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INVARIANCE FOR LINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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In studying the existence and smoothness of invariant manifolds arising from nonlinear, perturbed systems of ordinary differential equations, one encounters the study of certain linear (in x), perturbation problems of the type

$$\dot{\theta} = a + \varepsilon b(\theta, \varepsilon)$$
$$\dot{x} = (A + \varepsilon B(\theta, \varepsilon))x$$

where θ and x are vectors, A and B are matrices, b and B are multiply periodic in θ, and ε is a perturbation parameter. Assuming A is a constant matrix consisting of square submatrices on the diagonal,

$$A = \operatorname{diag}(A_{11}, \cdots, A_{nn}) ,$$

with the maximum of the real parts of the eigenvalues of A_{jj} less than the minimum of the real parts of the eigenvalues of A_{kk} for $1 \leqq j < k \leqq n$; we construct a change of variables which reduces B to similar diagonal form.

For perturbed systems of nonlinear ordinary differential equations in a neighborhood of an invariant manifold, the existence and smoothness of the center-stable, center, and center-unstable manifolds is proved in § 6 of [3]. The method of proof used will also show the existence of other invariant manifolds, but for nonlinear systems the situation is not as simple as the associated linear problem with regard to finding invariant manifolds. R. Venti [7] has given linearization results for nonlinear systems of differential equations near a critical point. The results of this paper can be regarded as a first step in obtaining similar linearization results for nonlinear systems near an invariant manifold.

The techniques of this paper are based on those used by Y. Sibuya [5], [6]. Sibuya treats time-varying perturbation problems where the perturbation parameter enters in an analytic way. In § 3 of this paper we consider $C^k(1 \leqq k < \infty)$, θ-varying perturbation problems with θ representing the many-dimensional coordinates of some invariant manifold. In § 4 we give a counter-example to an analytic change of variables procedure, and then modify the procedure appropriately.

For linear systems of ordinary differential equations of the type

$$\dot{x} = (A + \varepsilon B(t, \varepsilon))x$$

(see (1) below with $\dim \theta = 1, \dot{\theta} = 1$, for details), where the matrix B

is periodic in t, the theory presented in § 3 and § 4 below applies. However, for this periodic, time-dependent, perturbation problem, much more can be said. See, for example, Lemmas 4, 5, 6 in Sibuya [6] or Chapter 8 in Hale [2].

2. **Notation.** If $G = G(g)$ is a smooth, vector valued function of the vector g, then G_g represents the usual Jacobian matrix of partial derivatives. If $H = H(g)$ is a smooth, matrix valued function of the vector $g = (g_1, \cdots, g_n)$, then H_g represents an array of all possible first order derivatives,

$$H_g = \left(\frac{\partial}{\partial g_1}H, \frac{\partial}{\partial g_2}H, \cdots, \frac{\partial}{\partial g_n}H\right).$$

The norm $|\cdot|$ represents the euclidean norm on vectors and the operator norm on matrices; $\langle\cdot, \cdot\rangle$ represents the usual inner product on pairs of vectors. If ρ is an n-tuple of nonnegative integers, then $D_g^\rho = \partial^{|\rho|}/\partial^{\rho_1}g_1 \cdots \partial^{\rho_n}g_n$ where $|\rho| = \rho_1 + \cdots + \rho_n$. (The "norm" on ρ is not euclidean which we justify by not considering ρ to be a vector.)

3. **Invariance for linear systems.** Consider the real, C^1 system of ordinary differential equations

$$\begin{aligned} \dot\theta &= a + \varepsilon b(\theta, \varepsilon) \\ \dot x &= (A + \varepsilon B(\theta, \varepsilon))x \end{aligned} \tag{1}$$

where θ, x, a, b are real vectors; a is a constant vector; ε is a real perturbation parameter; b is defined and C^1 on

$$N_\delta = \{(\theta, \varepsilon) \mid \theta \text{ arbitrary}, |\varepsilon| < \delta\};$$

b and $b_{(\theta,\varepsilon)}$ are uniformly bounded on N_δ; A is a real, constant, square matrix partitioned as follows:

