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A RADICAL FOR LATTICE-ORDERED RINGS

J. E. DIEM

The main result of this paper states that for a lattice-ordered ring (l-ring) A with no nonzero nilpotent l-ideals the following are equivalent: (i) A is an f-ring; (ii) A is a subdirect union of totally-ordered rings with no nonzero divisors of zero; (iii) $x^+x^- = 0$ for all $x \in A$; (iv) $x^+ax^- = 0$ for all $x, a \in A$; and (v) $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$ for all $a, b, c \in A$ with $a \geqq 0$. In particular, the equivalence of (i) and (iii) implies that an l-ring which has an identity that is a weak order unit and which has no nonzero nilpotent l-ideals is necessarily an f-ring.

The basic tool in our considerations is the notion of prime l-ideal. Specifically, call a proper l-ideal P of an l-ring A prime if $I \subseteqq P$ or $J \subseteqq P$ wherever I and J are l-ideals of A with $IJ \subseteqq P$. Various conditions are obtained on A, each of which forces A modulo every prime l-ideal to be totally-ordered with no nonzero divisors of zero. Moreover the relationship between the join of all the nilpotent l-ideals of A and the intersection of all the prime l-ideals of A is investigated in order to obtain the theorem mentioned above.

The P-radical of an l-ring A is the intersection of all the prime l-ideals of A. In § 2 the general theory of the P-radical is considered. The results here are analogous to ring theoretic results found in McCoy [4] and Jacobson [2] (Chapter VIII).

In § 3 the general theory of the P-radical which is more or less independent of the order structure is tied together with the order. Specifically we investigate the relationship between the P-radical and the join of all of the nilpotent l-ideals for various classes of l-rings.

§ 4 contains a proof of the theorem mentioned above.

2. Prime l-ideals and the P-radical.

The results of this section are analogous to ring theoretic results found in McCoy [4] and Jacobson [2] (Chapter VIII). Consequently after proving a few of the results in detail, we sketch proofs indicating the idiosynrasies they take on in l-rings and note the analogous result in McCoy or Jacobson.

The reader is referred to Birkhoff and Pierce [1] and Johnson [3] for the general theory of l-rings. Our notation is the same as Johnson [3]. Also, the word l-ideal, unmodified means proper l-ideal.

DEFINITION 2.1. (i) An l-ideal P of an l-ring A is prime if $I \subseteqq P$ or $J \subseteqq P$ whenever I and J are I-ideals of A with $IJ \subseteqq P$.

(ii) A nonzero l-ring A is prime if $\{0\}$ is a prime l-ideal.

(iii) A nonzero l-ring A is an l-domain if $A^+\backslash\{0\}$ is closed under multiplication.

REMARK. If I and J are l-ideals of an l-ring A, then IJ denotes the ring theoretic product of the ideals I and J. Note that IJ is not, in general, an l-ideal. We can "make IJ into an l-ideal" by forming $\langle IJ\rangle$, the smallest l-ideal containing IJ. Birkhoff and Pierce [1] have denoted this by $I\cdot J$ and called it the l-product of I and J. As we shall have ocassion to use the notation $\langle S\rangle$ for the l-ideal generated by a subset S of an l-ring A, we use the notation $\langle IJ\rangle$ for the l-product of two l-ideals I and J. Note that if I, J, and P are l-ideals of A, then $IJ \subseteqq P$ if and only if $\langle IJ\rangle \subseteqq P$; and hence the definition of prime l-ideal is independent of the choice of IJ or $\langle IJ\rangle$.

To set the situation we note that a prime l-ideal need not be prime as a ring ideal. In fact, a prime l-ideal of an archimedean commutative l-ring in which the square of every element is positive need not be prime as a ring ideal (See 2.3 below.). However, Johnson [3] has shown.

THEOREM 2.2. *Let A be an f-ring,*[1] *and let P be an l-ideal of A. Then the following are equivalent:*

(i) *A/P is totally-ordered with no nonzero divisors of zero;*

(ii) *P is prime as a ring ideal; and*

(iii) *P is a prime l-ideal.*

In § 4 we generalize 2.2 to several classes of l-rings each of which properly contains the class of f-rings.

EXAMPLE 2.3. A prime l-ideal of an archimedean commutative l-ring in which the square of every element is positive which is not prime as a ring ideal.

