and* $|y| \leqq Br$

$$\left|\left(1-\frac{\sigma+iy}{a}\right)\left(1+\frac{\sigma+iy}{a}\right)\right|^2 \geqq 1 - K^{-1}r^2|a|^{-2} - (\min(0, (\operatorname{Re} a^2)\cdot|a|^{-2}))^2\,. \tag{3.4}$$

(b) *For* $|\sigma| \leqq Ar_1 \leqq Ar$, $|y| \geqq Br$ *and* $\delta > 0$

$$\left|\left(1-\frac{\sigma+iy}{a}\right)\left(1+\frac{\sigma+iy}{a}\right)\right|^2 \geqq \left(1+\frac{1}{4}y^4|a|^{-4}\right)(1 - 2(\min(0, \operatorname{Re} a^2/|a|^2))^2 - K^{-1}(r^2|a|^{-2} + r_1^2|a|^{-1-\delta} + |a|^{-2+2\delta}))\,. \tag{3.5}$$

*Proof.* To prove (3.4) we use (3.2) and (3.1) with $q = 1$ and obtain the result by choosing $B$ so that $6B^2 < K^{-1}$, and dropping some positive terms.

To prove (3.5) we use

$$4\sigma y(\operatorname{Im} a^2)|a|^{-4} \geqq -4|\sigma y|\,|a|^{-2} \geqq -\left(\frac{1}{\beta^2}y^2|a|^{-(3-\delta)} + 2\beta^2\sigma^2|a|^{-(1+\delta)}\right)$$

and

$$-\frac{1}{\beta^2}y^2 |a|^{-(3-\delta)} + \frac{1}{4}y^4 |a|^{-4} \geqq -\frac{4}{\beta^4}|a|^{-(2-2\delta)} .$$

Choosing $4/\beta^4 \leqq 1/K$ or $\beta \geqq \sqrt[4]{4K}$ and $A$ so that $2\beta^2 A^2 < K^{-1}$ or $A^2 < 1/4K\sqrt{K}$ or $A < 1/2K$ one can conclude the proof by using Lemma 3.1 (choosing there $q = 2$ in case $\operatorname{Re} a^2 < 0$) and dropping some positive terms.

LEMMA 3.4. *Let* $\operatorname{Re} a \neq 0$, *then for* $K > 1$ *there exist* $A$ *independent of* $a$, $0 < A < 1$, *such that for* $r < |\operatorname{Re} a|$ *and* $|\sigma| \leqq Ar$ *we have*

$$\begin{aligned}(3.6)\quad &\left|\left(1 - \frac{\sigma + iy}{a}\right)\left(1 - \frac{\sigma + iy}{\bar{a}}\right)\right|^2 \exp(4\sigma \operatorname{Re} a/|a|^2) \\ &\geqq \left(1 + \frac{1}{4}y^4 |a|^{-4}\right)(1 - 2(\min(0, \operatorname{Re} a^2/|a|^2))^2 - K^{-1}r^2|a|^{-2}) .\end{aligned}$$

*Proof.* Using (3.3) of Lemma 3.2, Lemma 3.1 with $q = 3/2$, the estimations

$$-4\sigma^3 |a|^{-4} \operatorname{Re} a \geqq -\sigma^4 |a|^{-4} - 4\sigma^2(\operatorname{Re} a)^2 |a|^{-4} ,$$
$$-4y^2\sigma |a|^{-4} \operatorname{Re} a \geqq -\frac{1}{2^5}y^4 |a|^{-4} - 4^3\sigma^2 |a|^{-2}$$

and dropping some positive terms we obtain

$$\begin{aligned}\left|\left(1 - \frac{\sigma + iy}{a}\right)\left(1 - \frac{\sigma + iy}{\bar{a}}\right)\right|^2 &\geqq 1 + \left(\frac{1}{3} - \frac{1}{2^4}\right)y^4 |a|^{-4} \\ &- \frac{3}{2}(\min(0, \operatorname{Re} a^2/|a|^2))^2 - 4^3A^2r^2|a|^{-2} - 4\sigma(\operatorname{Re} a)|a|^{-2} .\end{aligned}$$

Choosing $A$ so that $4^3A^2 < 1/4K$, which implies

$$-4^3A^2r^2|a|^{-2} > -\frac{1}{4K}r^2|a|^{-2} , \qquad 4|\sigma||a|^{-1} < \frac{1}{4} ,$$

and

$$\exp(4\sigma(\operatorname{Re} a)|a|^{-2}) \geqq 1 + 4\sigma(\operatorname{Re} a)|a|^{-2} - 4^2\sigma^2|a|^{-2}$$

from which (3.6) follows.

We shall define now two classes of convolution transforms by the function $E(s)$ and the sequence $\{a_k\}$.

DEFINITION 3.1. $\{a_k\} \in$ class $B(2, \delta)$ if

$$E(s) = \prod_{k=1}^{\infty} (1 - s^2 a_k^{-2}) , \tag{3.7}$$

$$\sum_{\mathrm{Re}\, a_k^2 < 0} | a_k |^{-4} (\mathrm{Re}\, a_k^2)^2 < \infty \tag{3.8}$$

and

$$\sum_{k=1}^{\infty} | a_k |^{-1-\delta} < \infty , \qquad \sum_{k=1}^{\infty} | a_k |^{-2+\delta} < \infty \qquad \text{for some } \delta > 0 . \tag{3.9}$$

DEFINITION 3.2. $\{a_k\} \in B(2)$ if there is $\delta > 0$ so that $\{a_k\} \in B(2, \delta)$.

DEFINITION 3.3. $\{a_k\} \in$ class $C(2)$ if

$$E(s) = \prod_{k=1}^{\infty} (1 - s a_k^{-1})(1 - s \cdot \bar{a}_k^{-1}) , \tag{3.10}$$

if condition (3.8) is satisfied and $\sum | a_k |^{-2} < \infty$.

REMARK. $S_{2m} = 2 \sum_{k=m+1}^{\infty} | a_k |^{-2}$ in case of class $B(2)$ and $C(2)$. We have to introduce some more notations before being able to prove the estimation on $E(s)$ for transforms of classes $B(2)$ and $C(2)$.

$$Q_m = \sum_{k=m+1}^{\infty} | a_k |^{-4} . \tag{3.11}$$

$$Q_m^{(j)} = Q_m - \max_{m < k(1) < \cdots < k(j)} \left\{ \sum_{r=1}^{j} | a_{k(r)} |^{-4} \right\} . \tag{3.12}$$

We shall state the estimations for classes $B(2)$ and $C(2)$ together and then outline the proofs.

