PACIFIC JOURNAL OF MATHEMATICS
Vol. 25, No. 1, 1968

ON RELATIVELY BOUNDED PERTURBATIONS OF ORDINARY DIFFERENTIAL OPERATORS

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This paper studies ordinary differential operators of the form

$$(-1)^m D^{2m} + Q_{2m-1}D^{2m-1} + \cdots + Q_0 ,$$

over a finite interval I. The coefficients Q_j are bounded operators in $L_2(I)$. This operator is treated as a perturbation $T + A$ of the operator T, which is generated by the leading term $(-1)^m D^{2m}$ plus suitable boundary conditions. The main hypothesis is that Q_{2m-1} can be written as the sum of a compact operator and a bounded operator of sufficiently small norm. Given that T is a discrete spectral operator, with eigenvalues $\{\lambda_n\}$, it is shown that $T + A$ is also a discrete spectral operator, with eigenvalues $\{\lambda'_n\}$ satisfying $|\lambda'_n - \lambda_n| = O(|\lambda_n|^{k/2m})$, where k is the largest integer $\leqq 2m - 1$ for which $Q_k \neq 0$. Proofs are based on the method of contour integration of resolvent operators.

If A and T are given, closed operators in a Hilbert space $\mathfrak{H}$, with $\mathfrak{D}(A) \supset \mathfrak{D}(T)$, we say that A is *bounded relative to* T if there are constants c_1, c_2 such that

$$(1.1)\qquad \|Au\| \leqq c_1 \|Tu\| + c_2 \|u\| , \qquad (u \in \mathfrak{D}(T)) .$$

The infimum of numbers c_1 such that (1.1) holds for some c_2 is called the *T-bound* of A, $|A|_T$. If $|A|_T = 0$, then for any $\varepsilon > 0$ one can find a constant C_ε such that

$$(1.2)\qquad \|Au\| \leqq \varepsilon \|Tu\| + C_\varepsilon \|u\| , \qquad (u \in \mathfrak{D}(T)) .$$

Operators A and T with $|A|_T = 0$ arise in the theory of differential operators, both ordinary and partial of elliptic type, T being generated by the highest order derivative terms, and A by the lower order terms.

In this paper we consider differential operators of the form

$$(1.3)\qquad (-1)^m D^{2m} + \sum_{j=0}^{2m-1} Q_j D^j \qquad (D = d/dx)$$

over a finite interval I. The Q_k are bounded operators in $L_2(I)$; with the exception of Q_{2m-1}, they can be completely arbitrary. The operator (1.3) is treated as a perturbation of an operator T generated by the leading term $(-1)^m D^{2m}$ together with suitable boundary conditions; T will be assumed to be a spectral operator in the sense of Dunford.

(See Kramer [6] and Dunford-Schwartz [2, Part III] for classification of boundary conditions under which $(-1)^m D^{2m}$ becomes spectral.) The perturbing operator A, given by

$$(1.4)\qquad Au = \sum_{j=0}^{2m-1} Q_j D^j u \qquad (u \in \mathfrak{D}(T)) ,$$

is bounded relative to T and satisfies (1.2) with

$$(1.5)\qquad C_\varepsilon = O(\varepsilon^{-k/(2m-k)}) \qquad (\varepsilon \to 0) ,$$

where the integer k is defined by

$$(1.6)\qquad Q_{k+1} = Q_{k+2} = \cdots = Q_{2m-1} = 0, \qquad Q_k \neq 0 .$$

Now suppose that the coefficient Q_{2m-1} can be written in the form

$$(1.7)\qquad Q_{2m-1} = B_1 + B_2$$

where B_1 is a bounded operator of sufficiently small norm, and B_2 is a compact operator. Under certain mild hypotheses about the eigenvalues of T, we will show that then

