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REPRODUCING KERNELS IN SEPARABLE HILBERT SPACES

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A theorem on the existence of a reproducing kernel in a separable Hilbert space of functions is proved. As an application of this theorem, a method of interpolation of the functions in a separable Hilbert space with a reproducing kernel is given. This method is used to construct the elements of the Hilbert space generated by a second order stochastic process, in case this space is separable.

Theorems 2, 3 and 4 of this paper, which were motivated by Parzen's work [2], [3], were originally proved in somewhat different form in collaboration with J. Ricatte [4]. In this paper it will be shown that these three theorems are the consequences of a more general statement given in what follows as Theorem 1.

1. **Preliminaries.** Let $\mathfrak{H}$ be a Hilbert space of real or complex functions defined on an arbitrary set T. The scalar product of any ordered pair of functions f, g in $\mathfrak{H}$ will be denoted by $\langle f, g\rangle$ and the norm of a function $f \in \mathfrak{H}$ by $\|f\|$. A two variable function K defined on the product set $T \times T = T^2$ is the reproducing kernel of $\mathfrak{H}$, if it satisfies the following two conditions:

(A) $K(t, \cdot) \in \mathfrak{H}, \forall t \in T$.

(B) $\langle f, K(t, \cdot)\rangle = f(t), \forall t \in T$ and $\forall f \in \mathfrak{H}$.

The last property is called reproduction property of K[1].

K is self-reproducing, i.e. $K(t, \tau) = \langle K(t, \cdot), K(\tau, \cdot)\rangle$. It is positive-semi-definite, i.e.

$$\sum_{i,j\in I} \lambda_i \bar{\lambda}_j K(t_i, t_j) = \left\|\sum_{i\in I} K(t_i, \cdot)\right\|^2 > 0, \lambda_i \in C, \forall_i \in I \subset N .$$

(where C is the set of complex numbers, I an arbitrary finite subset of the set N of positive integers and $\bar{\lambda}_j$ the conjugate of λ_j). In particular, K has the Hermitian symmetry $(K(t, \tau) = \bar{K}(\tau, t), \forall t, \tau \in T)$ and

$$0 \leqq \| K(t, \cdot) \|^2 = K(t, t) < \infty, \forall t \in T .$$

If $\mathfrak{H}$ has a reproducing kernel, this kernel is always unique, for if K and K' were two distinct reproducing kernels of $\mathfrak{H}$, their reproduction property would imply

[1] For a more general and detailed presentation of the Theory of Reproducing Kernels, see the article by Aronzajn [**1**].

$$K(t, \tau) = \langle K(t, \cdot), K'(\tau, \cdot)\rangle = \overline{\langle K'(\tau, \cdot), K(t, \cdot)\rangle} = \bar{K}'(\tau, t) = K'(t, \tau) .$$

The weak convergence (consequently the strong convergence) of a sequence $\{f_n\} \subset \mathfrak{H}$ to a function $f \in \mathfrak{H}$ implies its pointwise convergence to the same function f, for

$$\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \langle f_n, K(t, \cdot)\rangle = \langle f, K(t, \cdot)\rangle = f(t) .$$

If a topology is defined on T, then the continuity of K with respect to the product topology on T^2 implies the continuity of each function in $\mathfrak{H}$. This is the consequence of the Schwarz inequality applied to (B):

$$\begin{aligned} |f(t) - f(t_0)|^2 &= |\langle f, K(t, \cdot) - K(t_0, \cdot)\rangle|^2 \\ &\leqq \|f\|^2[K(t, t) - K(t, t_0) - K(t_0, t) + K(t_0, t_0)] . \end{aligned}$$

Given a finite and positive-semi-definite function K on T^2, there exists a uniquely defined Hilbert space of functions on T, whose reproducing kernel is K (Moore's Theorem). This space is obtained in the following way: Let L_K be the linear set generated by $\{K(t, \cdot), t \in T,\}$ i.e. the set of all finite linear combinations

$$\sum_i \lambda_i K(t_i, \cdot), \lambda_i \in C ,$$

Let a scalar product of any ordered pair of elements $f, g \in L_K$ be defined by

$$\langle f, g\rangle = \sum_{i,j} \lambda_i \bar{\mu}_j K(t_i, t_j)$$

where

$$f = \sum_i \lambda_i K(t_i, \cdot), g = \sum_j \mu_j K(t_j, \cdot) .$$

This scalar product induces a norm on L_K, so that L_K is a pre-Hilbert space. Obviously

$$f(t) = \langle f, K(t, \cdot)\rangle , \forall t \in T \text{ and } \forall f \in L_K .$$