Let S be the semigroup consisting of two elements a and b with multiplication $ab = ba = a^2 = b^2 = a$, and let $R(S)$ denote the semigroup ring on S with real coefficients. Make $R(S)$ into an archimedean commutative l-ring by decreeing that $\alpha a + \beta b \geqq 0$ if $\alpha \geqq 0$ and $\beta \geqq 0$ where α and β are real numbers. Then the square of every element of $R(S)$ is positive since $(\alpha a + \beta b)^2 = (\alpha + \beta)^2 a$. Now, $\{0\}$ is not prime as a ring ideal since $(a - b)^2 = 0$. However, it is easy to see that $R(S)$ is an l-domain, and hence $\{0\}$ is a prime l-ideal by the next result.

2.4. *If P is l-ideal of an l-ring A such that $A^+\backslash P$ is closed*

[1] An f-ring is an l-ring in which $a \wedge b = 0$ and $c \geqq 0$ imply $ca \wedge b = 0$ and $ac \wedge b = 0$. In [1] Birkhoff and Pierce showed that the class of f-rings is identical with the class of subdirect unions of totally-ordered rings.

under multiplication, then P is a prime l-ideal. The converse holds is A is commutative.

Proof. First suppose that I and J are l-ideals of A with $IJ \subseteqq P$. If I is not contained in P, then there is a non-zero positive element $a \in I \backslash P$. Let b be a positive element of J. Then $ab \in IJ \subseteqq P$, so that $b \in P$ since $a \notin P$. It follows that $J \subseteqq P$.

Now suppose that A is commutative, P is a prime l-ideal of A, and $a_1, a_2 \in A^+$ with $a_1 a_2 \in P$. Then $\langle a_1 a_2 \rangle \subseteqq P$. Let $z_i \in \langle a_i \rangle$, $i = 1, 2$. Then $|z_i| \leqq n_i a_i + r_i a_i$ $(i = 1, 2.)$ for suitable $r_i \in A^+$ and suitable nonnegative integers n_i. Thus

$$|z_1 z_2| \leqq |z_1||z_2| \leqq (n_1 a_1 + r_1 a_1)(n_2 a_2 + r_2 a_2)$$

which belongs to P since A is commutative and $\langle a_1 a_2 \rangle \subseteqq P$. It follows that $\langle a_1 \rangle \times \langle a_2 \rangle \subseteqq P$; and hence either $a_1 \in P$ or $a_2 \in P$.

The following characterization of prime l-ideals will be used repeatedly in the sequel.

2.5. *An l-ideal P of an l-ring A is prime if and only if $a, b \in A^+$ and $aA^+b \subseteqq P$ imply $a \in P$ or $b \in P$.*

Proof. *Necessity.* From $aA^+b \subseteqq P$ it follows that

$$\langle A^+aA^+ \rangle \times \langle A^+bA^+ \rangle \subseteqq P.$$

Thus either $\langle A^+aA^+ \rangle \subseteqq P$ or $\langle A^+bA^+ \rangle \subseteqq P$. Suppose that $\langle A^+aA^+ \rangle \subseteqq P$. Then $\langle a \rangle^3 \subseteqq P$, and hence $\langle \langle a \rangle \times \langle a \rangle \rangle \times \langle a \rangle \subseteqq P$. Thus either $\langle a \rangle^2 \subseteqq P$ or $\langle a \rangle \subseteqq P$. In either case we have that $a \in P$.

Sufficiency. If I and J are l-ideals of A which are not contained in P, then there is an $a \in I^+ \backslash P$ and a $b \in J^+ \backslash P$. If $IJ \subseteqq P$, then $aA^+b \subseteqq IJ \subseteqq P$; so that $a \in P$ or $b \in P$. Since this contradicts the choice of a and b, IJ is not contained in P; and we are done.

Note that 2.5 says that an l-ideal P of an l-ring A is prime if and only if $A^+ \backslash P$ is an m-system in the sense of

DEFINITION 2.6. A nonempty subset M of an l-ring A is an m-system if each element of M is positive and if for $a, b \in M$ there is an $x \in A^+$ such that $axb \in M$.