(1) *The eigenvalues* λ'_j *of* $T + A$ *are related to the eigenvalues* λ_j *of* T *by*

$$(1.8)\qquad |\lambda'_j - \lambda_j| = O(|\lambda_j|^{k/2m}) \qquad (j \to \infty)$$

where k *is determined by* (1.6), *and*

(2) $T + A$ *is a spectral operator.*

The first of these results seems to be new; the second has been obtained recently by R.E.L. Turner [11]. Special cases were treated by J. Schwartz [9] and H. P. Kramer [6]. Our method is a natural extension of the method used by Schwartz; it differs considerably from the method of Kramer, and bears virtually no resemblance to that of Turner. What we do is to construct a family of disjoint circles $\{C_j\}$ in the complex plane, centered at the original eigenvalues λ_j (for large j), and such that each C_j also contains exactly one eigenvalue λ'_j. We therefore have the formula

$$E'_j - E_j = \frac{1}{2\pi i}\int_{C_j} [R_\lambda(T + A) - R_\lambda(T)]d\lambda$$

for the spectral projections E'_j and E_j of $T + A$ and T respectively, corresponding to the eigenvalues λ'_j and λ_j. The proof that $T + A$ is a spectral operator depends on suitable estimates of these contour integrals, and is based on a new perturbation theorem due to T. Kato [5].

Section 2 is devoted to perturbation theorems of a general nature, without reference to differential operators; the latter are treated in § 3.

2. Relatively bounded perturbations. If A is an arbitrary linear operator in the (complex) Hilbert space $\mathfrak{H}$, we denote by $\rho(A)$ the *resolvent set* of A, that is the set of all complex numbers λ for which $R_\lambda(A) = (\lambda I - A)^{-1}$ exists as a bounded operator in $\mathfrak{H}$. The complement of $\rho(A)$ in the complex plane is the *spectrum* $\sigma(A)$. A closed operator A in $\mathfrak{H}$ is called *regular* if for some $\lambda \in \rho(A)$, the resolvent operator $R_\lambda(A)$ is completely continuous. The spectrum of a regular operator consists of a sequence $\{\lambda_n\}$ of eigenvalues of finite multiplicity, having no accumulation point in the complex plane.

The definition of spectral operator is given for example in Schwartz [9], where the following result is proved [9, Lemma 3].

LEMMA 1. *Let T be a regular spectral operator in the Hilbert space $\mathfrak{H}$. Assume that all but a finite number of the eigenvalues of T are simple poles of the resolvent, and also that $\sum E(\lambda_i) = 1$, where $E(\lambda_i)$ are the spectral projections of T. Then there exists a constant c such that for any point $\lambda \in \rho(T)$ not in a fixed neighborhood of the exceptional multiple eigenvalues, we have*

$$\|R_\lambda(T)\| \leqq c[\operatorname{dist}(\lambda, \sigma(T))]^{-1} . \tag{2.1}$$

LEMMA 2. *Let T and A be closed linear operators in $\mathfrak{H}$, with $\mathfrak{D}(A) \supset \mathfrak{D}(T)$, and suppose that $|A|_T = 0$. Define the operator $T + A$, with $\mathfrak{D}(T + A) = \mathfrak{D}(T)$, by $(T + A)u = Tu + Au$. Then $T + A$ is a closed operator, and moreover*

(i) *if $\lambda \in \rho(T) \cap \rho(T + A)$ then*

$$R_\lambda(T + A) - R_\lambda(T) = R_\lambda(T + A) \cdot AR_\lambda(T) ; \tag{2.2}$$

(ii) *if $\lambda \in \rho(T)$ and $\|AR_\lambda(T)\| < 1$, then $\lambda \in \rho(T + A)$ and*

$$R_\lambda(T + A) - R_\lambda(T) = R_\lambda(T)[I - AR_\lambda(T)]^{-1}AR_\lambda(T) . \tag{2.3}$$

The assertions of this lemma are easily verified. Note also that if A is T-bounded then for $\lambda \in \rho(T)$, $AR_\lambda(T)$ is a bounded operator in $\mathfrak{H}$:

$$\begin{aligned} \|AR_\lambda(T)u\| &\leqq c_1 \|(T + \lambda I - \lambda I)R_\lambda(T)u\| + c_2 \|R_\lambda(T)u\| \\ &\leqq \{(c_1 |\lambda| + c_2) \|R_\lambda(T)\| + c_1\} \|u\| \quad (u \in \mathfrak{H}) . \end{aligned} \tag{2.4}$$

THEOREM 1. *Let T be a regular spectral operator in $\mathfrak{H}$, and assume that its eigenvalues $\{\lambda_n\}$ satisfy*

$$\begin{aligned} \lambda_n &\sim an^\alpha \quad (n \to \infty) , \\ \lambda_{n+1} - \lambda_n &= a(n)n^{\alpha-1} , \end{aligned} \tag{2.5}$$

for some constants $a > 0$, $\alpha > 1$, *where*

$$0 < c_1 < a(n) < c_2 \qquad (\text{large } n) .$$

Assume also that $\sum E(\lambda_i) = 1$.