THEOREM 3.5. *If $\{a_k\} \in B(2, \delta)$, then for $m \geqq M$ and some $A$ and $B$ we have:*

(a) $| \sigma | \leqq A S_{2m}^{-1/2}$, $| y | \leqq B S_{2m}^{-1/2}$ *imply*

$$| E_{2m}(s) | \geqq 3/4 . \tag{3.13}$$

(b) $| \sigma | \leqq A (\sum_{k=m+1}^{\infty} | a_k |^{-1-\delta})^{-1/1+\delta}$ *and* $| y | \geqq B S_{2m}^{-1/2}$ *imply*

$$| E_{2m}(s) | \geqq \frac{3}{4} \left( 1 + \sum_{n=1}^{\infty} \frac{1}{n!} y^{4n} \prod_{l=0}^{n-1} Q_m^{(l)} \right)^{1/2} . \tag{3.14}$$

THEOREM 3.6. *If $\{a_k\} \in C(2)$ then for $m \geqq M$ there exists an $A$ so that for $| \sigma | \leqq A S_{2m}^{-1/2}$ (3.14) is valid.*

*Proof of Theorems* 3.5 *and* 3.6. The proof follows the proof of Theorems 2.3 and 2.4 Using Lemmas 3.3 and 3.4 we have to choose

$r = S_{2m}^{-1/2}$ and $r_1 = (2\sum_{k=m+1}^{\infty} | a_k |^{-1-\delta})^{-1/(1+\delta)}$ ($r_1$ necessary only in proving Theorem 3.5 from Lemma 3.3). Obviously $r_1 < \min_{k>m} | a_k |$, $r \leqq \min_{k>m} | a_k |$. Also we have

$$\begin{aligned}\sum_{k=m+1}^{\infty} | a_k |^{-1-\delta} &\geqq \Big(\sum_{k=m+1}^{\infty} | a_k |^{-2}\Big)\Big(\min_{k>m} | a_k |\Big)^{1-\delta} \\ &\geqq \Big(\sum_{k=m+1}^{\infty} | a_k |^{-2}\Big)^{(1+\delta)/2} \cdot \Big(\sum_{k=m+1}^{\infty} | a_k |^{-2}\Big)^{(1-\delta)1/2} \Big(\min_{k>m} | a_k |\Big)^{1-\delta} \\ &\geqq \Big(\sum_{k=m+1}^{\infty} | a_k |^{-2}\Big)^{(1+\delta)/2} .\end{aligned}$$

This implies

$$r_1 = \Big(2\sum_{k=m+1}^{\infty} | a_k |^{-1-\delta}\Big)^{-1/(1+\delta)} \leqq (S_{2m})^{-1/2} \leqq r .$$

Choose $m$ and $K$ so that $\sum_{k>m} (\min(0, \operatorname{Re} a_k^2/| a_k |^2))^2 < \varepsilon_1$, $1/K < \varepsilon_1$ ($K$ of Lemmas 3.3 and 3.4) and, for proving Theorem 3.5, $\sum_{k=m+1}^{\infty} | a_k |^{-2+2\delta} < \varepsilon_1$. The choice $\varepsilon_1 \leqq 1/16$ will yield the number 3/4 in (3.13) (every $1-\eta$ could be achieved by $\varepsilon_1$ small enough) and the coefficient 3/4 in (3.14).

To complete the proof we have to show

$$\prod_{k=m+1}^{\infty}\Big(1 + \frac{1}{4}y^4 | a_k |^{-4}\Big) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} y^{4n} \prod_{l=0}^{n-1} Q_m^{(l)} ,$$

the proof of which follows stepwise that of Theorem 2.4.

The classes in this section are not included in $A(2)$ since (1.6) may fail to be valid. The estimates in this section are weaker in the case where the transforms are also $A(2)$.

THEOREM 3.7. *Let $\{a_k\} \in B(2)$ or $C(2)$. Then for $\sigma$ satisfying $\operatorname{Re} a_k \neq 0$ for all $k > n$, and for $p, n = 0, 1, 2 \cdots$ there exist $k_1(p, \sigma, n)$ and $k_2(p, \sigma, n)$ such that when $\sigma \neq \operatorname{Re} a_k$*

$$| E_{2n}(\sigma + i\tau) |^2 \geqq k_1(p, \sigma, n) + k_2(p, \sigma, n)\tau^{2p} . \tag{3.15}$$

*Proof.* Deduced from Theorems 3.5 and 3.6 as Theorem 2.5 is deduced from Theorem 2.4 and 2.3.

4. **Estimates for $G_m(t)$.** We define $G_m(t)$, in the usual manner, by

$$G_m(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} [E_m(s)]^{-1} e^{st} ds , \qquad G_0(t) = G(t) . \tag{4.1}$$

We define also:

(4.2) $$\alpha(m) = \max\{\operatorname{Re} a_k, -\infty \mid \operatorname{Re} a_k < 0 \text{ and } k > m\} .$$

(4.3) $$\beta(m) = \min\{\operatorname{Re} a_k, \infty \mid \operatorname{Re} a_k > 0 \text{ and } k > m\} .$$

We recall that in the cases $\{a_k\} \in A(2)$, $\{a_k\} \in B(2)$ and $\{a_k\} \in C(2)$ we have

(4.4) $$| E_{2n}(\sigma + i\tau) |^2 \geqq k_1(p, \sigma, n) + k_2(p, \sigma, n)\tau^{2p} ,$$

for $n, p = 0, 1, 2 \cdots$ and $\alpha(2n) < \sigma < \beta(2n)$.

THEOREM 4.1. *Let* $E_n(s)$, $P_n(D)$ *and* $G_n(t)$ *be defined by* (2.1), (1.3) *and* (4.1); *let* (4.4) *be satisfied for* $m(l)$, *a subsequence of* $m$, *then:*

A. *For any* $\sigma$ *satisfying* $\sigma(m(l)) < \sigma < \beta(m(l))$ *we have*

(4.5) $$G_{m(l)}(t) = P_{m(l)}(D)G(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}[E_{m(l)}(s)]^{-1}e^{st}ds .$$

B. *Suppose in case* $\alpha(m(l)) \neq -\infty$ *that* $a_{k(1,1)} = \cdots = a_{k(1,m_1+1)}$, $a_{k(2,1)} = \cdots = a_{k(2,m_2+1)}, \cdots, a_{k(r,1)} = \cdots = a_{k(r,m_2+1)}$ *are all with indices greater than* $m(l)$ *and* $\alpha(m(l)) = \operatorname{Re} a_{k(1,1)} = \operatorname{Re} a_{k(2,1)} = \cdots = \operatorname{Re} a_{k(r,1)}$, *then*

(4.6) $$\frac{d^v}{dt^v}G_{m(l)}(t) = \sum_{i=1}^{r}\frac{d^v}{dt^v}\{P_i(t)e^{ta_{k(i,1)}}\} + 0(e^{k(t)}) \qquad t \longrightarrow \infty$$