Note that nonempty subset S of A^+ which is closed under multiplication is an m-system since $aab \in S$ whenever $a, b \in S$.

The next result, as did the proceeding, has its analogue in [4].

2.7. *Let M be an m-system of an l-ring A, and let I be an l-*

ideal of A that does not meet M. Then I is contained in a prime l-ideal that does not meet M.

Proof. The existence of an l-ideal P of A which is maximal with respect to the property of not meeting I is guaranteed by Zorn's Lemma. We show that P is prime. The proof of this is an in [4] (Lemma 4) once one knows that the l-ideal generated by P and a positive element a of A not in P is $\{z \in A: |z| \leqq p + na + ra + sa + tav$ where $r, s, t, v \in A^+, p \in P^+$, and n is a nonnegative integer$\}$.

DEFINITION 2.8. The P-radical, $P(A)$, of an l-ring A is the intersection of all of the prime l-ideals of A.

Recall that the l-radical of an l-ring A is the set $N(A) = \{a \in A$: there is a positive integer $n = n(a)$ such that

$$x_0 |a| x_1 |a| x_2 \cdots x_{n-1} |a| x_n = 0$$

for all $x_0, x_1, x_2, \cdots, x_n \in A\}$ ([1], p. 45.) If A is comutative, then $N(A) = \{a \in A: |a|$ is nilpotent$\}$ ([1], Corollary 1, p. 45). Moreover, for an arbitrary l-ring A, $N(A)$ is the join of all of the nilpotent l-ideals of A ([1], Th. 5).

Now suppose that $a \in A$ is not nilpotent. Then since $0 < |a^n| \leqq |a|^n$ for all n, $|a|$ is not nilpotent. Thus, by 2.7, there is a prime l-ideal P of A not meeting the m-system $\{|a|, |a|^2, \cdots, |a|^n, \cdots\}$. It follows that a does not belong to $P(A)$, and hence every element of $P(A)$ is nilpotent. Now note that every prime l-ideal of A contains every nilpotent l-ideal of A, and hence we have

2.9. *The P-radical of an l-ring A is a nil l-ideal of A containing the l-radical of A.*

The proof of the next result is as in [4] (Theorem 5).

2.10. *If A is an l-ring, then $P(A/P(A))$ is zero.*

The next result is useful in relating the l-radical to the P-radical.

2.11. *Let I be an l-ideal of an l-ring A such that $N(A/I)$ is zero, and let J be an l-ideal of A properly containing I. Then there is a prime l-ideal P of A containing I but but not containing J.*

Proof. (After Jacobson, [2], p. 196) Choose $a_0 \in J^+ \backslash I$. Then since $N(A/I)$ is zero, A/I has no nonzero nilpotent l-ideals; and hence $\langle a_0 \rangle^k$ is not contained in I for any positive integer k. Now, $\langle A^+ a_0 A^+ \rangle^2$ is not contained in I since $\langle a_0\rangle^3 \subseteqq \langle A^+a_0A^+\rangle$ and $\langle a_0\rangle^6$ is not contained in I. Now suppose that $a_0ba_0 \in I$ for all $b \in A^+$. Then for $z \in \langle A^+a_0A^+\rangle^2$, there are x_i, $y_i \in \langle A^+a_0A^+\rangle$ and $t_i, u_i, v_i, w_i, \in A^+$ such that

$$|z| \leqq \sum_{i=1}^{n} |x_i||y_i| \leqq \sum_{i=1}^{n} (t_ia_0u_i)(v_ia_0w_i) .$$

But $a_0u_iv_ia_0 \in I^+$, so that $z \in I$. Consequently there is a $b_0 \in A^+$ such that $a_1 = a_0b_0a_0 \in J^+\backslash I$. Similarly, there is a $b_1 \in A^+$ such that $a_2 = a_1b_1a_1 \in J^+\backslash I$. Containing inductively, we obtain two sequences: $\{a_i\}_{i=0}^{\infty} \subseteqq J^+\backslash I$ and $\{b_i\}_{i=0}^{\infty} \subseteqq A^+$ such that $a_n = a_{n-1}b_{n-1}a_{n-1} \in J^+\backslash I$ for all $n \geqq 1$. It follows that $\{a_i\}_{i=0}^{\infty}$ is an m-system that does not meet I. By 2.7 there is a prime l-ideal P of A containing I that does not meet $\{a_i\}_{i=0}^{\infty}$. Since $a_i \in J$ for $i \geqq 0$, we know that J is not contained in P; and hence P is as desired.