Let A *be a closed operator in* $\mathfrak{H}$, *with* $\mathfrak{D}(A) \supset \mathfrak{D}(T)$, *having the following property: for each* ε, $0 < \varepsilon < 1$, *there exists a real number* C_ε *such that*

$$\| Au \| \leqq \varepsilon \| Tu \| + C_\varepsilon \| u \| , \qquad (u \in \mathfrak{D}(T)) \tag{2.6}$$

and

$$C_\varepsilon = O(\varepsilon^{-\tau}) \quad as \quad \varepsilon \to 0^+ , \tag{2.7}$$

for some number τ, $0 \leqq \tau \leqq \alpha - 1$. *For values of* n *for which* $\lambda_n > 0$, *let* $\Gamma_n(\mu)$, $\mu > 0$, *be the circle with centre* λ_n *and radius* $\mu \cdot \lambda_n^{\tau/(1+\tau)}$.

Then the operator $T + A$ (*with* $\mathfrak{D}(T + A) = \mathfrak{D}(T)$) *is a closed regular operator in* $\mathfrak{H}$. *If* $\tau < \alpha - 1$ *then for sufficiently large* μ, *the eigenvalues* λ'_n *of* $T + A$ *can be enumerated so that* λ'_n *lies inside* $\Gamma_n(\mu)$, *with the possible exception of finitely many values of* n. *In case* $\tau = \alpha - 1$, *there exists* $\mu_0 > 0$ *such that the same is true provided the constant involved in* (2.7) *is sufficiently small, i.e. provided*

$$\xi_0 = \sup_{0<\varepsilon<1} \varepsilon^{\alpha-1} C_\varepsilon$$

is sufficiently small.

Proof. We will consider the case in which T is self-adjoint. The proof in the general case involves only slight modifications to cover the possibility of complex eigenvalues and non self-adjoint eigenprojections.

By Lemma 2, $T + A$ is closed. Since T is regular, $R_\lambda(T)$ is completely continuous for any $\lambda \in \rho(T)$. Identity (2.3) will then imply that $T + A$ is regular, provided we know that $\| AR_\lambda(T) \| < 1$ for some $\lambda \in \rho(T)$. By (2.6) and (2.7) we have, for $u \in \mathfrak{H}$, $0 < \varepsilon < 1$ and $\lambda \in \rho(T)$,

$$\| AR_\lambda(T) \| \leqq (\varepsilon | \lambda | + C\varepsilon^{-\tau}) \| R_\lambda(T) \| + \varepsilon$$

(cf. (2.4)). Choosing ε so as to minimize the expression in parentheses, we obtain

$$\begin{gathered} \| AR_\lambda(T) \| \leqq c_1 | \lambda |^{-1/(\tau+1)} + c_2 | \lambda |^{\tau/(\tau+1)} \| R_\lambda(T) \| \\ (\lambda \in \rho(T), | \lambda | > c\tau) ; \end{gathered} \tag{2.8}$$

here the constants c_1, c_2 depend only on τ; for $\tau = 0$ we can take $c_1 = 0$.

Since by Lemma 1, $||R_\lambda(T)|| \leqq (\operatorname{Im}\lambda)^{-1}$, we see that $||AR_\lambda(T)|| \leqq$ const. $|\lambda|^{-1/(\tau+1)}$ for purely imaginary λ, so that $||AR_\lambda(T)|| < 1$ for suitable $\lambda \in \rho(T)$. This ensures that $T + A$ is regular.

Consider now the case $\tau < \alpha - 1$. Then $\lambda_n^{\tau/(1+\tau)} = o(n^{\alpha-1}) = o(\min(\lambda_{n+1} - \lambda_n, \lambda_n - \lambda_{n-1}))$. It follows that for any $\mu > 0$, the circles $\Gamma_n(\mu)$ lie outside each other for $n \geqq N_1(\mu)$, and the only point of $\sigma(T)$ lying inside $\Gamma_n(\mu)$ is λ_n. Using (2.5), (2.8), and Lemma 1, we find that, for some $N(\mu) \geqq N_1(\mu)$,

$$(2.9)\quad ||AR_\lambda(T)|| \leqq c_1 |\lambda|^{-1/(\tau+1)} + c_2'\mu^{-1} \leqq c_3\mu^{-1} \qquad (\lambda \in \Gamma_n(\mu),\ n \geqq N(\mu))\ .$$

