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HOMOLOGICAL DIMENSIONS AND MACAULAY RINGS

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This paper shows some instances where properties of a local ring are closely connected with the homological properties of a single module. Particular stress is placed on conditions implying the regularity or the Cohen-Macaulay property of the ring.

First it is proved that the regularity of the local ring R is equivalent to the finiteness of the projective or injective dimensions of a nonzero module mA, where m is the maximal ideal of R and A a finitely generated R-module. Next it is shown that over Gorenstein rings the finiteness of the projective or injective dimension are equivalent notions. Then, using some change of rings, a theorem is strengthened on embedding modules of finite length into cyclic modules over certain Macaulay rings. Finally, to mimic the equivalent statement for projective dimension, it is shown that the annihilator of a module finitely generated and having finite inactive dimension must be trivial if it does not contain a nonzero divisor.

The rings considered in this paper will be assumed commutative and noetherian and as a general proviso all unspecified modules will be assumed finitely generated. For the notations and basic facts used here [2] is the standard reference.

1. A homological characterization of regular local rings. Among the local rings the regular ones are characterized as those having finite global dimension (see [2]). If the maximal ideal of the local ring R is denoted by m and $k = R/m$ is the corresponding residue field, it is even possible to test the regularity of R by looking at the projective dimensions of m or k only. Theorem 1 of this section shows that it is enough to consider any power of m, in fact any module of the form mA.

We take for granted the basic things on minimal projective resolutions of modules, just recalling that it means the following: An exact sequence

$$\cdots \xrightarrow{d} X_1 \xrightarrow{d} X_0 \longrightarrow A \longrightarrow 0$$

where the X_i are free $d(X_i) \subset mX_{i-1}$. It is then said to be a minimal projective resolution of the module A. It follows easily then that A has finite projective dimension if there exists an integer n so that $\operatorname{Tor}_i^R(k, A) = 0$ for all $i > n$.

We will prove Theorem 1 from the following lemma:

LEMMA. *Let C_i, $i = 0, 1, \cdots$ be a complex of R-modules such that $d(C_i) \subset mC_{i-1}$ for all i's and such that $H_p(mC) = H_{p-1}(mC) = 0$ for some integer p. Then $mC_p = 0$.*

Proof. Let L_p be the kernel of $d: C_p \to C_{p-1}$. Then the kernel of the induced map $mC_p \to mC_{p-1}$ is just $L_p \cap mC_p$. Hence

$$H_p(mC) = (L_p \cap mC_p)/md(C_{p+1}) .$$

By assumption $d(C_{p+1}) \subset L_p \cap mC_p$ but since $H_p(mC) = 0$, this says that $d(C_{p+1}) \subset md(C_{p+1})$ and thus, by the Nakayama lemma, $d(C_{p+1}) = 0$. Since $H_{p-1}(mC) = 0$, the same argument shows that $d(C_p) = 0$. So $H_p(mC) = mC_p = 0$.

THEOREM 1.1. *Let R be a local ring with maximal ideal m and suppose there exists an R-module A such that mA is different from 0 and has finite projective dimension. Then R is a regular local ring.*

Proof. Let $\{X_i\}$, $i = 0, 1, \cdots$ be a minimal resolution of k. Then

$$\operatorname{Tor}_i (k, mA) = H_i(X \otimes mA) .$$

Since the X_i are all free modules, $X \otimes mA$ may be regarded as a subcomplex of $X \otimes A$. In fact, $X \otimes mA = m(X \otimes A)$.

By definition of a minimal resolution, $d(X_i) \subset mX_{i-1}$ for all i, and thus $d \otimes 1(X_i \otimes A) \subset m(X_{i-1} \otimes A)$ for all i. We can then apply the lemma with $C = X \otimes A$. Namely, since mA has finite projective dimension, $\operatorname{Tor}_i(k, mA) = 0$ for large i. But

$$\operatorname{Tor}_i (k, mA) = H_i(X \otimes mA) = H_i(m(X \otimes A))$$

and so, $m(X_i \otimes A) = 0$ for large i, a hypothesis which is substained only if $X_i = 0$. The regularity of R follows then from [2].

Obviously the same procedures, using Ext now, work just as well for finite injective dimension.

2. Modules over Gorenstein rings and a special case of a conjecture. Let R be a local ring and A a nonzero, finitely generated R-module of finite injective dimension. This dimension is then necessarily equal to the codimension of R [2]. In [4] it was conjectured that R is then, a Macaulay ring, a contention we have been able to prove only in dimension one. This we shall do in this section, but first we make some remarks.