2.12. *If A is an l-ring, then $P(A) = \bigcap \{I$: I is an l-ideal of A and $N(A/I)$ is zero$\}$.*

Proof. Let $\mathscr{L}(A) = \bigcap\{I$: I is an l-ideal of A and $N(A\backslash I)$ is zero$\}$.

If P is a prime l-ideal of A, then $N(A\backslash P) \subseteqq P(A\backslash P) = \{0\}$. Thus $\mathscr{L}(A) \subseteqq P(A)$.

Now let $J/\mathscr{L}(A)$ be a nilpotent l-ideal of $A/\mathscr{L}(A)$, and let I be an l-ideal of A such that $N(A/I)$ is zero. Then $J^n \subseteqq \mathscr{L}(A)$ for some positive integer n; and since $\mathscr{L}(A) \subseteqq I$, we know that $J^n \subseteqq I$. It follows that $\langle I + J\rangle/I$ is a nilpotent l-ideal of A/I. Since $N(A/I)$ is zero, it follows that $J \subseteqq I$. Thus $J \subseteqq \mathscr{L}(A)$, so that $N(A/\mathscr{L}(A))$ is zero. Now if $\mathscr{L}(A)$ is properly contained in $P(A)$, then, by 2.11 there is a prime l-ideal containing $\mathscr{L}(A)$ but not containing $P(A)$. Since this contradicts the definition of $P(A)$, $\mathscr{L}(A) = P(A)$.

2.13. *If A is an l-ring, the $N(A/N(A))$ is zero if and only if $N(A) = P(A)$. Hence $N(A)$ is zero if and only if $P(A)$ is zero.*

Proof. If $N(A/N(A))$ is zero, then $P(A) = \bigcap\{I$: I is an l-ideal of A and $N(A/I)$ is zero$\} \subseteqq N(A) \subseteqq P(A)$.

If $N(A) = P(A)$, then $N(A/N(A)) = N(A/P(A)) \subseteqq P(A/P(A))$ which is zero.

The next result has its analogue in [4] (Theorem 6). It will be used in § 4 to obtain the theorem mentioned in the introduction.

2.14. *An l-ring A has zero l-radical if and only if it is a subdirect union of prime l-rings.*

Proof. The proof is immediate from 2.13.

The remaining results of this section will be useful in the next section where we determine various classes of l-rings for which the P-radical equals the l-radical.

2.15. *If A is an l-ring, then* $P(A) = \{a \in A$: *any m-system containing* $|a|$ *contains* 0.

Proof. Suppose that there is an m-system M containing $|a|$ that does not contain 0. Then, by 2.7, there is a prime l-ideal P of A that does not meet M. Thus $|a|$ does not belong to P, and it follows that a does not belong to $P(A)$.

Conversely, let $a \in A$ be such that any m-system containing $|a|$ contains 0, and let P be a prime l-ideal of A. If a does not belong to P, then $A^+ \backslash P$ is an m-system containing $|a|$. Thus $0 \in A^+ \backslash P$ which is clearly impossible. Hence $a \in P(A)$.

2.16. *If A is an l-ring, then* $N(A) = \{a \in A$: *there is a positive integer* $n = n(a)$ *such that* $(x|a|)^n x = 0$ *for all* $x \in A^+\}$.

Proof. It is clear from the definition of $N(A)$ that if $a \in N(A)$, then there is a positive integer n such that $(x|a|)^n x = 0$ for all $x \in A^+$.

Conversely, suppose that there is a positive integer n such that $(x|a|)^n x = 0$ for all $x \in A^+$, and let $x_0, x_1, \cdots, x_n \in A^+$. Then, since $x = x_0 \vee x_1 \vee \cdots \vee x_n \geqq x_i$ for all $i = 0, 1, \cdots, n$, it follows that $0 = (x|a|)^n x \geqq x_0 |a| x_1 \cdots x_{n-1} |a| x_n \geqq 0$. Since every element of A is the difference of two positive elements, the result follows.