Henceforth let μ satisfy

$$c_3\mu^{-1} \leqq \frac{1}{3}\ .$$

Let $E(\lambda_n)$ denote the eigenprojection of T corresponding to λ_n, and let $E'_{n,\mu}$ denote the sum of the eigenprojections of $T + A$ corresponding to eigenvalues of $T + A$ lying inside $\Gamma_n(\mu)$. Since $||AR_\lambda(T)|| < 1$ on $\Gamma_n(\mu)$, $n \geqq N(\mu)$, Lemma 2 (ii) shows that $T + A$ has no eigenvalues on $\Gamma_n(\mu)$, so that

$$E'_{n,\mu} - E(\lambda_n) = \frac{1}{2\pi i}\int_{\Gamma_n(\mu)} [R_\lambda(T + A) - R_\lambda(T)]d\lambda\ .$$

Hence by (2.1), (2.3) and (2.9),

$$||E'_{n,\mu} - E(\lambda_n)|| \leqq \frac{c_3\mu^{-1}}{1 - c_3\mu^{-1}} \leqq \frac{1}{2}\ .$$

Therefore ([2, p. 587]) the ranges of $E'_{n,\mu}$ and $E(\lambda_n)$ have the same dimension, namely 1; i.e. each circle $\Gamma_n(\mu)$, $n \geqq N(\mu)$, contains one simple eigenvalue λ_n' of $T + A$.

Next we construct a contour Γ_0 containing the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{N-1}$ only, such that the integral of $||R_\lambda(T + A) - R_\lambda(T)||$ over Γ_0 is small provided $N \geqq N(\mu)$ is sufficiently large. This will show that $T + A$ has $N - 1$ eigenvalues (counting possible multiplicities) inside Γ_0. Since also $R_\lambda(T + A)$ exists for λ outside Γ_0 and all $\Gamma_n(\mu)$, $n \geqq N(\mu)$, the assertion of the theorem about the eigenvalues λ_n' will be established.

For Γ_0 we take the rectangle with sides formed by the lines L_1: $\operatorname{Re}\lambda = \zeta_N = (1/2)(\lambda_{N-1} + \lambda_N)$, some $N \geqq N(\mu)$; L_2: $\operatorname{Re}\lambda = -\zeta_0 < 0$; L_3: $\operatorname{Im}\lambda = \eta_0 > 0$; L_4: $\operatorname{Im}\lambda = -\eta_0$. Consider first

$$\int_{L_1} ||R_\lambda(T + A) - R_\lambda(T)||\, d\lambda \leqq C\int_{-\infty}^{\infty}\Big\{\frac{1}{(x^2 + \zeta_N^2)^{1/(\tau+1)}} + \frac{(x^2 + \zeta_N^2)^{(1/2)\tau/(\tau+1)}}{(x^2 + \delta_N^2)^{(1/2)}}\Big\} \times \frac{dx}{(x^2 + \delta_N^2)^{1/2}}$$

where $\delta_N = (1/2)(\lambda_N - \lambda_{N-1})$. The integral of the first term is easily estimated; the second does not exceed

$$\zeta_N^{-1/(\tau+1)} \int_{-\infty}^{\infty} \frac{(t^2+1)^{(1/2)\tau/(\tau+1)}dt}{t^2+\delta_N^2 \cdot \zeta_N^{-2}} \leqq \zeta_N^{-1/(\tau+1)} \int_{-\infty}^{\infty} \frac{(t^2+1)^{(1/2)\tau/(\tau+1)}dt}{t^2+c\cdot N^{-2}} .$$

Treating separately the ranges $|t| \leqq 1$ and $|t| > 1$ in the latter integral, we readily verify that its value is small for large N. As for the rest of Γ_0, simple calculations show, for suitable choices of ζ_0, η_0, first that the contribution of L_2 is small, and then that the contribution from the sections L_3, L_4 lying between L_1 and L_2 is also small. Thus Γ_0 has the required property.

For the case $\tau = \alpha - 1$, notice that the constants c_1, c_2 in (2.8) are small provided ξ_0 is small. Thus this case can be dealt with in the same way as above, and the proof is complete.