Say $d =$ codimension $R =$ inj. dim. A. Then for an R-module B $\operatorname{Ext}^d (B, A) \neq 0$ if and only if m is associated to B. In fact, if x is an element of m which is a nonzero divisor with respect to B, the exact sequence

$$0 \longrightarrow B \xrightarrow{x} B \longrightarrow B/xB \longrightarrow 0$$

gives rise to the epimorphism $\mathrm{Ext}^d(B, A) \xrightarrow{x} \mathrm{Ext}^d(B, A) \longrightarrow 0$, and thus, by the Nakayama lemma, one has $\mathrm{Ext}^d(B, A) = 0$. Conversely, if m is associated to B we can see that $\mathrm{Ext}^d(B, A) \neq 0$. Reasoning along these lines we see that the length of the longest B-sequence, which we call the depth of B, is given by $d - r$ where r is the largest integer for which $\mathrm{Ext}^r(B, A) \neq 0$. In the current fashion we shall denote the depth of B by $\mathrm{Prof}_R B$. In particular we have that the depth of any module is at most equal to the codimension of the ring. Now it is a long standing conjecture that this happens only when R is a Macaulay ring. When codimension $R = 0$ it follows immediately that R is Artinian. We give a proof of the case $d = 1$ as a consequence of the following lemma.

LEMMA. *Let $f\colon R \to R'$ be a local homomorphism of local rings making R' a finitely generated R-module. For any R'-module A*

$$\mathrm{Prof}_R A = \mathrm{Prof}_{R'} A\ .$$

Proof. By induction on $\mathrm{Prof}_R A$. It is obvious that we may assume $R \subseteqq R'$. If m, the maximal ideal of R, is associated to A, i.e. if $\mathrm{Prof}_R A = 0$, then there exists $a \neq 0$ in A with $ma = 0$ or $mR'a = 0$. Since mR' is primary for the maximal ideal m' of R' we have that $(m')^n \subseteqq mR'$ for some integer n. Thus $(m')^n a = 0$ and obviously m' is associated to A, i.e. $\mathrm{Prof}_{R'} A$ is also 0. If m is not associated to A let $x \in m$, and so also an element of m', be a nonzero divisor of A. Thus by induction $\mathrm{Prof}_R A/xA = \mathrm{Prof}_{R'} A/xA$ and the conclusion follows.

THEOREM 2.1. *Let R be a local ring and A a finitely generated R-module of injective dimension one. Then R is a Macaulay ring.*

Proof. It is not hard to see that to test the finiteness of the injective dimension of a module A over a local ring R it is enough to check the nullity of $\mathrm{Ext}^n(k, A)$ for high n's. Thus it follows that $\mathrm{inj.\,dim.}_R A = \mathrm{inj.\,dim.}_{\bar{R}} \bar{A}$, where $\bar{R}$ and $\bar{A}$ denote the respective completions of R and A with respect to the m-adic topology. Assume thus that R is a complete local ring. Suppose, by way of contradiction, that Krull $\dim R > 1$ and let p be a prime ideal such that $\dim R/p > 1$. Let S be the integral closure of R/p. By [8] S is a finitely generated $R/p-$ and so R-module. Besides it is a local ring. Since S is integrally closed of $\dim > 1$, $\mathrm{Prof}_S S \geqq 2$, which is a contradiction by the previous lemma.

Before stating the next result we remark that if A is an R-module over the local ring R and $\operatorname{Prof} A <$ codimension R, and

$$0 \longrightarrow L \longrightarrow F \longrightarrow A \longrightarrow 0$$

is an exact sequence with F free, then $\operatorname{Prof} L = 1 + \operatorname{Prof} A$.

Gorenstein rings [4] in the local case are those rings R with finite self-injective dimension. Thus, clearly, any module of finite projective dimension has also finite injective dimension. The converse is proved in

THEOREM 2.2. *Let R be a local Gorenstein ring. Then any module of finite injective dimension also has finite projective dimension.*

Proof. Let A be such a module and map a free module over it

$$(1) \qquad 0 \longrightarrow L \longrightarrow F \longrightarrow A \longrightarrow 0 .$$

Thus L also has finite injective dimension. If $\operatorname{Prof} A <$ codimension R by the previous remark $\operatorname{Prof} L = 1 + \operatorname{Prof} A$. We can then assume that A already has maximum Prof. Our claim is then that (1) splits, i.e. that A is free. Taking $\operatorname{Hom}(A, \ )$ of the above sequence we get $0 \to \operatorname{Hom}(A, L) \to \operatorname{Hom}(A, F) \to \operatorname{Hom}(A, A) \to \operatorname{Ext}^1(A, L)$. Since L has finite injective dimension and $\operatorname{Prof} A =$ codimension R we get from the initial remarks of this section that $\operatorname{Ext}^1(A, L) = 0$ and the conclusion follows.