$$\begin{aligned} &A = (A_{jk}) \text{ with } A_{jk} = 0 \text{ for } j \neq k, \text{ equivalently} \\ &A = \operatorname{diag}(A_{11}, A_{22}, \cdots, A_{nn}); \end{aligned}$$

the sub-matrices $A_{jj} (j = 1, \cdots, n)$ on the diagonal are square but not necessarily of the same dimension; the sub-matrices on the diagonal have the property that the maximum of the real parts of the eigenvalues of A_{jj} is less than the minimum of the real parts of the eigenvalues of A_{kk} when $j < k$: symbolically

$$A_{11} < A_{22} < \cdots < A_{nn} \tag{2}$$

where the order relation $<$ reflects the ordering of the real parts of the eigenvalues of the sub-matrices; B is a real, C^1, square matrix defined on N_δ; B and $B_{(\theta,\varepsilon)}$ are uniformly bounded on N_δ.

THEOREM 1. *For system* (1) *there exists a unique C^1 change of variables*

$$x = (I + \varepsilon P(\theta, \varepsilon))y \tag{3}$$

such that

$$\dot{y} = (A + \varepsilon Q(\theta, \varepsilon))y \tag{4}$$

where I is the identity matrix; P and Q are real, C^1 matrices defined on N_{δ_1} for $0 < \delta_1 \leqq \delta$ sufficiently small; P, $P_{(\theta,\varepsilon)}$, Q, $Q_{(\theta,\varepsilon)}$ are uniformly bounded on N_{δ_1}; with P and Q partitioned into sub-matrices similar to A,

$$\begin{aligned} P_{jj}(\theta, \varepsilon) &\equiv 0 \qquad (j = 1, \cdots, n) \\ Q_{jk}(\theta, \varepsilon) &\equiv 0 \qquad (j, k = 1, \cdots, n;\ j \neq k); \end{aligned} \tag{5}$$

if system (1) *has multiple period ω in θ, then the change of variables* (3) *and the transformed system* (4) *also have multiple period ω in θ.*

Proof. Assuming the change of variables exists, we differentiate both sides of (3),

$$\dot{x} = \varepsilon \dot{P} y + (I + \varepsilon P)\dot{y} ,$$

and from (1), (3), (4) we have

$$(A + \varepsilon B)(I + \varepsilon P)y = \varepsilon \dot{P} y + (I + \varepsilon P)(A + \varepsilon Q)y ,$$

which leads to the matrix equation

$$\dot{P} = AP - PA + B - Q + \varepsilon BP - \varepsilon PQ . \tag{6}$$

Let $\psi = \psi(t, \theta, \varepsilon)$ be the unique solution of the θ-equation in (1) with initial condition θ at $t = 0$. This solution exists and is C^1 on

$$\tilde{N}_\delta = \{(t, \theta, \varepsilon) \mid -\infty < t < +\infty, (\theta, \varepsilon) \in N_\delta\} .$$

Rather than expressing $\dot{P}$ on the left side of (6) as $P_\theta\{a + \varepsilon b\,(\theta, \varepsilon)\}$, we consider

$$\dot{P} = \frac{d}{dt} P(\psi(t, \theta, \varepsilon), \varepsilon) ;$$

in words, we interpret $\dot{P}$ as the "derivative along the solution curve". (See the proof of Theorem 1 in [4] for a similar example of this notion.) From (5) and (6)

$$(7a)\qquad Q_{jj} = B_{jj} + \varepsilon \sum_{l=1}^{n} B_{jl} P_{lj} \qquad (j = 1, \cdots, n)$$

$$(7b)\qquad \begin{aligned} \dot{P}_{jk} &= A_{jj}P_{jk} - P_{jk}A_{kk} + B_{jk} \\ &\quad + \varepsilon \sum_{l=1}^{n} B_{jl}P_{lk} - \varepsilon P_{jk}Q_{kk} \qquad (j, k = 1, \cdots, n; j \neq k) . \end{aligned}$$

Conversely, we observe that a C^1 solution of (7) which is uniformly bounded on some N_{δ_1} will yield the change of variables (3) and the transformed system (4). To solve (7) by iteration define $P^0 \equiv 0$ and define Q^ν, H^ν, P^ν iteratively as follows. (Super-scripts designate steps of the iteration, not powers of the matrices.)