For our next result, the hypotheses about the perturbation A are of a slightly different nature than in Theorem 1. We will suppose that $A = BT^{(\alpha-1)/\alpha}$ where $B = B_1 + B_2$, the sum of a bounded operator B_1 of sufficiently small norm, and a compact operator B_2. Perturbations of this sort have been considered by Turner [11]. From Lemma 3 below we see that such an operator A is T-bounded, and satisfies (2.6) and (2.7), with $\tau = \alpha - 1$.

The operator T^θ (θ real) is defined by means of the functional calculus. Suppose T is a spectral operator with spectral family $\{E_j\}$, such that E_j is one-dimensional for $j \geqq 1$, and $E_0 = \sum_0^k E_{0i}$, each E_{0i} being a finite dimensional projection corresponding to an eigenvalue λ_{0i}. If f is a sufficiently smooth function which is uniformly bounded on the spectrum $\sigma(T)$, then $f(T)$ is defined by the formula (cf. [9])

$$f(T) = \sum_{i=0}^{k} \sum_{m=0}^{\mu_i} \frac{f^{(m)}(\lambda_{0i})}{m!} (T - \lambda_{0i})^m E_{0_i} + \sum_{j=1}^{\infty} f(\lambda_j) E_j \tag{2.10}$$

where μ_i is the algebraic multiplicity of λ_{0i}. In this expression, the first sum, being finite dimensional, plays a rather trivial role in analytic arguments, and we will generally omit details. The following is derived by a simple calculation.

LEMMA 3. *Let T satisfy the above conditions, and let $0 \leqq \theta \leqq 1$. Then there exists a constant $C = C(\theta)$ such that*

$$\| T^\theta u \| \leqq \varepsilon \| Tu \| + C\varepsilon^{-\theta/(1-\theta)} \| u \|$$

for all $u \in \mathfrak{D}(T^\theta)$ and $0 < \varepsilon \leqq 1$.

We also require the following recent result of Kato [5] concerning perturbation of spectral families. By a p-sequence we mean a sequence $\{P_j\}$ of (not necessarily self-adjoint) projections in a Hilbert space $\mathfrak{H}$, satisfying the orthogonality conditions

$$P_j P_k = \delta_{jk} \qquad (j, k \geqq 0) .$$

A p-sequence $\{E_j\}$ is *self-adjoint* if $E_j^* = E_j$ for all j. A self-adjoint p-sequence is *complete* if $\sum E_j = I$.

LEMMA 4 (*Kato*). *Let $\{P_j\}$ be a p-sequence and $\{E_j\}$ a complete self-adjoint p-sequence. Assume that*

(i) $\dim P_0 = \dim E_0 = m < \infty$,

(ii) $\sum_{j=1}^{\infty} \| E_j(P_j - E_j)u \|^2 \leqq c^2 \| u \|^2$

for all $u \in \mathfrak{H}$, where c is a constant, $0 \leqq c < 1$. Then $\{P_j\}$ is similar to $\{E_j\}$, i.e. there exists a nonsingular linear operator W such that for all $j \geqq 0$, $P_j = W^{-1} E_j W$.

The proof of this lemma is fairly simple: set $W = \sum_{j=0}^{\infty} E_j P_j$; one shows that W is well-defined and bounded, and using standard theorems about the index, that nullity $W =$ defect $W = 0$. We refer to [5] for details.

THEOREM 2. *Let T be a regular spectral operator in $\mathfrak{H}$, and suppose the eigenvalues of T satisfy the hypotheses* (2.5) *of Theorem* 1. *Let $A = (B_1 + B_2) T^{(\alpha-1)/\alpha}$ where B_1 is a bounded operator in $\mathfrak{H}$, of sufficiently small norm, and B_2 is a compact operator. Then $T + A$ is a regular spectral operator; moreover the eigenvalues $\{\lambda'_n\}$ of $T + A$ can be enumerated so that λ'_n lies inside the circle $\Gamma_n(\mu)$* (defined in Theorem 1) *for large n.*