If $\{f_n\}$ is a Cauchy sequence in L_K, then $\{f_n\}$ converges everywhere to a function f, for

$$|f_m(t) - f_n(t)|^2 \leqq \|f_m - f_n\|^2 K(t, t) .$$

If the norm of f is defined by $\|f\| = \lim_{n\to\infty} \|f_n\|$, the space obtained by the adjunction to L_K of pointwise limits of Cauchy sequences in L_K is a Hilbert space and K reproduces all functions of this space. The space generated by $\{K(t,\cdot), t \in T\}$ will be denoted by $\mathfrak{H}_K$.

Let $\mathfrak{H}$ be any Hilbert space whose reproducing kernel is K. Then the class $\{K(t, \cdot), t \in T\}$ is a basis for $\mathfrak{H}$, so that $\mathfrak{H}$ coincides with $\mathfrak{H}_K$. Consequently, if a closed subspace $\mathfrak{H}$ of a Hilbert space $\mathfrak{h}$ of functions on T has a reproducing kernel K, then for any function $h \in \mathfrak{h}$, the scalar product $\langle h, K(t, \cdot)\rangle$ gives the projection of h onto $\mathfrak{H}$. Also, if $\mathscr{L}$ is a closed subspace of $\mathfrak{H}_K$, then the reproducing kernel of $\mathscr{L}$ is the projection $\hat{K}(t, \cdot)$ of $K(t, \cdot)$ onto $\mathscr{L}$.

2. The case of separable Hilbert spaces. The following theorem gives a necessary and sufficient condition for a separable Hilbert space of functions to have a reproducing kernel.

THEOREM 1. *Let $\mathfrak{H}$ be a separable Hilbert space of functions defined on T and let $\{e_i\}$ be a countable class of linearly independent functions in $\mathfrak{H}$ forming a basis for $\mathfrak{H}$. Let $\{K_n\}$ be the sequence defined by*

$$K_n(t, \tau) = \sum_{i,j=1}^{n} \bar{e}_i(t)\, \gamma_{ijn} e_j(\tau) \tag{1}$$

where $(\gamma_{ijn})_{1 \leqq i,j \leqq n}$ is the inverse of the matrix $(\langle e_i, e_j\rangle)_{1 \leqq i,j \leqq n}$.

(C_1) If $\forall t \in T, \{K_n(t, t)\}$ converges as $n \to \infty$, then any Cauchy sequence $\{\sum_{i=1}^{n} \alpha_{n,i} e_i\} \subset \mathfrak{H}$ converges everywhere on T.

(C_2) If, moreover, pointwise limits of such Cauchy sequences coincide with their limits in norm,

then $K(t, \tau) = \lim_{n\to\infty} K_n(t, \tau)$, which exists $\forall t, \tau \in T$, is the reproducing kernel of $\mathfrak{H}$.

Conversely, if $\mathfrak{H}$ has a reproducing kernel K, then the conditions C_1 and C_2 are fulfilled and $\forall t, \tau \in T, K(t, \tau) = \lim_{n\to\infty} K_n(t, \tau)$.

Proof. To avoid all trivialities, $\mathfrak{H}$ can be supposed to be infinite dimensional.

Sufficiency of C_1 and C_2. Consequences of C_1. Let $\mathfrak{H}_n$ be the subspace generated by $\{e_i, 1 \leqq i \leqq n\}$. $K_n(t, \cdot)$ is obviously an element of $\mathfrak{H}_n$ and it reproduces all functions in $\mathfrak{H}$. Moreover, $\mathfrak{H}_n \subset \mathfrak{H}_m$ for $m > n$. Then $K_n(t, \cdot)$ is the projection of $K_m(t, \cdot)$ onto $\mathfrak{H}_n$. Consequently, the relations

$$\langle K_m(t, \cdot), K_n(\tau, \cdot)\rangle = K_n(t, \tau),\ m > n\ , \tag{2}$$

$$\| K_m(t, \cdot) - K_n(t, \cdot) \|^2 = K_m(t, t) - K_n(t, t),\ m > n\ , \tag{3}$$

hold. By the last relation, it can be seen that $\{K_n(t, t)\}$ is an increasing sequence which converges by hypothesis, so that $\{K_n(t, \cdot)\}$ is a Cauchy sequence in $\mathfrak{H}$ for every $t \in T$. Let $K(t, \cdot)$ be the limit of this sequence.