$$(8a)\qquad \begin{aligned} Q^\nu_{jj} &= B_{jj} + \varepsilon \sum_{l=1}^{n} B_{jl} P^\nu_{lj} \\ Q^\nu_{jk} &\equiv 0 \\ &\qquad (1 \leqq j, k \leqq n; j \neq k)(\nu = 0, 1, 2, \cdots) \end{aligned}$$

$$(8b)\qquad \begin{aligned} H^\nu &= BP^{\nu-1}P^{\nu-1}Q^{\nu-1}, \text{ so that in particular} \\ H^\nu_{jk} &= \sum_{l=1}^{n} B_{jl}P^{\nu-1}_{lk} - P^{\nu-1}_{jk}Q^{\nu-1}_{kk} \\ &\qquad (1 \leqq j, k \leqq n; j \neq k)(\nu = 1, 2, \cdots) \end{aligned}$$

$$(8c)\qquad \begin{aligned} P^\nu_{jj} &\equiv 0 \\ \dot{P}^\nu_{jk} &= A_{jj}P^\nu_{jk} - P^\nu_{jk}A_{kk} + B_{jk} + \varepsilon H^\nu_{jk} \\ &\qquad (1 \leqq j, k \leqq n; j \neq k)(\nu = 1, 2, \cdots) . \end{aligned}$$

The unique bounded solution P^ν_{jk} of the differential equation in (8c) is given by

$$(9)\qquad P^\nu_{jk}(\theta, \varepsilon) = \int_{\pm\infty}^{0} e^{-A_{jj}\sigma}\{B_{jk}(\psi(\sigma, \theta, \varepsilon), \varepsilon) + \varepsilon H^\nu_{jk}(\psi(\sigma, \theta, \varepsilon), \varepsilon)\}e^{A_{kk}\sigma}d\sigma$$

where the lower limit of integration is chosen $+\infty$ for $A_{kk} < A_{jj}$ and $-\infty$ for $A_{jj} < A_{kk}$ with $<$ the order relation in (2). If we assume that H^ν is a known C^1 function of (θ, ε) on some N_{δ_1} with H^ν, $H^\nu_{(\theta,\varepsilon)}$ uniformly bounded on N_{δ_1}, and if we assume that the infinite integral in (9) converges, then by replacing θ in (9) by $\psi(t, \theta, \varepsilon)$ we easily check that $P^\nu_{jk}(\psi(t, \theta, \varepsilon), \varepsilon)$ satisfies (8c). Beginning with $P^0 \equiv 0$ we want to show that (8) and (9) determine $Q^\nu, H^\nu, P^\nu(\nu = 1, 2, \cdots)$ iteratively as C^1 functions of (θ, ε) defined on some N_{δ_1} with $Q^\nu, H^\nu, P^\nu, (Q^\nu, H^\nu, P^\nu)_{(\theta,\varepsilon)}$ uniformly bounded on N_{δ_1}. We will need estimates on the rates of growth for $e^{A_{jj}t}(j = 1, \cdots, n)$, $\psi_\theta(t, \theta, \varepsilon)$, $\psi_\varepsilon(t, \theta, \varepsilon)$.

Associated with each A_{jj} is its real canonical form $\hat{A}_{jj}$,

$$(10)\qquad \hat{A}_{jj} = J^{-1}_{jj}A_{jj}J_{jj} \qquad (j = 1, \cdots, n) ,$$

with the "off-diagonalizable" terms of $\hat{A}_{jj}$ arbitrarily small. The ordering (2) means there exists real numbers

$$\mu_1 \leqq \tilde{\mu}_1 < \mu_2 \leqq \tilde{\mu}_2 < \cdots < \mu_n \leqq \tilde{\mu}_n \tag{11}$$

such that

$$\mu_j \, | p |^2 \leqq \langle \hat{A}_{jj} p, p \rangle \leqq \tilde{\mu}_j \, | p |^2 \qquad (j = 1, \cdots, n) \tag{12a}$$