*where* $p_i(t)$ *are polynomials of order* $m_i$ *and* $k$ *is any real number satisfying*

$$\max\{\operatorname{Re} a_k, -\infty \mid k > m(l), \operatorname{Re} a_k < \alpha(m(l))\} < k < \alpha(m(l)) .$$

C. *Suppose* $\alpha(m(l)) = -\infty$, *then*

(4.7) $$\frac{d^v}{dt^v}G_{m(l)}(t) = 0(e^{kt}) \qquad t \longrightarrow \infty \text{ for any real } k, k < 0 .$$

D. *Suppose in case* $\beta(m(l)) \neq \infty$ *that* $a_{r(1,1)} = \cdots = a_{r(1,m_1+1)}$, $\cdots a_{r(j,1)} = \cdots = a_{r(j,m_j+1)}$ *are all with indices greater than* $m(l)$ *and* $\beta(m(l)) = \operatorname{Re} a_{r(1,1)} = \cdots \operatorname{Re} a_{r(j,1)}$, *then*

(4.8) $$\frac{d^v}{dt^v}G_{m(l)}(t) = \sum_{i=1}^{j}\frac{d^v}{dt^v}\{q_i(t)e^{ta_{r(i,1)}}\} + 0(e^{kt}) \qquad t \longrightarrow -\infty$$

*where* $q_i(t)$ *are polynomials of order* $m_i$ *and* $k$ *is a real number satisfying* $\beta(m(l)) < k < \min\{\operatorname{Re} a_k, \infty \mid k > m(l), \operatorname{Re} a_k > \beta(m(l))\}$.

E. *Suppose* $\beta(m(l)) = \infty$, *then*

(4.9) $$\frac{d^v}{dt^v}G_{m(l)}(t) = 0(e^{kt}) \qquad t \longrightarrow -\infty$$

*where $k$ is any real positive number.*

F. *For $\alpha(m(l)) < \operatorname{Re} s < \beta(m(l))$ we have*

$$\frac{1}{E_{m(l)}(s)} = \int_{-\infty}^{\infty} e^{st} G_{m(l)}(t)dt \tag{4.10}$$

*which implies*

$$1 = \int_{-\infty}^{\infty} G_{m(l)}(t)dt \,. \tag{4.11}$$

*Proof.* The proof follows the method used in Hirschman and Widder's book "The convolution transform" [6, p. 108]. Formula (4.4), that was proved for class $A(2)$, $B(2)$ and $C(2)$, is used here instead of the theorems on $E_m(s)$ in [6].

The following result will estimate $G_{2m}(t)$ in the case when $m$ is large near the point $t = 0$ as well as when $|t| \to \infty$.

THEOREM 4.2. *Let $\{a_k\} \in A(2)$ and suppose that for some $n$ $S_{2m}^{(n+1)} \geqq L_n S_{2m}$ where $L_n > 0$ is independent of $m$, then there exist $M(n) > 0$ and $A > 0$ such that*

$$| G_{2m}^{(n)}(t) | \leqq M(n) S_{2m}^{-(n+1)/2} \exp(-A \cdot S_{2m}^{-1/2} | t |) \,. \tag{4.12}$$

*Proof.* By Theorem 4.1.A we have

$$G_{2m}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-(\sigma+iy)t}}{E_{2m}(\sigma + iy)} dy$$

and therefore

$$G_{2m}^{(n)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(\sigma + iy)^n e^{-(\sigma+iy)t}}{E_{2m}(\sigma + iy)} dy \,.$$

Remembering that $S_{2m}^{(n+1)} \geqq L_n S_{2m}$ implies $S_{2m}^{(k)} \geqq L_n S_{2m}$ for $k \leqq n + 1$, and using Theorems 2.3 and 2.4 we obtain, choosing $\sigma = AS_{2m}^{-1/2}$ for the case $t > 0$,

$$\begin{aligned} | G_{2m}^{(n)}(t) | &\leqq \frac{1}{2\pi} \exp(-AS_{2m}^{-1/2} t) \left\{ \int_{-BS_{2m}^{-1/2}}^{BS_{2m}^{-1/2}} \frac{(|\sigma| + |y|)^n}{| E_{2m}(\sigma + iy) |} dy \right. \\ &\quad \left. + \int_{|y| \geqq BS_{2m}^{-1/2}} \frac{(|\sigma| + |y|)^n dy}{| E_{2m}(\sigma + iy) |} \right\} \leqq \exp(-AS_{2m}^{-1/2} t) \left\{ \frac{\sqrt{2}}{2\pi} (A + B)^n 2BS_{2m}^{-(n+1)/2} \right. \\ &\quad \left. + 2 \sum_{k=0}^{n} \binom{n}{k} S_{2m}^{-k/2} \frac{2}{3\pi} \int_{BS_{2m}^{-1/2}}^{\infty} \frac{y^{n-k} dy}{\left( 1 + y^{2(n+2)} \frac{1}{(n+2)!} (L_{(n)})^{n+1} S_{2m}^{n+2} \left( \frac{\alpha}{4} \right)^{n+2} \right)^{1/2}} \right\} \\ &\leqq M(n) S_{2m}^{-(n+1)/2} \exp(-AS_{2m}^{-1/2} t) \,. \end{aligned}$$

The result for $t < 0$ is achieved choosing $\sigma = -AS_{2m}^{-1/2}$.

REMARK. When $a_{2k-1} = -a_{2k}$ we have $S_{2m}^{(1)} \geqq (1/2)S_{2m}$ and therefore Theorem 4.2 for $n = 0$ includes Lemma 2.4 of [1, p. 432]. Whenever the connection between pair is $0 < \theta_1 \leqq |a_{2k-1}/a_{2k}| \leqq \theta_2 < \infty$, where $\theta_1, \theta_2$ are fixed for all $m$, we have $S_{2m}^{(1)} \geqq L_1 S_{2m} L_1 > 0$. But in case of $n = 0$ the restriction $S_{2m}^{(1)} \geqq L_1 S_{2m}$ is not necessary as is proved by the following.