Proof. Expressing $AR_\lambda(T)$ by means of the functional calculus, we obtain

$$AR_\lambda(T) = B(\lambda) + (B_1 + B_2) \sum_{j=1}^{\infty} \frac{\lambda_j^{(\alpha-1)/\alpha}}{\lambda_j - \lambda} E(\lambda_j) ,$$

where $\| B(\lambda) \| = O(| \lambda |^{-1})$ as $\lambda \longrightarrow \infty$. (We are assuming, without loss of generality, that no λ_j vanishes.) We will express the sum in two parts, $\sum_1^p + \sum_{p+1}^{\infty}$. In the second of these, we can replace $(B_1 + B_2)$ by $(B_1 + B_2)\widetilde{E}_p$ where $\widetilde{E}_p = \sum_{p+1}^{\infty} E(\lambda_j)$. Since B_2 is a compact operator we have $\| B_2 \widetilde{E}_p \| = \varepsilon_p \longrightarrow 0$ as $p \longrightarrow \infty$. The sum $\sum_1^p$ can be combined with $B(\lambda)$, and we reach the following estimate:

$$\text{(2.11)} \qquad \| AR_\lambda(T) \|^2 \leqq c(\| B_1 \| + \varepsilon_p)^2 \sum_{j=p+1}^{\infty} \frac{| \lambda_j |^{2(\alpha-1)/\alpha} \| E(\lambda_j) \|^2}{| \lambda_j - \lambda |^2} + C_p | \lambda |^{-2} .$$

For $\lambda \in \Gamma_n(\mu)$, the sum in (2.11) is bounded independently of p (a more detailed estimate for this sum appears below). Hence with $\|B_1\| + \varepsilon_p$ sufficiently small, we can choose N so that $\|AR_\lambda(T)\| \leqq \delta < 1$ for $\lambda \in \Gamma_n(\mu)$, $n \geqq N$. By (2.3) this implies that $\|R_\lambda(T + A)\| <$ const. r_n^{-1}. Therefore (with the notation of Theorem 1) we have

$$\|(E'_{n,\mu} - E(\lambda_n))u\| = \left\| \frac{1}{2\pi i}\int_{\Gamma_n(\mu)} R_\lambda(T + A)[I - AR_\lambda(T)]^{-1}AR_\lambda(T)u d\lambda \right\|$$

$$\leqq c \sup_{\lambda \in \Gamma_n(\mu)} \|AR_\lambda(T)u\| \leqq \frac{1}{2}\|u\|$$

provided $\|B_1\|$ is sufficiently small and n sufficiently large. This proves the assertion about the eigenvalues λ'_n.

We pass now to the proof that $T + A$ is spectral. If E_0, $E(\lambda_1)$, $E(\lambda_2)$, $\cdots$ are the spectral projections for T ($E(\lambda_i)$ being one-dimensional), then according to the theorem of Lorch-Mackey-Wermer [12], this family is similar to a complete self-adjoint p-sequence $\{E_j\}$. There is no loss of generality in supposing the similarity to be the identity transformation. By taking $\dim E_0$ large enough we may also suppose that the circles $C_n = \Gamma_n(\mu)$, $n > 0$, are separated, and that their radii satisfy $r_n \geqq c \cdot n^{\alpha - 1}$ (with $c > 0$).

Let P_n denote the eigenprojection of $T + A$ corresponding to λ'_n. We wish to verify that the hypotheses of Kato's lemma are satisfied. First we can show that $\dim E_0 = \dim P_0$ provided sufficiently many of the eigenprojections E_j are included in E_0. The proof is the same as in Theorem 1, modified to utilize the compactness of B_2 in the same way as above.

Next, it is obviously sufficient to show that for some integer N we have

$$\sum_{n=N}^{\infty} \|E_n(P_n - E_n)u\|^2 \leqq c^2 \|u\|^2 \qquad (c^2 < 1)\ .$$

Using (2.11) we have for any integer $p > 1$

$$\begin{aligned}
&\sum_{n=N}^{\infty} \|E_n(P_n - E_n)u\|^2 \\
&\quad \leqq c \sum_{n=N}^{\infty} \sup_{\lambda \in C_n} \Big(\|B_p(\lambda)u\|^2 + (\|B_1\| + \varepsilon_p)^2 \\
&\qquad \cdot \sum_{k=p+1}^{\infty} |\lambda_k|^{2(\alpha-1)/\alpha} |\lambda_k - \lambda|^{-2} \|E_k u\|^2 \Big) \\
&\quad \leqq c_p \Big(\sum_{n=N}^{\infty} |\lambda_n|^{-2} \Big) \|u\|^2 \\
&\qquad + c'(\|B_1\| + \varepsilon_p)^2 \Big[\sum_{n=N}^{\infty} \sum_{p+1 \leqq k \neq n} |\lambda_k|^{2-2/\alpha} |\lambda_k - \lambda_n|^{-2} \|E_k u\|^2 \\
&\qquad + \sum_{n=N}^{\infty} r_n^{-2} |\lambda_n|^{2-2/\alpha} \|E_n u\|^2 \Big]\ .
\end{aligned}$$