3. Change of local rings and a generalization of a theorem of Auslander. Given a homomorphism of two rings, $f: R \to S$, the S-modules can be considered, via f, as R-modules. For an S-module A the change of rings problem consists in comparing the various homological invariants attached to A (e.g. projective dimension, injective dimension) relative to both R and S. Usually it is the case that any such information relative to S is easier to relate to R than the other way around. Here we examine a case where nevertheless a complete answer is possible and use the method employed to generalize a theorem of [1].

Let R be a local ring and x a nonzero divisor in $m - m^2$. Put $S = R/(x)$ and let A be a finitely generated S-module.

THEOREM 3.1. $\text{inj. dim.}\,{}_R A = 1 + \text{inj. dim.}\,{}_S A$.

Proof. Since $\text{proj. dim.}\,{}_R S = 1$, the spectral sequence [5]

$$\operatorname{Ext}_S^p(\operatorname{Tor}_q^R(S, m), A) \longrightarrow \operatorname{Ext}_R^n(m, A)$$

yields the exact sequence

$$\cdots \operatorname{Ext}_S^{n-2}(\operatorname{Tor}_1^R(S, m), A) \longrightarrow \operatorname{Ext}_S^n(S \otimes m, A) \longrightarrow \operatorname{Ext}_R^n(m, A) \longrightarrow \operatorname{Ext}_S^{n-1}(\operatorname{Tor}_1^R(S, m), A) \cdots .$$

But $\operatorname{Tor}_1^R(S, m) = 0$ for x is not a zero divisor and thus we have the isomorphism

$$(2) \qquad \operatorname{Ext}_S^n(m/xm, A) = \operatorname{Ext}_R^n(m, A) .$$

Before proceeding with the proof we need a lemma to be found in [6]:

LEMMA. *If $x \in m - m^2$ we have the decomposition*

$$m/xm = m/(x) \oplus (x)/xm .$$

Proof. Let $x, x_2, \cdots, x_r$ be a minimal generating set for m and let $U = xm + (x_2, \cdots, x_r)$. Then $U + (x) = m$ and $U \cap (x) = xm$ by the independence mod m of the x's. This shows that the inclusion $0 \rightarrow (x)/xm \rightarrow m/xm$ splits, whence the conclusion.

Replacing m/xm by this direct sum decomposition in (1) and bearing in mind that to test injective dimension it is enough to consider m or k in the Ext functors, we get that the injective dimensions of A (with respect to R and S) are both finite or both infinite. The equality mentioned follows from the identification with the codimensions.

REMARKS. By considering a similar spectral sequence we could prove the corresponding statement for projective dimension. This is however done, by elementary means, in Nagata's book [8].

If R is a local Gorenstein ring and

$$0 \longrightarrow R \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_d \longrightarrow 0$$

is a minimal injective resolution of R, it is proved in [4] that E_d is really the injective envelope of the R-module k. This fact, coupled with a theorem of Bourbaki, is used by Auslander in [1, Th. C] to embed modules of finite length into cyclic modules over certain Gorenstein rings. He proves that this can always be done over integrally closed Gorenstein rings of Krull dimension > 1. We will give a construction which enables us to substitute Macaulay for Gorestein in the previous statement.

We shall construct now, for any local Macaulay ring R, a finitely generated module A with finite injective dimension and such that the last nonzero term in a minimal resolution of A is simply $E(k)$, the injective envelope of k.

Let $R_i = R/(x_1, \cdots, x_i)$, $i = 0, 1, \cdots, d$, where $x_1, \cdots, x_d$ form a system of parameters contained in $m - m^2$. Let A be the injective envelope of k over the ring R_d. Then A is finitely generated and has finite injective dimension over R, by repeated application of Theorem 3.1. We claim that A has the other desired property which is equivalent to saying that $\operatorname{Ext}_R^d(k, A)$ is one-dimensional over k. Using the formula (1) we get

$$\operatorname{Ext}_R^n(k, A) = \operatorname{Ext}_{R_1}^n(k, A) \oplus \operatorname{Ext}_{R_1}^{n-1}(k, A) .$$

Letting $n = d$ we get for arbitrary i

$$\operatorname{Ext}_R^d(k, A) = \operatorname{Ext}_{R_i}^{d-i}(k, A)$$

because $\operatorname{inj\,dim}_{R_i} A = d - i$. This works as long as $d - i > 0$ and for the remaining case it is easy to verify directly, using the injectiveness of A over R_d and the decomposition provided by the lemma, that $\operatorname{Ext}_{R_{d-1}}^1(k, A) = \operatorname{Hom}(k, A) = k$, and we have what we wanted.