THEOREM 4.3. *Let $\{a_k\} \in A(2)$, then for some $A > 0$ we have*

$$|G_{2m}(t)| \leqq MS_{2m}^{-1/2} \exp(-AS_{2m}^{-1/2}|t|) . \tag{4.12}$$

*Proof.* Following the proof of Theorem 4.2 and using Theorem 2.4 we have for $t > 0$ ($t < 0$ can be treated similarly)

$$|G_{2m}(t)| \leqq \exp(-AS_{2m}^{-1/2}t)\left\{\frac{\sqrt{2}}{\pi}BS_{2m}^{-1/2} + \frac{4}{3\pi}\int_{BS_{2m}^{-1/2}}^{\infty}\frac{dy}{\left(1 + \frac{1}{2}y^4 S_{2m}S_{2m}^{(1)}\left(\frac{\alpha}{4}\right)^2\right)^{1/2}}\right\} .$$

$$\int_{BS_{2m}^{-1/2}}^{\infty}\frac{dy}{(1 + Ly^4 S_{2m}S_{2m}^{(1)})^{1/2}} = S_{2m}^{-1/2}(S_{2m}/S_{2m}^{(1)})^{1/4}\int_{(B/L^{1/4})(S_{2m}/S_{2m}^{(1)})^{1/4}}^{\infty}\frac{1}{(1 + y^4)^{1/2}}dy$$
$$\leqq 2^d S_{2m}^{-1/2}(S_{2m}/S_{2m}^{(1)})^{1/4}\int_{B_1(S_{2m}/S_{2m}^{(1)})^{1/4}}^{\infty}\frac{dy}{1 + y^2}$$
$$\leqq 2S_{2m}^{-1/2}(S_{2m}/S_{2m}^{(1)})^{1/4}\lim_{\zeta\to\infty}(\operatorname{arc} tg\zeta - \operatorname{arc} tgB_1(S_{2m}/S_{2m}^{(1)})^{1/4})$$
$$\leqq 2S_{2m}^{-1/2}(S_{2m}/S_{2m}^{(1)})^{1/4}\lim_{\zeta\to\infty}\operatorname{arc} tg\left\{\frac{1 - B_1(S_{2m}/S_{2m}^{(1)})^{1/4}\zeta}{\zeta^{-1} + B_1(S_{2m}/S_{2m}^{(1)})^{1/4}}\right\} \leqq B_2 S_{2m}^{-1/2} .$$

From this the proof can be easily concluded.

Lemma 3.2A of [1, p. 434] is generalized by Theorem 4.2 in case $S_{2m}^{(2)} \geqq L_2 S_{2m}$ for some $L_2$. Case $B$ is covered only in part. The following theorem generalizes Lemma 3.2B [1, p. 434].

THEOREM 4.4. *Let $a_k \in A(2)$, $b_{2m} = 0$ and suppose $0 < \theta_1 < |a_{2k}/a_{2k-1}| < \theta_2 < \infty$ where $\theta_1, \theta_2$ are independent of $k$ and $|\operatorname{Re} a_k|/|a_k| > \eta$, then for some $A_1 > 0$ and $M_1$ we have:*

$$|G'_{2m}(t)| \leqq M_1 S_{2m}^{-1} \exp(-A_1 S_{2m}^{-1/2}|t|) . \tag{4.13}$$

*Proof.* Let us split the proof into two cases

(a) $$S_{2m} - \max_{k>m} (| a_{k-1} |^{-2} + | a_{2k} |^{-2}) \geqq \frac{1}{K} S_{2m}$$

and

(b) $$S_{2m} - \max_{k>m} (| a_{2k-1} |^{-2} + | a_{2k} |^{-2}) < \frac{1}{K} S_{2m} .$$

In case (a) (4.13) was proved by Theorem 4.2 for any arbitrary $K$. We shall choose $K > 2$. To prove (4.13) in case (b) we define $k_0$ by

$$\max_{k>m} (| a_{2k-1} |^{-2} + | a_{2k} |^{-2}) = | a_{2k_0-1} |^{-2} + | a_{2k_0} |^{-2} .$$

(In case (b) the choice of $k_0$ is unique.) Define $g^*_{k_0}(t)$ and $G_{2m+2}(t)$ by:

(4.14) $$g^*_{k_0}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [(1 - iy/a_{2k_0-1})(1 - iy/a_{2k_0})]^{-1} e^{-iyt} dy .$$

(4.15) $$G^*_{2m+2}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [(1 - iy/a_{2k_0-1})(1 - iy/a_{2k_0})](E_{2m}(iy))^{-1} e^{-iyt} dy .$$

By [9, p. 255] we have

$$G_{2m}(t) = g^*_{k_0}(t) * G^*_{2m+2}(t) .$$

One can calculate $g^*_{k_0}(t)$:

$$g^*_{k_0}(t) = \frac{a_{2k_0-1} a_{2k_0}}{a_{2k_0-1} - a_{2k_0}} \begin{cases} e^{a_{2k_0} t} & t \geqq 0 \\ e^{a_{2k_0-1} t} & t < 0 \end{cases}$$

when $\operatorname{Re} a_{2k_0-1} > 0$, $\operatorname{Re} a_{2k_0} < 0$.

$$g^*_{k_0}(t) = \begin{cases} \dfrac{a_{2k_0-1} a_{2k_0}}{a_{2k_0} - a_{2k_0-1}} [e^{a_{2k_0-1} t} - e^{a_{2k_0} t}] & t < 0 \\ 0 & t > 0 \end{cases}$$

when $\operatorname{Re} a_{2k_0-1} > 0$, $\operatorname{Re} a_{2k_0} > 0$, $a_{2k_0} \neq a_{2k_0-1}$.

$$g^*_{k_0}(t) = \begin{cases} -a^2_{2k_0} t e^{a_{2k} t} & t < 0 \\ 0 & t > 0 \end{cases}$$

when $a_{2k_0} = a_{2k_0-1}$, $\operatorname{Re} a_{2k_0} > 0$.

Either $g^*_{k_0}(t)$ or $g^*_{k_0}(-t)$ is of the above form.

$$G^*_{2m+2}(t - \operatorname{Re} (a^{-1}_{2k_0+1} + a^{-1}_{2k_0}))$$

satisfies the assumptions of Theorem 4.3 with $S^*_{2m+2} = S_{2m} - | a_{2k_0-1} |^2 - | a_{2k_0} |^2$ and therefore

$$|G^*_{2m+2}(t)| \leqq M(S^*_{2m+2})^{1/2}\exp(-AS^{*-1/2}_{2m+2}|t + \mathrm{Re}(a^{-1}_{2k_0+1} + a^{-1}_{2k_0})|) .$$

Integrating by parts

$$\begin{aligned} G'_{2m}(t) &= \int_{-\infty}^{\infty} g^*_{k_0}(u)\frac{d}{dt}G^*_{2m+2}(t-u)du \\ &= \left\{\int_{-\infty}^{0} + \int_{0}^{\infty}\right\}\left(\frac{d}{du}g^*_{k_0}(u)\right)G^*_{2m+2}(t-u)du . \end{aligned}$$