For a given function $f \in \mathfrak{H}$, the function f_n defined by

$$\hat{f}_n(t) = \langle f, K_n(t, \cdot)\rangle = \sum_{i,j=1}^{n} \beta_i \gamma_{ijn} e_j(t), \beta_i = \langle f, e_i\rangle \tag{4}$$

is the projection of f onto $\mathfrak{H}_n$. Thus, the relations

$$||f - \hat{f}_n||^2 = ||f_n||^2 - ||\hat{f}_n||^2 \tag{5}$$

$$||\hat{f}_m - \hat{f}_n||^2 = ||\hat{f}_m||^2 - ||\hat{f}_n||^2 , \qquad m > n \tag{6}$$

$$||\hat{f}_n|| \leqq ||\hat{f}_m|| \leqq ||f|| , \qquad m > n . \tag{7}$$

hold. Consequently, $\{||\hat{f}||\}$ is a nondecreasing sequence bounded by $||f||$, therefore it converges. Then, according to (6), $\{f_n\}$ is a Cauchy sequence in $\mathfrak{H}$.

Let us suppose that

$$f_n = \sum_{k=1}^{n} \alpha_{n,i} e_i \tag{8}$$

is a sequence converging to f. Since $f_n \in \mathfrak{H}_n$, the relation

$$\langle f, f_n\rangle = \langle f - \hat{f}_n + \hat{f}_n, f_n\rangle = \langle \hat{f}_n, f_n\rangle$$

holds. Then, $\lim_{n\to\infty} \langle \hat{f}_n, f_n\rangle = \lim_{n\to\infty} \langle f, f_n\rangle = ||f||^2$, and according to (7),

$$0 \leqq \lim_{n\to\infty} ||\hat{f}_n - f_n||^2 = \lim_{n\to\infty} (||\hat{f}_n||^2 - \langle \hat{f}_n, f_n\rangle - \langle f_n, \hat{f}_n\rangle + ||f_n||^2)$$
$$= \lim_{n\to\infty} ||\hat{f}_n||^2 - ||f||^2 \leqq 0 .$$

Consequently, $\lim_{n\to\infty} ||\hat{f}_n|| = ||f||$. Then the relation (5) shows that $\{\hat{f}_n\}$ converges to f in norm.

Since the strong convergence of $\{\hat{f}_n\}$ implies its weak convergence, one has

$$\begin{aligned} \lim_{n\to\infty} \hat{f}_n(t) &= \lim_{n\to\infty} \langle \hat{f}_n, K(t, \cdot)\rangle \\ &= \langle f, K(t, \cdot)\rangle = g(t) . \end{aligned} \tag{9}$$

Thus, $\{\hat{f}_n\}$ converges everywhere. From this, it is easy to see that any Cauchy sequence of the type (8) also converges everywhere. In fact,

$$\hat{f}_n(t) - f_n(t) = \langle f - f_n, K_n(t, \cdot)\rangle .$$

By applying the Schwarz inequality and taking into account the fact that $K_n(t,t) < K(t,t)$, one can write

$$|\hat{f}_n(t) - f_n(t)|^2 \leqq ||f - f_n||^2 K_n(t,t) \leqq ||f - f_n||^2 K(t,t) .$$

Since $\{f_n\}$ converges to f in norm, it is seen that $\lim_{n\to\infty}|\hat{f}_n(t) - f_n(t)| = 0$. Finally, the inequality

$$|g(t) - f_n(t)| \leqq |g(t) - \hat{f}_n(t)| + |\hat{f}_n(t) - f_n(t)|$$

shows that $\{f_n(t)\}$ converges to the same limit $g(t)$ as $\{\hat{f}_n(t)\}$.

Consequences of C_2. In case the pointwise limit and the limit in norm of Cauchy sequences of the type (8) coincide, then by (9) the reproduction property $g(t) = f(t) = \langle f, K(t,\cdot)\rangle$ is obtained. Also, the sequence $\{K_n(t,\tau)\}$ converges to $K(t,\tau)$, $\forall t, \tau \in T$. Hence, $K(t,\tau) = \lim_{n\to\infty} K_n(t,\tau)$ is the reproducing kernel of $\mathfrak{H}$.