2.17. *If I is a right (resectively, left) l-ideal of an l-ring A, then* $P(I) = P(A) \cap I$.

Proof. Let $a \in P(I)$ and let M be an m-system in A containing $|a|$. We show that $M \cap I$ is an m-system in I. Let $x, y \in M \cap I$. Then there is a $z \in A^+$ such $xzy \in M \cap I$. Again there is a $z_1 \in A^+$ such that $xzyz_1xzy \in M \cap I$. But $zyz_1xz \in I^+$ since I is a right (respectively, left) l-ideal; hence $M \cap I$ is an m-system in I. By 2.15, $0 \in M \cap I$ since $|a| \in M \cap I$ and $a \in P(I)$. Again, by 2.15, it follows that $a \in P(A) \cap I$.

Conversely, let $a \in P(A) \cap I$, and let M be an m-system in I containing $|a|$. Then M is an m-system in A containing $|a|$. By 2.15, M contains 0; and hence $a \in P(I)$.

2.18. *If I is a right (respectively, left) l-ideal of an l-ring A,* *then* $N(I) = N(A) \cap I$.

Proof. If $a \in N(I)$, then, by 2.16, there is a positive integer n such that $(x\,|\,a\,|)^n x = 0$ for all $x \in I^+$. But for $y \in A^+$ we know that $y\,|\,a\,|\,y \in I^+$, and hence $0 = (y\,|\,a\,||\,y\,|\,x)^n y = (y\,|\,a\,|)^{2n+1} y$; so that

$$y \in N(A) \cap I$$

by 2.16. That $N(A) \cap I \subseteqq N(I)$ is clear from the definition of $N(A)$.

3. The P-radical equals the *l*-radical. Birkhoff and Pierce ([1], p. 45, Example 8) have given an example of an l-ring A such that $N(A/N(A))$ is not zero. By 2.13, the l-radical of such an l-ring is properly contained in its P-radical. However, there are many l-rings for which the l-radical is equal to the P-radical. In this section we identify some of them and prove some results about l-rings in which the square of every element is positive.

THEOREM 3.1. *If A is an l-ring which is commutative, or satisfies either the ascending or descending chain condition on l-ideals, or is an f-ring, then* $N(A) = P(A)$.

Proof. Birkhoff and Pierce ([1], p. 46, Corollary 4; and [1], p. 63, Corollary 1) have shown that if an l-ring A is commutative, or satisfies either the ascending or descending chain condition on l-ideals, or is an f-ring, then $N(A/N(A))$ is zero. The result follows from 2.13.

COROLLARY 3.2. *If A is an l-ring, and if $P(A)$ is commutative, or satisfies either the ascending or descending chain condition on l-ideals, or is an f-ring, then* $N(A) = P(A)$.

Proof. Using 2.9, 2.17, 2.18, and 3.1, we have

$$N(A) = N(A) \cap P(A) = N(P(A)) = P(P(A)) = P(A) \cap P(A) = P(A)\ .$$

In [1] Birkhoff and Pierce show that is A is an l-ring with an identity element 1 that is a weak order unit[2], then every nilpotent of A is, in absolutive value, $\leqq 1$. We generalize this result to

THEOREM 3.3. *Let A be an l-ring with an identity element* 1, *and suppose that the square of every element of A is positive. Then each nilpotent x of A is, in absolute value,* $\leqq 1$.

Proof. (We are indebted to the referee for this proof.) The

[2] A positive element e of an l-ring A is a weak order unit if $e \wedge x = 0$ and $x \in A$ imply $x = 0$.

proof is by induction on the nilpotency index k of x. For $k = 1$ the result is trivial. For $k \geqq 1$ nilpotency index of x^2 is less than k. Thus $x^2 = |x^2| \leqq 1$. Since $0 \leqq (x-1)^2 = x^2 - 2x + 1$ and $0 \leqq (x+1)^2 = x^2 + 2x + 1$, we have that $-(1 + x^2) \leqq 2x \leqq 1 + x^2$. Thus $2|x| = |2x| \leqq 1 + x^2 \leqq 2.\ 1$, and hence $|x| \leqq 1$.

COROLLARY 3.4. *Let A be an l-ring with an identity element* 1, *and suppose that the square of every element of A is positive. Then* $N(A) = P(A)$.