Since

$$\theta_1^2|a_{2k_0-1}|^2 \leqq |a_{2k_0}|^2 \quad \text{and} \quad |a_{2k_0}|^2 \leqq \theta_2^2|a_{2k_0-1}|^2$$

we have

$$(\theta_1^{-2}+1)|a_{2k_0}|^{-2} \geqq \frac{1}{2}S_{2m} \quad \text{and} \quad (\theta_2^2+1)|a_{2k_0-1}|^{-2} \geqq \frac{1}{2}S_{2m} ;$$

therefore

$$\max(|a_{2k_0}|, |a_{2k_0-1}|) \leqq [(2\theta_1^{-2}+2)^{1/2} + (2\theta_2^2+2)^{1/2}]S_{2m}^{-1/2} = R_1S_{2m}^{-1/2} .$$

By the same method $(\theta_2^{-2}+1)|a_{2k_0}|^{-2} \leqq S_{2m}$ and $(\theta_1^2+1)|a_{2k_0-1}|^2 \leqq S_{2m}$, from which we deduce

$$\begin{aligned} |\mathrm{Re}\, a_{2k_0}| &\geqq \eta|a_{2k_0}| \geqq \eta(\theta_2^{-2}+1)^{-1/2}S_{2m}^{-1/2} , \\ |\mathrm{Re}\, a_{2k_0-1}| &\geqq \eta(\theta_1^2+1)^{-1/2}S_{2m}^{-1/2} \end{aligned}$$

and

$$\min(|\mathrm{Re}\, a_{2k_0}|, |\mathrm{Re}\, a_{2k_0-1}|) \geqq R_2S_{2m}^{-1/2} > 0$$

where

$$R_2 = \eta\cdot\min((\theta_2^{-2}+1)^{-1/2}, (\theta_1^2+1)^{-1/2}) .$$

One has to estimate $G_{2m}(t)$ for different cases of $g^*_{k_0}(t)$ of which the case where $\mathrm{Re}\, a_{2k_0} > 0$, $\mathrm{Re}\, a_{2k_0-1} > 0$ and $a_{2k_0} \neq a_{2j-1}$ will be done here. The other cases are similar and simpler.

$$\frac{dg^*(u)}{du} = \begin{cases} a_{2k_0-1}a_{2k_0}\dfrac{(a_{2k_0-1}\exp(a_{2k_0-1}u) - a_{2k_0}\exp(a_{2k_0}u))}{a_{2k_0} - a_{2k_0-1}} & u < 0 \\ 0 & u > 0 . \end{cases}$$

Let us recall from [8, p. 203] that if $f'(t)$ is continuous and $f(t)$ is complex valued, then

$$\frac{f(a) - f(b)}{a - b} = \lambda f'(t_1) + (1-\lambda)f'(t_2) \qquad t_1, t_2 \in (a, b) \ 0 < \lambda < 1$$

from which it is obvious that

$$\frac{f(\zeta_1)-f(\zeta_2)}{\zeta_1-\zeta_2}=\lambda f'(\zeta_3)+(1-\lambda)f'(\zeta_4)\qquad 0<\lambda<1$$

where $\zeta_i=\alpha_i\zeta_1+(1-\alpha_i)\zeta_2$, $0\leqq\alpha_i\leqq 1$ and $i=3,4$. Substituting $f(\zeta)=\zeta e^{\zeta}$, $f'(\zeta)=e^{\zeta u}+\zeta u e^{\zeta u}$, we obtain the following estimate for $(d/du)g^*(u)$ when $u<0$:

$$\left|\frac{dg^*(u)}{du}\right|\leqq|a_{2k_0-1}a_{2k_0}|(\exp(R_{k_0}u)+\max(|a_{2k_0-1}|,|a_{2k_0}|)|u|\exp(R_{k_0}u))$$

where $R_{k_0}=\min(\operatorname{Re}a_{2k_0-1},\operatorname{Re}a_{2k_0})$. Therefore we obtain

$$\begin{aligned}|G'_{2m}(t)|=|a_{2k_0-1}a_{2k_0}|\Big|\int_{-\infty}^{0}&\exp(R_{k_0}u)\{1+|u|\max(|a_{2k_0-1}|,|a_{2k_0}|)\}\\&\cdot S^{*-1/2}_{2m+2}\exp(-AS^{*-1/2}_{2m+2}|t-u+\operatorname{Re}(a^{-1}_{2k_0-1}+a^{-1}_{2k_0})|)du\,.\end{aligned}$$

Using relations among $S^*_{2m+2}$, $S_{2m}$, $a_{2k_0}$ and $a_{2k_0-1}$ one obtains

$$\exp(-AS^{*-1/2}_{2m+2}|t-u+\operatorname{Re}(a^{-1}_{2k_0-1}+a^{-1}_{2k_0})|)\leqq M_2\exp(-AS^{-1/2}_{2m+2}|t-u|)\,.$$

Using this and the definition of $R_1$ and $R_2$ one derives

$$\begin{aligned}|G'_{2m}(t)|\leqq M_2R_1^2S^{-1}_{2m}\int_{-\infty}^{0}&\exp(R_2S^{-1/2}_{2m}u)\{1+|u|R_1S^{-1/2}_{2m}\}S^{*-1/2}_{2m+2}\\&\cdot\exp(-AS^{*-1/2}_{2m+2}|t-u|)du\,.\end{aligned}$$

We have to distinguish two cases $t<0$ and $t\geqq 0$. Let us prove first the theorem in case $t<0$:

$$\begin{aligned}|G'_{2m}(t)|\leqq{}& M_2R_1^2S^{-1}_{2m}\exp(-AtS^{*-1/2}_{2m+2})\int_{-\infty}^{t}\{1-uR_1S^{-1/2}_{2m}\}S^{*-1/2}_{2m+2}\\&\cdot\exp\{(R_2S^{-1/2}_{2m}+AS^{*-1/2}_{2m+2})u\}du\\&+M_2R_1^2S^{-1}_{2m}\exp(AtS^{*-1/2}_{2m+2})\int_{t}^{0}\{1-uR_1S^{-1/2}_{2m}\}S^{*-1/2}_{2m+2}\\&\cdot\exp\{(R_2S^{-1/2}_{2m}-AS^{*-1/2}_{2m+2})u\}du\,.\end{aligned}$$

Choosing $K$ so that $AS^{*-1/2}_{2m+2}>2R_2S^{-1/2}_{2m}$ we have

$$|G'_{2m}(t)|\leqq M_1S^{-1}_{2m}\exp(R_2tS^{-1/2}_{2m})\,.$$

For $t>0$

$$\begin{aligned}|G'_{2m}(t)|\leqq{}& M_2R_1^2S^{-1}_{2m}\exp(-AtS^{*-1/2}_{2m+2})\cdot\int_{-\infty}^{0}\{1-uR_1S^{-1/2}_{2m}\}\cdot S^{*-1/2}_{2m+2}\\&\cdot\exp\{(R_2S^{-1/2}_{2m}+AS^{*-1/2}_{2m+2})u\}du\leqq M_1S^{-1}_{2m}\exp(-AtS^{*-1/2}_{2m+2})\\&\leqq M_1S^{-1}_{2m}\exp(-R_2S^{-1/2}_{2m}t)\,.\end{aligned}$$

Estimations similar to those achieved in Theorem 4.2 for $\{a_k\}\in B(2,\delta)$ and $\{a_k\}\in C(2)$ are developed in the following theorems.