Necessity of C_1 *and* C_2. Suppose that $\mathfrak{H}$ possesses a reproducing kernel K. The relation (3) which is still valid, together with the relation

$$||K(t,\cdot) - K_n(t,\cdot)||^2 = K(t,t) - K_n(t,t) ,$$

obtained from (5) by replacing $f(\cdot)$ by $K(t,\cdot)$, imply that

$$K_n(t,t) < K_m(t,t) < K(t,t) \qquad \text{for } m > n .$$

Thus, $\{K_n(t,t)\}$ is an increasing sequence bounded by $K(t,t) < \infty$. Hence, it converges, so that the condition C_1 is fulfilled. On the other hand, since $\mathfrak{H}$ possesses a reproducing kernel, the condition C_2 is automatically fulfilled.

Consequently, $\lim_{n\to\infty} K_n(t,\tau)$ is a reproducing kernel of $\mathfrak{H}$. Reproducing kernel being always unique, one has $K(t,\tau) = \lim_{n\to\infty} K_n(t,\tau)$.

REMARK. If only the condition C_1 holds, then the space $\mathfrak{H}$ can be made isomorphic to a Hilbert space whose reproducing kernel is $\Gamma(t,\tau) = \langle K(t,\cdot), K(\tau,\cdot)\rangle$ with $K(t,\cdot)$ as the strong limit of $\{K_n(t,\cdot)\}$ in $\mathfrak{H}$. In fact, any Cauchy sequence of the type (8) converging to $f \in \mathfrak{H}$ converges everywhere in T to a function g. As in the theorem of Moore, if the set of all linear combinations of the functions $\{e_i\}$ is completed by the adjunction of pointwise limits of Cauchy sequences of this set with respect to the topology of $\mathfrak{H}$, and if the limit of the norms for each sequence is assigned as the norm of the pointwise limit of the sequence, then a Hilbert space $\mathfrak{H}_\Gamma$ is obtained. The reproducing kernel of $\mathfrak{H}_\Gamma$ turns out to be Γ. This latter space is obviously isomorphic to $\mathfrak{H}$. This isomorphism can be represented by

$$g(t) = \langle f, K(t, \cdot)\rangle, \forall t \in T, f \in \mathfrak{H} \text{ and } g \in \mathfrak{H}_\Gamma .$$

It can be proved also, that the class of functions $\{K(t, \cdot), t \in T\}$ generates $\mathfrak{H}$, in the sense that it is a basis for $\mathfrak{H}$, that is, any function $f \in \mathfrak{H}$ for which $\langle f, K(t, \cdot)\rangle = 0$ for all $t \in T$, has its norm equal to zero. In fact, let f be such a function. Then the function $g \in \mathfrak{H}_\Gamma$ corresponding to f in the isomorphism between $\mathfrak{H}$ and $\mathfrak{H}_\Gamma$ is the null function in $\mathfrak{H}_\Gamma$. Consequently, its norm and the norm of f equal zero.

It is worth mentionning that in view of this remark and the following theorem, there exists a countable subset S of T such that $K = \Gamma$ on both $S \times T$ and $T \times S$.

In what follows, a separable Hilbert space $\mathfrak{H}_K$ of functions on T, with reproducing kernel K, will be considered. Since the class $\{K(t, \cdot), t \in T\}$ generates $\mathfrak{H}_K$, there exists a countable subset S of T such that $\{K(t_i, \cdot), t_i \in S, i \in N\}$ is a class of linearly independent functions forming a basis for $\mathfrak{H}_K$. The matrix $(\gamma_{ijn})_{1 \leq i, j \leq n}$ will denote the inverse of the matrix $(K(t_i, t_j))_{1 \leq i, j \leq n}$ and S_n will denote $\{t_1, t_2, \cdots, t_n\} \subset S$.

THEOREM 2. *For any function $f \in \mathfrak{H}_K$, the sequence of functions defined by*

$$\hat{f}_n(\cdot) = \sum_{i,j=1}^{n} f(t_i)\gamma_{ijn}K(t_j, \cdot) \tag{10}$$

converges to f, as $n \longrightarrow \infty$, (both in norm and everywhere).