Now we can copy the proof of [1, Th. C] to get

THEOREM 3.2. *Let R be an integrally closed Macaulay ring such that any maximal ideal has rank > 1. Then any module M of finite length can be embedded in a cyclic module of which it is an essential extension.*

4. The annihilator of a module of finite injective dimension. It was established in [3] that the annihilator of a finitely generated module over a local ring, with finite projective dimension, is either trivial or contains a nonzero divisor. The corresponding statement for injective dimensions is still true although the proof is considerably more involved. The difficulty stems from the diversity of injective modules compared with the projective ones. The purpose of this section is to prove the following

THEOREM 4.1. *Let R be a local ring and A a finitely generated R-module. If A has finite injective dimension then its annihilator is either trivial or contains a nonzero divisor.*

Proof. We are going to show that a reduction can be made to the dimension one case but first let us take care of a trivial situation. Denote by I the annihilator of A. Assume I to be nontrivial and consisting entirely of zero divisors. This means that the annihilator J of I in R is nonzero. Suppose some prime ideal p of (0) contains both I and J. In this case $J_p \neq R_p$ and so $I_p \neq (0)$ also. But A_p is a nonzero module of finite injective dimension over R_p, a ring of codimension 0, i.e. A_p is an injective module and thus R_p is Artinian by a previous remark. But then it is clear that A_p is a faithful R_p-module being a direct sum of faithful modules. Now we make the reduction to the dimension one case. For that we need a lemma.

LEMMA (*Abhyankar-Hartshorne*). *Let I and J be nonzero ideals in a commutative ring such that $I. J = (0)$. Then the length of the maximal R-sequence in $I + J$ is at most one.*

Proof (Kaplansky). In general we call the length mentioned above for an arbitrary ideal K, the grade of K. In the above conditions the lemma says that grade $(I + J) \leqq 1$. Can assume that $I \cap J = (0)$ for otherwise, if $0 \neq x \in I \cap J$ then $x(I + J) = (0)$ thus showing grade $(I + J) = 0$. Can even assume that R has a unique maximal ideal. Let $a = i + j$, $i \in I$ and $j \in J$, be a nonzero divisor; clearly $i \neq 0$, $j \neq 0$. Also, $i \notin R(i + j)$ for an equation $i = r(i + j)$ gives $(1 - r)i = rj$, a contradiction whether r is a unit or not. Finally $(I + J)i \subseteqq R(i + j)$, i.e. grade $(I + J) \leqq 1$.

In order to apply this to our question let J' be the annihilator of J in R. Then $I \subseteqq J'$, and $J. J' = (0)$. By the lemma, grade $(J + J') \leqq 1$, in fact $= 1$, for otherwise I and J would be inside the same minimal prime. Let p be a grade one prime containing $J + J'$; it is easily seen that R_p has codimension one. We claim that $J_p \neq (0)$—thus implying that I_p, which in not zero, consists entirely of zero divisors. Otherwise $J'_p = R_p$ which is impossible. This is the required reduction. We can then make a fresh start and assume that A has injective dimension one over the local ring R, of maximal ideal m. We also know that, by Theorem 2.1 that R has Krull dimension one. This fact will be useful to see what is happening. There are two cases to cope with.

Case 1. m is not associated to A.

Here the only primes associated to A are the minimal primes of R containing I. Let

$$0 \longrightarrow A \longrightarrow E_0 \longrightarrow E \longrightarrow 0 \tag{1}$$

be a minimal injective resolution of A. E_0 is a direct sum of copies of $E(R/p) =$ the injective envelope of R/p, for the various primes of rank 0 containing I. E on the other hand is a direct sum of copies of $E(R/m)$. Let $p_1, \cdots, p_r$ be the above mentioned minimal primes. We can then pick x in some other minimal prime (one containing J) but not in $p_1 \cdots p_r$. Map the exact sequence (1) into itself by the multiplication induced by x and look at the kernels and cokernels sequence (where ${}_xN$ denotes the kernel of $N \xrightarrow{x} N$)

$$0 \longrightarrow {}_xA \longrightarrow {}_xE_0 \longrightarrow {}_xE \longrightarrow A/xA \longrightarrow E_0/xE_0 \longrightarrow E/xE \longrightarrow 0 .$$

Since x is not in any of the p's it acts as a unit on any $E(R/p)$ and thus

$${}_xE = A/xA .$$

But ${}_xE$ is an injective $R/(x)$ module [4] and thus $0 \neq A/xA$ is a finitely generated injected module over $R/(x)$. But as was remarked before, $R/(x)$ is then an Artinian ring, which is a contradiction for x was taken in a minimal prime.