THEOREM 4.6. *Let* $\{a_k\} \in B(2, \delta)$ *and* $Q_m^{(j)} \geqq L(j)Q_m$ *for some* $j$, *then there exist* $A > 0$ *and* $M > 0$ *(independent of* $m$*) so that for* $k \leqq 2j$:

$$(4.14) \qquad |G_{2m}^{(k)}(t)| \leqq MQ_m^{-k/4} \exp\left(-A\left(2\sum_{k=m+1}^{\infty}|a_k|^{-1-\delta}\right)^{-1/(1+\delta)}|t|\right).$$

THEOREM 4.7. *Let* $\{a_k\} \in C(2)$ *and* $Q_m^{(j)} \geqq L(j)Q_m$ *for some* $j$, *then there exist* $A > 0$ *and* $M > 0$ *(independent of* $m$*) so that for* $k \leqq 2j$:

$$(4.15) \qquad G_{2m}^{(k)}(t) \leqq MQ_m^{-k/4} \exp(-AS_{2m}^{-1/2}|t|).$$

One can note that in case $k = 0$ no condition of the form $Q_m^{(j)} \geqq L(j)Q_m$ is needed.

*Proof of Theorems* 4.6 *and* 4.7. Using Theorems 3.5 and 3.6 (for Theorems 4.6 and 4.7 respectively) we obtain by Theorem 4.1

$$|G_{2m}^{(k)}(t)| \leqq \left| e^{-\sigma t}\int_{\sigma+i\infty}^{\sigma-i\infty}\frac{(\sigma+iy)^k e^{-iyt}}{E_{2m}(\sigma+iy)}dy\right|, \qquad \beta(2m) < \sigma < \alpha(2m).$$

Using the fact that $Q_m^{-1/4} < ((1/2)S_{2m})^{-1/2}$, as $S_m^2 > Q_m$ (which is achieved by dropping many positive terms) and recalling that

$$S_{2m}^{-1/2} < \left(2\sum_{k=m+1}^{\infty}|a_k|^{-1-\delta}\right)^{1/1+\delta},$$

we obtain

$$Q_m^{-1/4} < \left(\sum_{k=m+1}^{\infty}|a_k|^{-1-\delta}\right)^{1/1+\delta}.$$

The completion of the proof is similar to the proof of Theorem 4.2.

## 5. Some inversion theorems.

In this section we shall show that inversion formulae can be given for $\{a_k\} \in A(2)$, $\{a_k\} \in B(2, \delta)$ and $\{a_k\} \in C(2)$.

THEOREM 5.1. *Suppose*: (1) $G(t)$ *and* $E(s)$ *are defined by* (1.2) *and* $\{a_k\} \in A(2)$.

(2) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt.$

(3) *For some* $M$ *and* $K$, $|\varphi(t)| \leqq Ke^{M|t|}$, *where* $M < \min|\operatorname{Re} a_n|$.

(4) $b_{2m} = o(1) \quad m \uparrow \infty$.

*Then*

$$(5.1) \qquad \lim_{m\to\infty} P_{2m}(D)f(x) = \lim_{m\to\infty}\exp((b-b_{2m})D)\prod_{k=1}^{m}\left(1-\frac{D}{a_{2k-1}}\right)\left(1-\frac{D}{a_{2k}}\right).$$

$\exp((\mathrm{Re}\, a_{2k-1}^{-1} + a_{2k}^{-1})D)f(x) = \varphi(x)$ *at any point of continuity of* $\varphi(t)$.

*Proof.* By steps following those of [1; p. 433]

$$|P_{2m}(D)f(x) - \varphi(x)| \leqq \sup_{|t|<\delta} |\varphi(x-t) - \varphi(x)| \int_{-\infty}^{\infty} |G_{2m}(t)|\, dt + M_0 \int_{|t|>\delta} |G_{2m}(t)|\, e^{M|t|} dt\,.$$

Using Theorem 4.3, the conditions of which are satisfied by the kernel $G_{2m}(t + b_m)$, choosing $m$ so big that $|b_m| < \delta/2$ and $AS_{2m}^{-1/2} > 4M$, we conclude the proof of the theorem.

THEOREM 5.2. *Suppose: Assumptions* (1) *and* (2) *of Theorem* 5.1 *are satisfied*

(3) *For* $\alpha(t) = \int_0^t \varphi(u)du$ *there exist positive* $M$ *and* $K$ *so that* $|\alpha(t)| \leqq ke^{M|t|}$ *where* $M < \mathrm{Min}\, |\mathrm{Re}\, a_k|$.

(4) $b_{2m} = o(S_{2m}^{1/2})$ $m \to \infty$.

(5) $\int_0^h [\varphi(x+y) - \varphi(x)]dy = o(h)$ $h \to 0$.

(6) *Either* $S_{2m}^{(2)} \geqq L(2)S_{2m}$ *or* $0 < \theta_1 < |a_{2k-1}/a_{2k}| < \theta_2 < \infty$ *and* $|\mathrm{Re}\, a_k/|a_k|| > \eta$.

*Then* $\lim_{m\to\infty} P_{2m}(D)f(x) = \varphi(x)$.

*Proof.* Integrating by parts and since $\int_{-\infty}^{\infty} G_{2m}(t)dt = 1$ we obtain

$$|P_{2m}(D)f(x) - \varphi(x)| > \int_{|x-t|\leqq\delta} |\beta(t)|\, |G'_{2m}(x-t)|\, dt + \int_{|x-t|\geqq\delta} |G'_{2m}(x-t)|\, |\beta(t)|\, dt\,,$$

where $\beta(t) = \int_x^t [\varphi(x+u) - \varphi(x)]du$ and therefore $\beta(x+t) = o(t)$ $t \to 0$ and $|\beta(t)| \leqq K_1 e^{M|t|}$.

To obtain the inversion result for the case $S_{2m}^{(2)} \geqq L(2)S_{2m}$ we use the estimation from Theorem 4.2; while for $|\mathrm{Re}\, a_k/a_k| > \eta$, $0 < \theta_1 < |a_{2k-1}/a_{2k}| < \theta_2 < \infty$ we use Theorem 4.4, both are applicable to $G_{2m}(t + b_{2m})$.

