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POINT NORMS IN THE CONSTRUCTION OF HARMONIC FORMS

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Let V be an arbitrary Riemannian n-space, and V_1 a regular neighborhood of its ideal boundary. Given a harmonic field σ in $\bar{V}_1$, necessary and sufficient conditions are known for the existence in V of a harmonic field ρ which imitates the behavior of σ in V_1 in the sense $\int_{V_1}(\rho-\sigma)\wedge *(\rho-\sigma)<\infty$. In the present paper we give the solution of the corresponding problem for harmonic forms in locally flat spaces.

One aspect of our treatment which may have possibilities for generalization is the use of the point norm defined by $|\varphi|^2 = \varphi_{i_1\cdots i_p}\varphi^{i_1\cdots i_p}$. Another approach to generalizations is discussed in [3].

1. Throughout our presentation the symbol V shall stand for a locally flat Riemannian space. Since the curvature tensor vanishes in V, there exists a covering $\{\bar{U}_a \mid a \in V\}$ of V such that $\bar{U}_a$ is the carrier of local coordinates $x_a = (x_a^1, \cdots, x_a^n)$ with $x_a(a) = 0$ and

$$|x_a| = \sqrt{|x_a^1|^2 + \cdots + |x_a^n|^2} \leqq r_a \qquad (0 < r_a < \infty)$$

in U_a with the following property:

$$g_{ij}(x_a) \equiv \delta_{ij} \qquad (x_a \in \bar{U}_a)\,. \tag{1}$$

We moreover require that V is *parallel* in the sense that the above $\{U_a\}$ can be chosed so as to satisfy

$$x_a^i = x_b^i + c_{ab}^i \qquad (i = 1, \cdots, n) \tag{2}$$

in $\bar{U}_a \cap \bar{U}_b$ with constants c_{ab}^i. We call $(\bar{U}_a \mid a \in V\}$ *a parallel coordinate covering* and each U_a *a distinguished coordinate neighborhood.*

2. The space of harmonic p-forms φ, defined by $d\delta\varphi + \delta d\varphi = 0$, will be denoted by H_p. For a set $E \subset V$, the notation $\varphi \in H_p(E)$ shall mean that φ is a harmonic p-form in an open set containing E.

Let $\bar{V}_1$ be the complement in V of a regular subregion [4] of V. Suppose $\sigma \in H_p(\bar{V}_1)$ is given. The problem is to construct a corresponding $\rho \in H_p(V)$, to be called the *principal form*, characterized by the existence of a constant M such that

$$|\rho - \sigma| < M < \infty \tag{3}$$

on V_1.

The space V is called *hyperbolic* or *parabolic* according as it does or does not possess Green's functions [4].

THEOREM 1. *If V is hyperbolic, then the principal form ρ always exists.*

THEOREM 2. *If V is parabolic, then a necessary and sufficient condition for the existence of a principal form ρ is that*

$$\int_\beta *d < \sigma, c > = 0 \tag{4}$$

for every constant form c. The principal form is unique up to an additive constant form.

Here $\langle\varphi, \psi\rangle = \varphi_{i_1\cdots i_p}\psi^{i_1\cdots i_p}$, and β stands for the ideal boundary of V. For constant forms see No. 4 below.

The above theorems will be consequences of the main existence theorem for harmonic forms (No. 7), which we shall first establish.

Theorem 1 is known to be valid without the assumption that V is parallel ([3]).