holds for all vectors p (restricted for each j to the appropriate dimension $(\dim \hat{A}_{jj} = \dim p \times \dim p)$). See [1] page 341 for details concerning the real canonical form of a matrix. From (12a) we conclude for $(j = 1, \cdots, n)$

$$\begin{aligned} e^{\mu_j t} &\leqq | e^{\hat{A}_{jj} t} | \leqq e^{\tilde{\mu}_j t} \qquad (0 \leqq t < +\infty) \\ e^{\mu_j t} &\geqq | e^{\hat{A}_{jj} t} | \geqq e^{\tilde{\mu}_j t} \qquad (-\infty < t \leqq 0) \, . \end{aligned} \tag{12b}$$

From (1) for $(t, \theta, \varepsilon) \in \tilde{N}_\delta$

$$\begin{aligned} \dot{\psi}_\theta(t, \theta, \varepsilon) &= \varepsilon b_\theta(\psi(t, \theta, \varepsilon), \varepsilon) \psi_\theta(t, \theta, \varepsilon) \\ \dot{\psi}_\varepsilon(t, \theta, \varepsilon) &= b(\psi(t, \theta, \varepsilon), \varepsilon) + \varepsilon b_\theta(\psi(t, \theta, \varepsilon), \varepsilon) \psi_\varepsilon(t, \theta, \varepsilon) \\ &\quad + \varepsilon b_\varepsilon(\psi(t, \theta, \varepsilon), \varepsilon) \\ \psi_\theta(0, \theta, \varepsilon) &\equiv I \text{ (identity)}, \ \psi_\varepsilon(0, \theta, \varepsilon) \equiv 0 \end{aligned}$$

which yields

$$\begin{aligned} | \dot{\psi}_\theta | &\leqq | \varepsilon | K_1 | \psi_\theta | \\ | \dot{\psi}_\varepsilon | &\leqq K_1 + | \varepsilon | K_1 | \psi_\varepsilon | \\ | \psi_\theta(0, \theta, \varepsilon) | &\equiv 1, \quad | \psi_\varepsilon(0, \theta, \varepsilon) | \equiv 0 \, , \end{aligned} \tag{13}$$

where K_1 is a sufficiently large positive constant; more specifically we may take

$$K_1 = (1 + \delta) \max \{ \sup_{(\theta, \varepsilon) \in N_\delta} | b(\theta, \varepsilon) |, \sup_{(\theta, \varepsilon) \in N_\delta} | b_{(\theta, \varepsilon)}(\theta, \varepsilon) | \} \, .$$

From (13) and the Hale inequality (see Lemma 2 in [3])

$$\begin{aligned} | \psi_\theta(t, \theta, \varepsilon) | &\leqq e^{|\varepsilon| K_1 |t|} \\ | \psi_\varepsilon(t, \theta, \varepsilon) | &\leqq K_1 \, | t | \, e^{|\varepsilon| K_1 |t|} \end{aligned} \tag{14}$$

holds for all $(t, \theta, \varepsilon) \in \tilde{N}_\delta$.

In what follows $K_2, K_3, \cdots$ (a finite number of K's) will designate sufficiently large positive constants. If $j < k$, we have from (9), (10), (12)

$$\begin{aligned} | P^\nu_{jk}(\theta, \varepsilon) | \leqq \int_{-\infty}^0 & | J_{jj} | \, | e^{-\hat{A}_{jj} \sigma} | \, | J^{-1}_{jj} | \\ & \cdot \{ | B_{jk}(\psi(\sigma, \theta, \varepsilon), \varepsilon) | + | \varepsilon | \, | H^\nu_{jk}(\psi(\sigma, \theta, \varepsilon), \varepsilon) [\} \\ & \cdot | J_{kk} | \, | e^{\hat{A}_{kk} \sigma} | \, | J^{-1}_{kk} | \, d\sigma \\ \leqq \int_{-\infty}^0 & K_2 e^{-\tilde{\mu}_j \sigma} \{ \cdots \} K_2 e^{\mu_k \sigma} d\sigma \, , \end{aligned} \tag{15}$$

where K_2 is chosen sufficiently large so that

$$|J_{jj}|\,|J_{jj}^{-1}| \leqq K_2 \qquad (j = 1, \cdots, n)\,.$$

A similar inequality holds for the case $j > k$. From (11), (15) we conclude that (8), (9) determine Q^ν, H^ν, P^ν iteratively as continuous functions of (θ, ε) defined and uniformly bounded on N_δ.