REMARK 1. In case $a_{2k-1} = -a_{2k}$ (from some $k$ onward) we can drop (5) and write instead

$$\int_0^h [\varphi(x \pm y) - \varphi(x \pm 0)]dy = o(h) \qquad h \to 0+$$

(if the numbers $\varphi(x \pm 0)$ exist) and then if we write $b_{2m} = 0$ instead of (4) and drop (6), we shall obtain

$$(5.2)\qquad \lim_{m\to\infty} P_{2m}(D)f(x) = \frac{1}{2}[\varphi(x+0)+\varphi(x-0)] .$$

The proof is similar if we remember that $G_{2m}(t) = G_{2m}(-t)$ and therefore $\int_{-\infty}^{0} G_{2m}(t)dt = 1/2$.

REMARK 2. The condition (3) of Theorem 5.2 seems too strong since for the case where $a_k$ are real the assumption could be dropped. We hope that at least for some classes of $\{a_k\}$ Theorem 5.2 could be proved without (3).

THEOREM 5.3. *Suppose*: (1) $G(t)$ *and* $E(s)$ *are defined by* (1.2) *and* $\{a_k\} \in B(2, \delta)$.

(2) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$.

(3) *For some* $M$ *and* $K$ $|\varphi(t)| \leqq Ke^{M|t|}$ *where* $M = \min|\operatorname{Re} a_k|$ .

(4) $\{(\sum_{k=m+1}^{\infty}|a_k|^{-1-\delta})^{1/1+\delta}\}^{\beta} \leqq KQ_m^{1/4}$ *for some* $\beta \geqq 1$.

(5) $\varphi(x) - \varphi(t) = 0(|t-x|^{\beta-1})$ $t \to x$.

*Then*

$$(5.3)\qquad \lim_{m\to\infty} P_{2m}(D)f(x) = \varphi(x) .$$

*Proof.* We have

$$\begin{aligned}
|P_{2m}(D)f(x) - \varphi(x)| &= \left|\left\{\int_{|x-t|\geqq\delta} + \int_{|x-t|\leqq\delta}\right\}G_{2m}(x-t)[\varphi(t)-\varphi(x)dx\right| \\
&\leqq K_1\int_{|t-x|\geqq\delta} |G_{2m}(x-t)|e^{Mt}dt + \varepsilon\int_{-\infty}^{\infty}|G_{2m}(t)||t|^{\beta-1}dt \\
&\leqq o(1) + \varepsilon K_2\int_{-\infty}^{\infty} Q_m^{-1/4}|t|^{\beta-1}\exp\left(-A\left(\sum_{k=m+1}^{\infty}|a_k|^{-1-\delta}\right)^{-1/1+\delta}|t|\right)dt \\
&\leqq o(1) + \varepsilon K_2K\int_{-\infty}^{\infty}|u|^{\beta-1}\exp(-Au)du \leqq o(1) + \varepsilon K_2K
\end{aligned}$$

$$m \to \infty .$$

THEOREM 5.4. *Suppose*: (1) $G(t)$ *and* $E(s)$ *are defined by* (1.2) *and* $\{a_k\} \in C(2)$.

(2) $f(x) = \int_{-\infty}^{\infty} G(x-t)\varphi(t)dt$.

(3) *For some* $K$ *and* $M$ $|\varphi(X)| \leqq Ke^{M|t|}$ *where* $M = \min|\operatorname{Re} a_k|$.

(4) $S_{2m}^{\beta/2} \leqq KQ_m^{1/4}$ *for some* $\beta \geqq 1$.

(5) $\varphi(x) - \varphi(t) = o(|t-x|^{\beta-1})$ $t \to x$.

*Then*

(5.3) $$\lim_{m\to\infty} P_{2m}(D)f(x) = \varphi(x) .$$

*Proof.* Similar to that of Theorem 5.3.

REMARK. When $\beta$ of condition (4) of Theorem 5.4 and Theorem 5.3 is equal to one, the condition on $\varphi(t)$ is mere continuity at a point $t = x$.

LEMMA 5.5. *If an integer $r$ exists such that for all $n$ $|a_{n+r}| > q|a_n|$ for $q > 1$, then*

$$\left(\sum_{k=m+1}^{\infty} |a_k|^{-1-\delta}\right)^{1/1+\delta} \leqq KQ_m^{1/4} \qquad 0 \leqq \delta \leqq 1$$

*(for $\delta = 1$ we have $S_{2m}^{1/2} \leqq KQ_m^{1/4}$).*

*Proof.* Obvious.

COROLLARY 5.5. *If the kernel is defined by $a_{2k} = 2^{k-1}(1+i)$ and $a_{2k-1} = -2^{k-1}(1+i)$ then* (5.3) *is valid at any point of continuity.*

**6. Examples, remarks and generalizations.** In this section we shall show some examples of convolution transform by giving its related sequence $\{a_k\}$. When we say $G(t) \in A(2), B(2, \delta)$ or $C(2)$ we mean that there is an order for which $\{a_k\} \in A(2), B(2, \delta)$ and $C(2)$ respectively.

EXAMPLE 6.1. $\{a_k\}$ defined by $a_{2k-1} = k, a_{2k} = q^k e^{i\theta_k}$ for $q > 1$, $0 < \delta < \pi(1/2), |\theta_k - (\pi/2)| > \delta, |\theta_k + (\pi/2)| > \delta$. $\{a_k\} \in A(2)$. The kernel $G(t)$ related to $\{a_k\}$ is not necessarily one of those discussed in [6]; for instance in case $\theta_k = (2/5)\pi$ the result of Theorem 5.2 can be applied as $S_{2m}^{(j)} \geqq L(j)S_{2m}$ for all $j$ ($j = 2$ is needed).

EXAMPLE 6.2. $G(t)$ defined by $a_{2k-1} = (2k-1)!$ $a_{2k} = (2k)!\ e^{i\theta_k}$ where $-\pi < \theta_k < \pi, |\theta_k - (\pi/2)| > \delta, |\theta_k + (\pi/2)| > \delta$ for some $0 < \delta < \pi/2$ where the $a_k$'s are arranged in the order of $|a_k|$. Theorem 5.2 does not apply here as one can easily verify that $S_{2m}^{(j)} = o(S_{2m})$ $m \to \infty$ for all $j > 0$. We can apply Theorem 5.1 and get an inversion formula.