Proof. By 3.3, $B(A) = \{x \in A\colon |x| \leqq n1$ for some positive integer $n\}$ contains all of the nilpotents of A, and hence it contains $P(A)$. Now, Birkhoff and Pierce [1] have shown (and it is easy to see) that $B(A)$ is a sub-l-ring of A which is an f-ring. Consequently $P(A)$ is a sub-f-ring of A, so-that, by 3.2, $N(A) = P(A)$.

We now turn our attention to finding a sufficient condition for the P-radical of an l-ring A in which the square of every element is positive to be equal to $\{x \in A;\ |x|$ is nilpotent$\}$.

LEMMA 3.5. *Let A be an l-ring in which the square of every element is positive. Then for $a, b \in A^+$ with $a^2 = b^2 = 0$, we have that $ab = ba = 0$.*

Proof. Since ab, ba, and $(a-b)^2$ are positive, we know that $0 \leqq (a-b)^2 = -ba - ab \leqq 0$. Thus $ab + ba = 0$, and the lemma follows.

LEMMA 3.6. *Let A be a prime l-ring in which the square of every element is positive. Then A is an l-domain if and only if $a, b \in A$, $a \wedge b = 0$, and $ab = 0$ imply $ba = 0$.*

Proof. Necessity is clear since if A is an l-domain and $a, b \in A^+$ are such that $ab = 0$, then either $a = 0$ or $b = 0$.

Conversely, we first show that A has no nonzero positive nilpotents of index 2. Suppose that $a \in A^+$ and $a^2 = 0$, and let $z \in A^+$. We will show that $aza = 0$. There are three cases.

1. $0 \leqq za \leqq az$. Then $0 \leqq aza \leqq a^2z = 0$, so that $aza = 0$.
2. $0 \leqq az \leqq za$. Then $0 \leqq aza \leqq za^2 = 0$, so that $aza = 0$.
3. $(za - az) \notin A^+ \cup -(A^+)$. Then $(za - az)^+ > 0$ and $(za - az)^- > 0$.

Now $0 \leqq (za - az)^+(za - az)^- = (za - az)^+(az - za)^+ \leqq za^2z = 0$. Thus $(za - az)^+(za - az)^- = 0$, and hence $(za - az)^-(za - az)^+ = 0$ since $(za - az)^+ \wedge (za - az)^- = 0$. Now $(za - az)^+y(za - az)^-$ is a positive nilpotent of index 2 for any $y \in A^+$; so that, by 3.5, $a(za - az)^+y(za -$

$az)^- = 0$. Since A is a prime l-ring and $(za - az)^- > 0$, we know that $a(za - az)^+ = 0$ by 2.5. Similiarly, $a(za - az)^- = 0$. Consequently, we have that $0 = a[(za - az)^+ - (za - az)^-] = a(za - az) = aza$ for all $z \in A^+$. Again using 2.5, it follows that $a = 0$.

Now let $a, b \in A^+$ with $ab = 0$. Then for any $z \in A^+$, bza is a nilpotent of index 2 and hence is 0. Thus, by 2.5, $a = 0$ or $b = 0$; and the proof is complete.

REMARK. We do not know if every prime l-ring A in which the square of every element is positive satisfies: $a, b \in A$, $a \wedge b = 0$, and $ab = 0$ imply $ba = 0$.

THEOREM 3.7. *Let A be an l-ring in which the square of every element is positive, suppose that disjoint elements of A commute, and suppose that A has zero l-radical. Then A is a subdirect union of l-domains in which all squares are positive and disjoint elements commute.*

Proof. B 2.14, A is a subdirect union of a family $\{A_\alpha;\ \alpha \in \Gamma\}$ of prime l-rings. Since both of the properties of disjoint elements commuting and all square being positive are preserved under homomorphisms, each A_α has these properties and hence is an l-domain by 3.6.

COROLLARY 3.8. *Let A be an l-ring in which the square of every element is positive, and suppose that disjoint elements of A commute. Then $P(A) = \{x \in A\colon |x| \text{ is nilpotent}\}$. Moreover, if A has an identity element 1, then $P(A) = \{x \in A\colon x \text{ is nilpotent}\}$.*

Proof. Since $P(A/P(A))$ is zero, $A/P(A)$ is a subdirect union of l-domains by 3.7. It follows that $A/P(A)$ has no nonzero positive nilpotents, and hence all of the positive nilpotents of A are in $P(A)$. The first part of the corollary now follows since $P(A)$ is a nil l-ideal.