3. Take a p-form φ on V:

$$\varphi = {}_a\varphi_{i_1\cdots i_p} dx_a^{i_1} \wedge \cdots \wedge dx_a^{i_p} .$$

In $U_a \cap U_b$, $dx_a^i = dx_b^i$ and therefore

$$ {}_a\varphi_{i_1\cdots i_p} = {}_b\varphi_{i_1\cdots i_p} .$$

For this reason there exists a *global function* $\varphi_{i_1\cdots i_p}$ in V such that

$$\varphi_{i_1\cdots i_p} \equiv {}_a\varphi_{i_1\cdots i_p}$$

in $\bar{U}_a$. Conversely, given functions $\varphi_{i_1\cdots i_p}$, there exists a p-form $\varphi = {}_a\varphi_{i_1\cdots i_p} dx_a^{i_1} \wedge \cdots \wedge dx_a^{i_p}$ with $\varphi_{i_1\cdots i_p} \equiv {}_a\varphi_{i_1\cdots i_p}$ in each $\bar{U}_a$.

4. We call φ a *constant p-form* if

$$\Delta\varphi = 0 , \tag{5}$$

$$|\varphi| = \text{const.}, \tag{6}$$

and we denote by K^p the class of constant p-forms. It is easy to see that

$$d\varphi = 0, \delta\varphi = 0$$

for $\varphi \in K^p$, i.e., constant forms are harmonic fields. If $\varphi \in H_p(V)$ and $|\varphi|$ is constant in some open set $D \subset V$, than $\varphi \in K^p(V)$. In fact, let

$$\varphi = \varphi_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} .$$

Then $\Delta\varphi = (\Delta\varphi_{i_1 \cdots i_p}) dx^{i_1} \wedge \cdots \wedge dx^{i_p} = 0$, and we see that each $\varphi_{i_1 \cdots i_p}$ is harmonic. Consequently $(\varphi_{i_1 \cdots i_p})^2$ is subharmonic, and so is

$$|\varphi|^2 = \sum_{i_1 < \cdots < i_p} (\varphi_{i_1 \cdots i_p})^2 .$$

Since $|\varphi|^2 = c$ (const.) in D, we have

$$c - (\varphi_{i_1 \cdots i_p})^2 = \sum_{j \neq i} (\varphi_{j_1 \cdots j_p})^2$$

in D. The left-hand member is subharmonic and superharmonic and the same is true of $(\varphi_{i_1 \cdots i_p})^2$. But $\Delta(\varphi_{i_1 \cdots i_p})^2 = |\operatorname{grad} \varphi_{i_1 \cdots i_p}|^2$, and for this reason $\varphi_{i_1 \cdots i_p}$ must be constant.

Clearly K^p is an $\binom{n}{p}$ -dimensional vector space.

5. Let L^p be the operator in the space of p-forms on $\alpha_1 = \partial V_1$ into the space of continuous p-forms in $\bar{V}_1$, harmonic in V_1, such that $L^p\varphi | \alpha_1 = \varphi$ and

$$L^p(\lambda\varphi_1 + \mu\varphi_2) = \lambda L^p\varphi_1 + \mu L^p\varphi_2 , \tag{7}$$

$$|L^p\varphi| \leqq \sup_{\alpha_1} |\varphi| , \tag{8}$$

$$\int_{\alpha_1} * d < L^p\varphi, c > = 0 \quad \text{for every} \quad c \in K^p . \tag{9}$$

We call L^p a *normal operator.*

A normal operator L for 0-forms induces one for p-forms:

$$L^p\varphi = (L\varphi_{i_1 \cdots i_p}) dx^{i_1} \wedge \cdots \wedge dx^{i_p} .$$

More interesting is the following. Let ${}_{i_1 \cdots i_p}L$ be normal operators for 0-forms, with $i_1 < \cdots < i_p$. We define one for p-forms by setting

$$L^p = {}_{i_1 \cdots i_p}L dx^{i_1} \wedge \cdots \wedge dx^{i_p} ,$$

that is

$$L^p\varphi = ({}_{i_1 \cdots i_p}L\varphi_{i_1 \cdots i_p}) dx^{i_1} \wedge \cdots \wedge dx^{i_p} .$$

In particular, if ${}_{i_1 \cdots i_p}L = L_0$ or L_1 for all $i_1 < \cdots < i_p$, we denote the corresponding L^p by L_0^p or L_1^p.