Suppose we have shown $P^{\nu-1}$ to be C^1 on some N_{δ_1} (δ_1 yet to be determined) with $P^{\nu-1}, P^{\nu-1}_{(\theta,\varepsilon)}$ uniformly bounded on N_{δ_1}. Then $Q^{\nu-1}$, H^ν given by (8a, b) will also be C^1 on N_{δ_1} with $Q^{\nu-1}, H^\nu, (Q^{\nu-1}, H^\nu)_{(\theta,\varepsilon)}$ uniformly bounded on N_{δ_1}. Let $\theta = (\theta_1, \cdots, \theta_m)$ so that θ_l represents the l^{th}-component of θ and $m = \dim\theta$. If $\partial/\partial\theta_l$ commutes with integration, then differentiating the right side of (9) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\theta_l}\int_{\pm\infty}^{0} e^{-A_{jj}\sigma}[B_{jk}(\psi, \varepsilon) + \varepsilon H^\nu_{jk}(\psi, \varepsilon)]e^{A_{kk}\sigma}d\sigma \\
&\quad = \int_{\pm\infty}^{0} e^{-A_{jj}\sigma}\Big[\Big\{\Big(\frac{\partial}{\partial\theta_1}B_{jk}\Big)(\psi, \varepsilon)\Big\}\frac{\partial}{\partial\theta_l}\psi_1 + \cdots \\
(16)\qquad &\quad + \Big\{\Big(\frac{\partial}{\partial\theta_m}B_{jk}\Big)(\psi, \varepsilon)\Big\}\frac{\partial}{\partial\theta_l}\psi_m + \varepsilon\Big\{\Big(\frac{\partial}{\partial\theta_1}H^\nu_{jk}\Big)(\psi, \varepsilon)\Big\}\frac{\partial}{\partial\theta_l}\psi_1 \\
&\quad + \cdots + \varepsilon\Big\{\Big(\frac{\partial}{\partial\theta_m}H^\nu_{jk}\Big)(\psi, \varepsilon)\Big\}\frac{\partial}{\partial\theta_l}\psi_m\Big]e^{A_{kk}\sigma}d\sigma\,.
\end{aligned}$$

If $\partial/\partial\varepsilon$ commutes the integration, then differentiating the right side of (9) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\varepsilon}\int_{\pm\infty}^{0} e^{-A_{jj}\sigma}[B_{jk}(\psi, \varepsilon) + \varepsilon H^\nu_{jk}(\psi, \varepsilon)]e^{A_{kk}\sigma}d\sigma \\
&\quad = \int_{\pm\infty}^{0} e^{-A_{jj}\sigma}\Big[\Big\{\Big(\frac{\partial}{\partial\theta_1}B_{jk}\Big)(\psi, \varepsilon)\Big\}\frac{\partial}{\partial\varepsilon}\psi_1 + \cdots \\
(17)\qquad &\quad + \Big\{\Big(\frac{\partial}{\partial\theta_m}B_{jk}\Big)(\psi, \varepsilon)\Big)\Big\}\frac{\partial}{\partial\varepsilon}\psi_m + \Big\{\Big(\frac{\partial}{\partial\varepsilon}B_{jk}\Big)(\psi, \varepsilon)\Big\} + H^\nu_{jk}(\psi, \varepsilon) \\
&\quad + \varepsilon\Big\{\Big(\frac{\partial}{\partial\theta_1}H^\nu_{jk}\Big)(\psi, \varepsilon)\Big\}\frac{\partial}{\partial\varepsilon}\psi_1 + \cdots + \varepsilon\Big\{\Big(\frac{\partial}{\partial\theta_m}H^\nu_{jk}\Big)(\psi, \varepsilon)\Big\}\frac{\partial}{\partial\varepsilon}\psi_m \\
&\quad + \varepsilon\Big\{\Big(\frac{\partial}{\partial\varepsilon}H^\nu_{jk}\Big)(\psi, \varepsilon)\Big\}\Big]e^{A_{kk}\sigma}d\sigma\,.
\end{aligned}$$