Finally, if A has a positive identity 1, then every nilpotent of A is contained in the sub-f-ring $B(A) = \{x \in A\colon |x| \leqq n1$ for some non-negative integer $n\}$ of A by 3.3. But an element of an f-ring is nilpotent if and only if its absolute value is. Thus, by the first part, $P(A) = \{x \in A\colon x$ is nilpotent$\}$.

THEOREM 3.9. *Let A be an archimedean l-ring in which the square of every element is positive. Then*

(i) *if $x \in A^+$ and $x^2 = 0$, then $xA = Ax = \{0\}$;*

(ii) *every positive nilpotent of A has index $\leqq 3$;*

(iii) $P(A)A^2 = A^2P(A) = P(A)^3 = \{0\}$;

(iv) $N(A) = P(A) = \{x \in A\colon |x| \text{ is nilpotent}\}$;

(v) *if A has no nonzero positive left or right annihilators, then A has no nonzero positive nilpotents*; *and*

(vi) *if A has an identity element* 1, *then A has no nonzero nilpotents.*

Proof. The proof is broken up into several steps.

(1) If $x \in A^+$ and $x^2 = 0$, then $xA = Ax = \{0\}$.

Proof. Let $y \in A^+$, and let n be an integer. Then $0 \leqq (nx - y)^2 = n^2x^2 - nxy - nyx + y^2$; and hence $n(xy + yx) \leqq y^2$. Since A is archimedean, $xy + yx = 0$. Since xy and yx are positive, $xy = yx = 0$. Since every element of A is the difference of two positive elements, $xA = Ax = \{0\}$.

(2) Every positive nilpotent of A has index $\leqq 3$.

Proof. Let x be a positive nilpotent of index $n \geqq 4$. Then $2n - 4 \geqq n$, so that $(x^{n-2})^2 = 0$. Hence, by (1), $x^{n-1} = x(x^{n-2}) = 0$; and the result follows.

(3) Let $\eta(A) = \{x \in A: |x| \text{ is nilpotent}\}$. Then $N(A) = P(A) = \eta(A)$.

Proof. Let $x \in \eta(A)$. For $y \in A^+$ and n an integer, we have that $0 \leqq (n|x| - y)^2 = n^2|x|^2 - n|x|y - ny|x| + y^2$; so that $n(|x|y + y|x|) \leqq n^2|x|^2 + y^2$. But $|x|^3 = 0$ by (2), so that $|x|^2$ is both a left and right annihilator of A by (1). Hence for $z \in A^+$ we have that $(|x|yz + y|x|z) \leqq y^2z$. Since A is archimedean, it follows that $|x|yz = y|x|z = 0$; and; hence $|x|yz = y|x|z = 0$ for all $y, z \in A$. Since $y|x|z = 0$ for all $y, z \in A$, we have that $x \in N(A)$; and hence

$$N(A) \subseteqq P(A) \subseteqq \eta(A) \subseteqq N(A) .$$

Note that since $|x|yz = 0$ and $\eta(A) = P(A)$, we have that $P(A)A^2 = P(A)^3 = \{0\}$. Moreover, if the inequality $n(|x|y + y|x|) \leqq n^2|x| + y^2$ is multiplied on the left by $z \in A^+$, then it follows that $A^2P(A) = \{0\}$. We have now completed the proofs of parts (i) through (iv).

Part (v) is an immediate consequence of part (i); and part (vi) follows from part (i) and (v) since if A has an identity element, then x is nilpotent if and only if $|x|$ is.

4. Subdirect unions of totally-ordered rings with no nonzero divisors of zero. In this section we prove the theorem mentioned in the introduction. It is a consequence of the following three propositions.

PROPOSITION 4.1. *Let A be an l-ring which satisfies the identity $x^+ax^- = 0$. Then an l-ideal P of A is prime if and only if A/P is totally-ordered with no nonzero divisors of zero.*

Proof. If A/P has no nonzero divisors of zero, then P is a prime l-ideal by 2.4.