The three sums here (from N to ∞) are fairly easily estimated. Assume that p has been chosen, and $\|B_1\| + \varepsilon_p$ is suitably small. Since $\lambda_k \sim ak^\alpha$, the first sum in square brackets can be approximated by

$$\text{const.}\left\{\sum_{k=1}^{\infty} k^{-2}\left[\sum_{1\leqq n\neq k} |1-(n/k)^\alpha|^{-2}\right]\cdot \|E_k u\|^2\right\} \leqq \text{const.}\sum_{k=1}^{\infty}\|E_k u\|^2 ,$$

because by an elementary calculation, the sum in the square brackets here is $O(k^2)$. Since the first and last sums above are trivial to estimate, we finally obtain

$$\sum_{n=N}^{\infty}\|E_n(P_n - E_n)u\|^2 \leqq c^2\|u\|^2$$

where $c^2 < 1$ provided $\|B_1\|$ is small and N large. This completes the proof.

COROLLARY. *Suppose that A and T satisfy the hypotheses of Theorem 1, and that $\tau < \alpha - 1$. Then $T + A$ is a spectral operator.*

Proof. It follows from (2.6) and (2.7) that

$$\|Au\| \leqq C\|Tu\|^{\tau/(\tau+1)}\|u\|^{1/(\tau+1)}, \qquad u \in \mathfrak{D}(T). \tag{2.12}$$

If we assume, as we may without loss of generality, that $\sigma(T)$ lies entirely in the open right half-plane, we can apply a theorem of Krasnoselsky and Sobolevsky [7, Th. 5] to conclude that $AT^{-\sigma}$ is a bounded operator, for any $\sigma > \tau/(\tau+1)$. In particular, we can choose σ such that $\tau/(\tau+1) < \sigma < (\alpha-1)/\alpha$, and write

$$A = BT^{(\alpha-1)/\alpha} \quad \text{with} \quad B = (AT^{-\sigma})(T^{\sigma-/(\alpha-1)/\alpha}).$$

Since T^μ is compact for any $\mu < 0$ (see [7]), we see that B is a compact operator. It follows from the Theorem, therefore, that $T + A$ is spectral.

REMARK. If $\tau < \alpha - 1$ is given, the proof of Theorem 1 will yield explicit constants $C(\tau)$ and $N(\tau)$ such that

$$|\lambda_n' - \lambda_n| < C(\tau)|\lambda_n|^{\tau/(\tau+1)}$$

for $n \geqq N(\tau)$. The same information cannot be derived via the above Corollary, since $\|AT^{-\sigma}\|$ may approach infinity in an unspecified fashion as $\sigma \to \tau/(\tau+1)^+$. The case $\tau = \alpha - 1$ is, of course, not covered at all by the Corollary.

3. Application. Let $I = [x_0, x_1]$ be a finite closed interval, $x_0 < x_1$, and consider the Sobolev space $H^m(I)$ consisting of all $f \in L_2(I)$ having generalized derivatives $D^j f$ also in $L_2(I)$, for $j \leqq m$. The norm in $H^m(I)$ is given by

$$\| f \|_m = \left\{ \sum_{j=0}^{m} \int_I | D^j f(x) |^2 \, dx \right\}^{1/2} .$$

We denote by $H_0^m(I)$ the closure in $H^m(I)$ of $C_0^\infty(I^0)$, the space of infinitely differentiable functions whose support is a compact subset of the open interval (x_0, x_1). If W is any closed subspace such that

$$H_0^{2m}(I) \subset W \subset H^{2m}(I) ,$$

we define an operator T_W in $\mathfrak{H} = L_2(I)$ by

$$\begin{aligned} \mathfrak{D}(T_W) &= W \\ T_W f &= (-1)^m D^{2m} f . \end{aligned} \tag{3.1}$$

Explicit forms of boundary conditions determining W have been studied extensively, cf. [2, Ch. XIII]. In particular, it is known that under quite general conditions T_W is a regular spectral operator, with eigenvalues satisfying (2.5) for $\alpha = 2m$; see [2], [6], and [8] for details.