6. Given a compact set E in V let $F_E^p \subset H^p$ be the class of harmonic p-forms φ in V such that $\langle\varphi, c\rangle$ is not of constant sign in

E except for being identically zero for every $c \in K^p$. Observe that F_E^p is closed with respect to uniform convergence in terms of $|\cdot|$ on compact sets. In fact,

$$|\langle\varphi_n, c\rangle - \langle\varphi_m, c\rangle| = |\langle\varphi_n - \varphi_m, c\rangle| \leqq |c|\,|\varphi_n - \varphi_m|\,.$$

We shall need the following generalization of the q-lemma for 0-forms [4]:

LEMMA. *There exists a constant* q_E $(0 < q_E < 1)$ *such that*

$$\max_E |\varphi| \leqq q_E \sup_V |\varphi|$$

for all $\varphi \in F_E^p$.

We only have to consider forms φ with $\sup_V |\varphi| = 1$. Suppose there existed a sequence with $\max_E |\varphi_n| \nearrow 1$. Then since $\{\varphi \mid \sup_V |\varphi| = 1\}$ is a normal family, we would have $\varphi = \lim \varphi_n$ with $\max_E |\varphi| = 1$. By the subharmonicity of $|\varphi|^2$, φ would be a constant form c on V. The contradiction $\langle\varphi, c\rangle = \langle\varphi, \varphi\rangle = 1$ completes the proof.

7. With the scene so set for $p \geqq 0$, we can state the following generalization to p-forms of the main existence theorem known thus far for 0-forms only [4]:

THEOREM 3. *The principal form* $\rho \in H_p(V)$ *characterized by*

$$L(\rho - \sigma) = \rho - \sigma \tag{10}$$

exists if and only if

$$\int_\beta * d\langle\sigma, c\rangle = 0 \tag{11}$$

for all $c \in K^p$. *The principal form is unique up to an additive constant form.*

The proof is analogous to that for 0-forms [4] and we can restrict ourselves to a brief outline.

Let $V_0 \subset V$ be a regular region with $\partial V_0 \subset V_1$ and $\partial V_1 \subset V_0$. Denote by L' the Dirichlet operator for V_0. We only have to establish the convergence of $\varphi = \sum_{n=0}^{\infty}(LL')^n \sigma_0$, where $\sigma_0 = \sigma - L\sigma$ and $L = L^p$.

Observe that condition (11) means that $\int_\alpha * d\langle\sigma, c\rangle = 0$ for every α homologous to α_1, since $\langle\sigma, c\rangle$ is a harmonic function. We conclude that

$$\int_{\partial V_1} \langle L'(LL')^n \sigma_0, c\rangle * dh = 0,$$

where h is the harmonic measure of ∂V_0 in $\bar{V}_0 \cap \bar{V}_1$. For this reason $L'(LL')^n\sigma_0 \in F^p_{\partial V_1}(V_0)$, the lemma applies in V_0, and we have the convergence.

Theorem 2 is a consequence of Theorem 3.

8. To prove Theorem 1 suppose V is hyperbolic. The form $\sigma \in H^p(\bar{V}_1)$ may or may not satisfy (11). We set

$$\psi = \sum\left[\left(-\int_{\partial V_1} * d\sigma_{i_1\cdots i_p}\right)\Big/\left(\int_{\partial V_1} * d\omega\right)\right]\omega dx^{i_1} \wedge \cdots \wedge dx^{i_p} ,$$

where $\sigma = \sigma_{i_1\cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ is the global expression in $\bar{V}_1$ and ω is the harmonic measure of the ideal boundary β of V with respect to V_1. Clearly $|\psi|$ is bounded in V_1. Consequently, $\tilde{\sigma} = \sigma + \psi$ satisfies (11) and the solution ρ satisfies

$$\rho - \sigma = L^p(\rho - \tilde{\sigma}) + \psi$$

on V_1. We infer that $|\rho - \sigma|$ is bounded in V_1.

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