Case 2. m is associated to A.

With the same notations of Case 1, a minimal injective resolution of A now looks like

$$(2) \qquad 0 \longrightarrow A \longrightarrow E_0 \oplus E(k)^r \longrightarrow E(k)^s \longrightarrow 0$$

where the integers r and s are determined by $r = \dim_k \operatorname{Hom}(k, A)$ and $s = \dim_k \operatorname{Ext}^1(k, A)$. Since m is associated to A, $r > 0$. Let p be a prime not represented in E_0, i.e. p is a minimal prime containing J. From (2) we get the exact sequence

$$(3) \qquad \begin{aligned} 0 \longrightarrow \operatorname{Hom}(R/p, A) \longrightarrow \operatorname{Hom}(R/p, E_0 \oplus E(k)^r) \\ \longrightarrow \operatorname{Hom}(R/p, E(k)^s) \longrightarrow 0 , \end{aligned}$$

for $\operatorname{Ext}^1(R/p, A) = 0$ since $\operatorname{Prof} R/p = 1$. Another way to write (3) is

$$0 \longrightarrow {}_pA \longrightarrow {}_pE_0 \oplus {}_pE(k)^r \longrightarrow {}_pE(k)^s \longrightarrow 0 .$$

But ${}_pE_0 = 0$ for p is not contained in any of the primes of E_0. Thus we get

$$0 \longrightarrow {}_pA \longrightarrow {}_pE(k)^r \longrightarrow E(k)^s \longrightarrow 0$$

or, in other words, that ${}_pA$ is a nonzero module of finite length and injective dimension one over the ring R/p. From the duality theory [7] it follows that $r = s$.

Define $\rho(C)$ for any R-module of finite length to be

$$\rho(C) = \text{length} \operatorname{Hom}(C, A) - \text{length} \operatorname{Ext}^1(C, A) .$$

It is easy to see that if $0 \to C' \to C \to C'' \to 0$ is an exact sequence of modules of finite length, then $\rho(C) = \rho(C') + \rho(C'')$. So since any such module is an extension of simple modules, i.e. of k's, we have that $\rho(C) = 0$ or that length $\operatorname{Hom}(C, A) =$ length $\operatorname{Ext}^1(C, A)$.

Let A_0 be the largest submodule of A of finite length, i.e. the largest submodule of A annihilated by some power of m. Also let x be a nonzero divisor in R such that $xA_0 = 0$. Consider the exact sequence

$$(4) \qquad 0 \longrightarrow A_0 \longrightarrow A \longrightarrow \bar{A} \longrightarrow 0 .$$

By hypothesis $\bar{A} \neq 0$ for A is not of finite length and it is clear that m is not associated to $\bar{A}$. The exact sequence induced by multiplications by x gives

$$0 \longrightarrow A_0 \longrightarrow {}_xA \longrightarrow {}_x\bar{A} \longrightarrow A_0 \longrightarrow A/xA \longrightarrow \bar{A}/x\bar{A} \longrightarrow 0 .$$

But ${}_x\bar{A} = (0)$ and thus length (A_0) + length $(\bar{A}/x\bar{A})$ = length (A/xA). On the other hand the exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0$$

gives easily that $\mathrm{Hom}\,(R/(x), A) = A_0$ and $\mathrm{Ext}^1\,(R/(x), A) = A/xA$. But $R/(x)$ has finite length and so $\rho(R/(x)) = 0$ which implies length $(\bar{A}/x\bar{A}) = 0$, a contradiction by the Nakayama lemma.

REFERENCES

1. M. Auslander, *Remarks on a theorem of Bourbaki*, Nagoya Math. J. **27** (1966), 361-369.
2. M. Auslander and D. A. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc. **85** (1957), 390-405.
3. ———, *Codimension and multiplicity*, Ann. of Math. **68** (1958), 625-657.
4. H. Bass, *On the ubiquity of Gorenstein rings*, Math. Zeitchr. **82** (1963), 8-28.
5. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, 1956.
6. I. Kaplansky, *Homological dimensions of rings and modules*, Univ. of Chicago, 1959 (mimeographed notes).
7. E. Matlis, *Injective modules over noetherian rings*, Pacific J. Math. **8** (1958), 511-528.
8. M. Nagata, *Local rings*, Interscience, New York, 1962.

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