Proof. To prove the theorem, it suffices to replace e_i by $K(t_i, \cdot)$ in the preceding theorem. Then $K_n(t, \tau)$ becomes

$$K_n(t, \tau) = \sum_{i,j=1}^{n} K(t, t_i)\gamma_{ijn}K(t_j, \tau) \tag{11}$$

and the function (4) reduces to (10).

Notice that K_n coincides with K on $S_n \times T$ and $T \times S_n$, and consequently, $\hat{f}_n = f$ on S_n. According to the second part of Theorem 1, $K_n(t, \cdot)$ converges to $K(t, \cdot)$ in norm and everywhere, and the first part of the proof of the same theorem shows that the sequence (10) converges to f in norm and everywhere.

So, it appears that $\hat{f}_n$ gives an approximation of f in norm and everywhere in terms of the values taken by f on the finite subset S_n of S.

COROLLARY. *The scalar product of any pair of functions $f, g \in \mathfrak{H}_K$ is given by*

$$\langle f, g\rangle = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(t_i)\gamma_{ijn}\bar{g}(t_j)\ . \tag{12}$$

Consequently, the norm of any function $f \in \mathfrak{H}_K$ *is given by*

$$||f||^2 = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(t_i)\gamma_{ijn}\bar{f}(t_j)\ . \tag{13}$$

THEOREM 3.[1] *Let* f *be an arbitrary function defined on* T, *such that*

$$\lim_{n\to\infty} \sum_{i,j=1}^{n} f(t_i)\gamma_{ijn}\bar{f}(t_j) < \infty,\ t_i,\ t_j \in S,\ \forall i, j \in N\ . \tag{14}$$

Then the sequence of functions defined by

$$f_n(\cdot) = \sum_{i,j=1}^{n} f(t_i)\gamma_{ijn}K(t_j,\ \cdot) \tag{15}$$

is a Cauchy sequence in $\mathfrak{H}_K$, *whose limit* f' *coincides with* f *on* S.

Proof. The relation

$$||f_m - f_n||^2 = ||f_m||^2 - ||f_n||^2\ , \qquad m > n$$

holds for the sequence (15), with

$$||f_n||^2 = \sum_{i,j=1}^{n} f(t_i)\gamma_{ijn}\bar{f}(t_j)\ .$$

It is then seen that $||f_n||^2$ is a nondecreasing sequence converging to (14), so that $\{f_n\}$ is a Cauchy sequence in $\mathfrak{H}_K$. Let f' be its limit. Since $f' \in \mathfrak{H}_K$, according to Theorem 2, the sequence

$$\hat{f}'_n(t_i) = \sum_{i,j=1}^{n} f'(t_i)\gamma_{ijn}K(t_j,\ \cdot)$$

is also a Cauchy sequence converging to f' and therefore $\{f_n - \hat{f}'_n\}$ converges to the null function in $\mathfrak{H}_K$. Since the relation

$$||f_m - \hat{f}'_m||^2$$

$$||(f_m - \hat{f}'_m) - (f_n - \hat{f}'_n)||^2 = ||f_m - \hat{f}'_m||^2 - ||f_n - \hat{f}'_n||^2\ , \qquad m > n$$

holds, one has

$$0 \leqq ||f_n - \hat{f}'_n|| \leqq \lim_{m\to\infty} ||f_m - \hat{f}'_m|| = 0$$

so that $\forall n \in N$, $||f_n - \hat{f}'_n|| = 0$. Consequently $\forall t \in T$ and $\forall n \in N$, $f_n(t) = \hat{f}'_n(t)$. In particular $\forall i \leqq n, f(t_i) = f_n(t_i) = \hat{f}'_n(t_i) = f'(t_i)$. Thus, $f(t) = f'(t)$, all $t \in S$.

[1] This extension was suggested to the author by Professor H. L. Royden.

It follows from the last theorem that the set $\mathscr{F}$ of all functions satisfying the condition (14) is a Hilbert space in which the scalar product of f by g is given by

$$(16) \qquad \lim_{n\to\infty} \sum_{i,j=1}^{n} f(t_i)\gamma_{ijn}\bar{g}(t_j) , \qquad t_i, t_j \in S, \forall i, j \in N .$$

In this space all the functions coinciding on S belong to the same equivalence class defined by the relation

$$f \sim g \Leftrightarrow \lim_{n\to\infty} \sum_{i,j=1}^{n} [f(t_i) - g(t_i)]\gamma_{ijn}[\bar{f}(t_j) - \bar{g}(t_j)] = 0 .$$

In particular, the function $f \in \mathscr{F}$ and the function $f' \in \mathfrak{H}_K$ corresponding to f as the limit of the sequence (15) are equivalent.