The perturbing operator A is now defined as the closure of the operator A_0:

$$\begin{aligned} \mathfrak{D}(A_0) &= W \\ A_0 f &= \sum_{k=0}^{2m-1} Q_k(D^k f) , \end{aligned} \tag{3.2}$$

the Q_k denoting arbitrary bounded operators in $\mathfrak{H}$.

LEMMA 5. *Let j, k be nonnegative integers, $j < k$, $k \geqq 2$. Then there exists a constant $C = C_{jk}$ such that for all ε, $0 < \varepsilon < 1$, and all $f \in H^k(I)$,*

$$\begin{aligned} &\left\{ \int_I | D^j f(x) |^2 \, dx \right\}^{1/2} \\ &\qquad \leqq \varepsilon \left\{ \int_I | D^k f(x) |^2 \, dx \right\}^{1/2} + C \varepsilon^{-j/(k-j)} \left\{ \int_I | f(x) |^2 \, dx \right\}^{1/2} . \end{aligned} \tag{3.3}$$

This result can be proved by elementary but tedious calculations; a complete proof (in n dimensions) is given in [1, pp. 17–25]. The following is obvious.

COROLLARY. *There exists a constant C, independent of the operators Q_k, such that for $0 < \varepsilon_i < 1$ $(i = 1, 2, \cdots, 2m-1)$ and $f \in W$,*

$$\begin{aligned} \| Af \| \leqq &\left(\sum_{k=0}^{2m-1} \varepsilon_k \| Q_k \| \right) \| Tf \|_0 \\ &+ C \left(\sum_{k=0}^{2m-1} \| Q_k \| \varepsilon_k^{-k/(2m-k)} \right) \| f \|_0 . \end{aligned} \tag{3.4}$$

THEOREM 3. *Let T_W and A be given by* (3.1) *and* (3.2) *respectively, and assume that T_W is a spectral operator, with eigenvalues $\{\lambda_n\}$ satisfying* (2.5). *Let $\{\lambda'_n\}$ be the eigenvalues of the regular operator $T_W + A$. Assume that $Q_{2m-1} = B_1 + B_2$ where $\|B_1\|$ is sufficiently small and B_2 is a compact operator, and that the remaining coefficients Q_j are bounded operators. Then for large n,*

$$|\lambda'_n - \lambda_n| \leqq c\,|\lambda_n|^{k/2m}\,, \tag{3.5}$$

where k is defined by (1.6). *Moreover $T_W + A$ is a spectral operator.*

Proof. Suppose first that $k \leqq 2m - 2$. Letting $\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon < 1$ in (3.4) we obtain

$$\|Af\| \leqq c_1\varepsilon\,\|Tf\| + c_2\varepsilon^{-k/(2m-k)}\,\|f\|$$

for $f \in \mathfrak{D}(T_W)$. Hence the hypotheses of Theorem 1 are satisfied, with $\tau = k/(2m-k)$, i.e. $\tau + 1 \leqq m = \alpha/2 \leqq \alpha - 1$. Hence the results in this case are immediate consequences of Theorem 1 and the Corollary to Theorem 2.

For the case $k = 2m - 1$, let us write $A_0 = Q_{2m-1}D^{2m-1}$ and $A = A_0 + A_1$. By the first part of the proof, $T_W + A_1$ is a spectral operator with eigenvalues $\{\lambda_{n1}\}$ satisfying (3.5) for $k = 2m - 2$. The eigenvalues $\{\lambda_{n1}\}$ therefore satisfy the hypotheses (2.5) of Theorem 1.

Now we can write $A_0 = (B'_1 + B'_2)T^{(2m-1)/2m}$, where

$$B'_i = B_iD^{2m-1}T^{-(2m-1)/2m}\,.$$

Since $T^{-(2m-1)/2m}$ is a continuous linear map from $L_2(I)$ to $H^{2m-1}(I)$ (cf. [2, Ch. XIII]) and D^{2m-1} is continuous from $H^{2m-1}(I)$ to $L_2(I)$, we see that B'_1 is a bounded operator in $L_2(I)$ with $\|B'_1\| \leqq c\,\|B_1\|$; also B'_2 is compact. An application of Theorem 2 to the operator $T_W + A = (T_W + A_1) + A_0$ then yields the desired conclusions, and the proof is complete.

The author wishes to thank Professor Tosio Kato for several valuable suggestions. The general idea of the proof of Theorem 2 is due to him.

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Received November 29, 1966. Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant Nr. AF-AFOSR 379-65.

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