EXAMPLE 6.3. Let $c_k$ be real, $\sum c_k^{-2} < \infty$ and

$$a_{2k-1} = c_k(\sin\theta_1)^{-1}e^{i\theta_2} , \qquad a_{2k} = c_k(\sin\theta_2)^{-1}e^{-i\theta_1}$$

where $0 < \theta_1, \theta_2 < \pi/2, 0 < \delta_1 < \theta_1 + \theta_2 < \pi/2 - \delta_2$. (1.5) is easily verified. (1.6) is valid also since $\sin^2\theta_1 + \sin^2\theta_2 - 4\sin^2\theta_1\sin^2\theta_2 \geqq \eta$ and $\cos^2\theta_1 > \sin^2\theta_2$ and therefore $\sin^2\theta_1 + \sin^2\theta_2 < 1$ implies

$$\left(1 - \frac{\eta}{2}\right)(\sin^2\theta_1 + \sin^2\theta_2) - 4\sin^2\theta_1\sin^2\theta_2 \geqq \frac{\eta}{2} .$$

Using $\sin^2\theta_1 < \cos^2\theta_2$ and $\sin^2\theta_2 < \cos^2\theta_1$ we get after some calculations that $\sin^2(\theta_1+\theta_2)\sin^2(\theta_1-\theta_2) < \sin^2\theta_1 + \sin^2\theta_2 - 4\sin^2\theta_1\sin^2\theta_2$ which implies (1.7). It should be noted that the class defined by $a_{2k-1} = a_{2k}$ and $\min(|\arg a_{2k}|\,|\arg - a_{2k}|) \leqq \pi/4 - \delta$ which includes Garder's class of transforms [5] as a very special case, is a special case of this example. Theorem 5.2 can be applied here.

EXAMPLE 6.4. Let $c_k$ be real, $\sum c_k^{-2} < \infty$ and $a_{2k-1} = c_k(\sin\theta_1)^{-1}e^{i\theta_2}$. $a_{2k} = -c_k(\sin\theta_2)^{-1}e^{i\theta_1}$ where either $0<\theta_1,\ \theta_2<\pi/2,\ 0<\delta,\ <\theta_1+\theta_2<\pi/2-\delta_2$ or $-(\pi/2) < \theta_1,\ \theta_2 < 0,\ -(\pi/2) + \delta_2 < \theta_1 + \theta_2 < \delta_1 < 0$.

The inqualities used in Example 6.3 for the validity of $\{a_k\} \in A(2)$ can also be used here. It should be noted that the class of transforms defined by Dauns and Widder [1] is the case $\theta_1 = \theta_2$ here.

EXAMPLE 6.5. Let $a_{2n-1} = n^\gamma(1+i)$, $a_{2n} = n^\gamma(1-i)$, $\gamma > 1/2$. In this case $\{a_k\} \notin A(2)$ (since (1.6) is not satisfied) but clearly $\{a_k\} \in C(2)$. In this case $\beta$ of Theorem 5.4 is easily computed as $S_{2m} = (1+o(1))4\gamma m^{-2\gamma+1}$, $Q_m = (1+o(1))4\gamma m^{-4\gamma-1}$ $m \to \infty$ and therefore

$$\left(-\gamma + \frac{1}{2}\right)\beta \leqq -\gamma + \frac{1}{4}$$

that is $\beta \geqq 1 + 1/2(2\gamma - 1)$. From this one can see easily that: (a) When $\gamma = 1$ it is enough to have at $t = x$ $\varphi(t) - \varphi(x) = o(|t-x|^{1/2})$ for Theorem 5.4.

(b) When $\gamma > 3/4$ it is enough to have $\varphi(t) - \varphi(x) = 0(t-x)$ $t \to x$ or it is enough for $\varphi(t)$ to have a left and right derivative at $t = x$.

EXAMPLE 6.6. $a_{2n-1} = n^\gamma(1+i)$, $a_{2n} = -n^\gamma(1+i)$. For $\gamma > 3/4$ $\{a_k\} \in B(2, 1/3)$. The following remarks will constitute generalizations of the Theorems of § 5 in various directions.

REMARK 6.1. In Theorem 5.1 $|\varphi(t)| \leqq Ke^{M|t|}$ can be replaced by $\left|\int_0^t \varphi(t)dt\right| \leqq Ke^{M|t|}$ if for every $\delta > 0$ if

$$(S_{2m}S_{2m}^{(1)}S_{2m}^{(2)})^{-1/2}\exp(-\delta S_m^{-1/2}) = o(1) \qquad m \to \infty . \tag{6.1}$$

This result can be achieved by a proper change of Theorem 4.2 that will yield now

(6.2) $$|G'_{2m}(t)| \leqq M(S_{2m}S_{2m}^{(1)}S_{2m}^{(2)})^{-1/2} \exp(-AS_{2m}^{-1/2}|t|) .$$

REMARK 6.2. In Theorems 5.3 and 5.4 condition (3) can be replaced by $\left|\int_0^t \varphi(t)dt\right| \leqq Ke^{M|t|}$ if either $Q_m^{(1)} \geqq LQ_m$ or if for all $\eta > 0$

$$(Q_m^{(1)}Q_m)^{-1/4} \exp\left(-\eta \sum_{m+1}^{\infty} |a_k|^{-1-\delta}\right)^{-1/(1+\delta)}\Big) = o(1) \qquad m \to \infty$$

for Theorem 5.3 and $(Q_m^{(1)}Q_m)^{-1/4} \exp(-\eta S_{2m}^{-1/2}) = o(1)$ $m \to \infty$ for Theorem 5.4. For the above generalization slight improvements of Theorems 4.6 and 4.7 are needed in case $Q_m^{(1)} \geqq LQ_m$ is not satisfied.

REMARK 6.3. If $S_{2m}^{-1/2} \leqq KQ_m^{1/4}$, then in Theorem 5.4 $\varphi(t) - \varphi(x) = 0(1)$ $t \to x$ can be replaced by

$$\int_x^{x+h} [\varphi(t) - \varphi(x)]dt = o(h) \qquad h \to 0 .$$

REMARK 6.4. If in Theorem 5.3 (5) is replaced by

$$\varphi(t) - \varphi(x+) = o(|t - x|^{\beta-1}) \qquad t \to x+$$

and

$$\varphi(t) - \varphi(x-) = o(|t - x|^{\beta-1}) \qquad t \to x- ,$$

then

$$\lim_{m\to\infty} P_{2m}(D)f(x) = \frac{1}{2}[\varphi(x+) + \varphi(x-)] .$$

REMARK 6.5. If in Theorem 5.3 we have

$$\left(\sum_{m+1} |a_k|^{-1-\delta}\right)^{1/1+\delta} \leqq K_1Q_m ,$$

then $\varphi(t) - \varphi(x) = o(1)$ $t \to x$ can be replaced by

$$\int_x^{x+h} [\varphi(x \pm t) - \varphi(x \pm 0)dt = o(h) \qquad h \to 0+$$

and then

$$\lim_{m\to\infty} P_{2m}(D)f(x) = \frac{1}{2}[\varphi(x + 0) + \varphi(x - 0)] .$$

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