3. Hilbert Space generated by a second order random process. Let (Ω, Σ, P) be a probability space, where Ω is a sample space, Σ is the σ-algebra generated by a class of subsets of Ω and P a probability measure defined on Σ. Let $\{X_t, t \in T\}$ be a class of complex valued random variables defined on Ω and measurable with respect to Σ. The symbol E will denote the mathematical expectation with respect to the probability measure P. It will be supposed that $\forall t \in T, E(X_t) = 0$ and $E(|X_t|^2) < \infty$. The covariance function $E(X_t\bar{X}_\tau)$ of thus defined second order stochastic process will be denoted by $K(t, \tau)$.

Let L_X be the linear set of all finite linear combinations

$$\sum_i \lambda_i X_{t_i}, \; t_i \in T, \lambda_i \in C .$$

A scalar product on L_X can be defined for any ordered pair of elements

$$Y = \sum_i \lambda_i X_{t_i}, \; Z = \sum_j \mu_j X_{t_j}$$

by the bilinear form

$$E(Y\bar{Z}) = \sum_{i,j} \lambda_i \bar{\mu}_j K(t_i, t_j)$$

which induces, for any element $Y \in L_X$, a norm whose square is defined by

$$E(|Y|^2) = \sum_{i,j} \lambda_i \bar{\lambda}_j K(t_i, t_j) .$$

The Hilbert space which is the closure of L_X in the topology induced by this norm will be denoted by $\mathfrak{H}_X$ and will be said to be generated by the process $\{X_t, t \in T\}$.

The theorem of Moore says that there exists a uniquely defined Hilbert space $\mathfrak{H}_K$ of functions on T, admitting K as its reproducing kernel. The construction of $\mathfrak{H}_X$ and of $\mathfrak{H}_K$ shows that these two spaces are isomorphic if K is the covariance function of $\{X_t, t \in T\}$. Under this isomorphism, the random variable X_t corresponds obviously to $K(t, \cdot)$. Consequently, the two spaces are simultaneously separable and if $\{K(t_i, \cdot), t_i \in S\}$ is a basis for $\mathfrak{H}_K$ in the sense given in Theorem 1, then $\{X_{t_i}, t_i \in S\}$ is a basis for $\mathfrak{H}_X$.

Given an element Z in $\mathfrak{H}_X$, the element f_Z in $\mathfrak{H}_K$ corresponding to Z is given by

$$f_Z(t) = \langle f_Z, K(t, \cdot)\rangle = E(Z\bar{X}_t) \, .$$

For separable $\mathfrak{H}_K$ (or equivalently $\mathfrak{H}_X$) the following theorem gives a representation of the element of $\mathfrak{H}_X$ corresponding to any given function f in $\mathfrak{H}_K$. The symbols have exactly the same meaning as in the two preceding theorems.

THEOREM 4. *For any function $f \in \mathfrak{H}_K$, the stochastic element $X(f) \in \mathfrak{H}_X$ corresponding to f under the isomorphism between $\mathfrak{H}_K$ and $\mathfrak{H}_X$, is given by the limit in the quadratic mean of*

$$X(\hat{f}_n) = \sum_{i,j=1}^{n} f(t_i)\gamma_{ijn}X_{t_j} \tag{17}$$

as $n \to \infty$.

Proof. By replacing $X(t_j)$ by $K(t_j, \cdot)$ in (17), it is seen that $X(\hat{f}_n)$ is the element of $\mathfrak{H}_X$ corresponding to (10). Since $\{\hat{f}_n\}$ is a Cauchy sequence in $\mathfrak{H}_K$ converging to f. Then $\{X(\hat{f}_n)\}$ is a Cauchy sequence converging to $X(f)$.

In view of the analogy between (12) and (17), the element $X(f)$ can be represented, following Parzen, as $\langle f(\cdot), \bar{X}_{(\cdot)}\rangle$. But this is not really a scalar product because, almost surely, $X_{(\cdot)}$ does not belong to $\mathfrak{H}_K$.

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