Conversely, we may suppose that A is a prime l-ring since the identity $x^+ax^- = 0$ is preserved under homomorphisms. But if $x^+ax^- = 0$ for all $a \in A^+$, then either $x^+ = 0$ or $x^- = 0$ by 2.5. It follows that A is totally-ordered. By 2.2, A has no nonzero divisors of zero.

In the next proposition we shall call an l-ring in which $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$ for $a \geqq 0$ a distributive l-ring. Note that a distributive l-ring also satisfies $a(b \wedge c) = ab \wedge ac$ and $(b \wedge c)a = ba \wedge ca$ for $a \geqq 0$.

PROPOSITION. Let A be a distributive l-ring. Then an l-ideal P of A is prime if and only if A/P is totally-ordered with no nonzero divisors of zero.

Proof. Sufficiency is a restatement of 2.4.

Conversely, let P be a prime l-ideal of A. Since A/P is a distributive l-ring, we may assume that A is a prime l-ring. If $a \in A^+$ is either a left or right annihilator, then $aA^+a = \{0\}$; so that, since A is a prime l-ring, $a = 0$ by 2.5. But ([1], Th. 14) a distributive l-ring with no nonzero left or right positive annihilators is an f-ring. Hence A is totally-ordered with no nonzero divisors of zero by 2.2.

PROPOSITION 4.3. Let A be an l-ring which satisfies the identity $x^+x^- = 0$. Then an l-ideal P of A is prime if A/P is totally-ordered with no nonzero divisors of zero.

Proof. Sufficiency is a restatement of 2.4.

Conversely, we may assume that A is a prime l-ring since the identity $x^+x^- = 0$ is preserved under homomorphisms. Then ([1], p. 59, Lemma 2) all squares of A are positive. Also, disjoint elements of A commute since $x^+x^- = 0$ for all $x \in A$. Thus, by 3.6, A is an l-domain. Since $x^+x^- = 0$ for all $x \in A$, it follows that A is totally-ordered; and hence A has no nonzero divisors of zero by 2.2.

THEOREM 4.4. *Let A be an l-ring with zero l-radical. Then the following are equivalent*:

(i) *A is an f-ring;*

(ii) *A is a subdirect union of totally-ordered rings with no nonzero divisors of zero;*

(iii) *$x^+ax^- = 0$ for all $x, a \in A$;*

(iv) *if $a, b, c \in A$ with $a \geqq 0$, then $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$; and*

(v) *$x^+x^- = 0$ for all $z \in A$.*

Proof. The equivalence of (i) and (ii) was proved by Pierce ([1], Th. 4) Also see Johnson [3](Theorem I. 4.8).

Since (iii), (iv), and (v) hold in any totally-ordered ring and are preserved under the formation of subdirect unions, it is clear that (i) implies (iii), (i) implies (iv), and (i) implies (v).

Now let A be an l-ring with zero l-radical. Then, by 2.14, A is subdirect union of a family $\{A_\alpha: \alpha \in \Gamma\}$ of prime l-rings. If A satisfies (iii) [(iv), (v)], then each A_α satisfies (iii) [(iv), (v)] since (iii) [(iv), (v)] is preserved under homomorphisms. By Proposition 4.1[4.2, 4.3], each A_α is totally-ordered with no nonzero divisors of zero, and the proof is complete.

The following corollary of 4.4 answers affirmatively the question of Birkhoff and Pierce originally asked in [1].

COROLLARY 4.5. *Let A be an l-ring with an identity element* 1, *and suppose that A has zero l-radical. Then A is an f-ring if and only if* 1 *is a weak order unit.*

Proof. Since ([1], Th. 15) 1 is a weak order unit if and only if $x^+x^- = 0$ for all $x \in A$, the corollary follows from the equivalence of (i) and (v) above.

Finally we note

COROLLARY 4.6. *Let A be an l-ring which satisfies either* (iii), (iv), *or* (v) *of* 4.4. *Then* $P(A) = \{x \in A$: *x is nilpotent*$\}$.

Proof. $A/P(A)$ is a subdirect union of totally-ordered rings with no nonzero divisors of zero. Hence all of the nilpotents of A are in $P(A)$. Since $P(A)$ is a nil l-ideal, the corollary follows.

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