Define

$$\begin{aligned}
&\|\cdot\|_0 = \sup_{(\theta,\varepsilon)\in N_{\delta_1}} |\cdot| \\
(18)\qquad &\|\cdot\|_1 = \max_{0\leqq|\rho|\leqq 1}\ \sup_{(\theta,\varepsilon)\in N_{\delta_1}} |D^\rho_{(\theta,\varepsilon)}(\cdot)|\,.
\end{aligned}$$

For the case $j < k$ we have from (14), (16), (17)

$$\left|\frac{\partial}{\partial\theta_l}\int_{-\infty}^{0}\cdots d\sigma\right| \leqq K_2^2\int_{-\infty}^{0} e^{(\mu_k-\tilde{\mu}_j)\sigma} m\{\|B\|_1 + |\varepsilon|\,\|H^\nu\|_1\} e^{|\varepsilon|K_1|\sigma} d\sigma$$

$$\text{(19)}\quad \left|\frac{\partial}{\partial\varepsilon}\int_{-\infty}^{0}\cdots d\sigma\right| \leqq K_2^2\int_{-\infty}^{0} e^{(\mu_k-\tilde{\mu}_j)\sigma}[m\{\|B\|_1 + |\varepsilon|\,\|H^\nu\|_1\}$$
$$\cdot K_1|\sigma|e^{|\varepsilon|K_1|\sigma|} + \|B\|_1 + \|H^\nu\|_0 + |\varepsilon|\,\|H^\nu\|_1]d\sigma$$

uniformly in $(\theta, \varepsilon) \in N_{\delta_1}$. Similar inequalities hold for the case $j > k$. If we restrict δ_1 so that

$$\text{(20)}\quad \begin{aligned} &\mu_k - \tilde{\mu}_j - \delta_1 K_1 > 0 \qquad (1 \leqq j < k \leqq n)\\ &\tilde{\mu}_k - \mu_j + \delta_1 K_1 < 0 \qquad (1 \leqq k < j \leqq n)\,, \end{aligned}$$

then inequality (19) shows that $\partial/\partial\theta_l (l = 1, \cdots, m), \partial/\partial\varepsilon$ in fact do commute with integration and that (8), (9) determine $Q^\nu, H^\nu, P^\nu(\nu = 1, 2, \cdots)$ iteratively as C^1 functions of (θ, ε) defined on N_{δ_1}, δ_1 restricted by (20), with $Q^\nu, H^\nu, P^\nu, (Q^\nu, H^\nu, P^\nu)_{(\theta,\varepsilon)}$ uniformly bounded on N_{δ_1} for each ν fixed.

We now consider the problem of convergence. With δ_1 further restricted, if necessary, we will show that $(Q^\nu, H^\nu, P^\nu) \to (Q, H, P)$ in the C^1 topology on N_{δ_1} as $\nu \to +\infty$. From (8a)

$$\text{(21)}\quad \begin{aligned} &\|Q^{\nu+1} - Q^\nu\|_0 \leqq \delta_1 K_3 \|P^{\nu+1} - P^\nu\|_0 \qquad (\nu = 0, 1, \cdots)\\ &\|Q^{\nu-1} - Q^\nu\|_1 \leqq K_3 \|P^{\nu+1} - P^\nu\|_1 \qquad (\nu = 0, 1, \cdots)\,. \end{aligned}$$

From (8b)

$$\text{(22)}\quad \begin{aligned} &H^{\nu+1} - H^\nu = B(P^\nu - P^{\nu-1}) - P^\nu(Q^\nu - Q^{\nu-1}) - (P^\nu - P^{\nu-1})Q^{\nu-1}\\ &\|H^{\nu+1} - H^\nu\|_0 \leqq \{\|B\|_0 + \|P^\nu\|_0 + \|Q^{\nu-1}\|_0\}\{\|P^\nu - P^{\nu-1}\|_0\\ &\qquad\qquad + \|Q^\nu - Q^{\nu-1}\|_0\}\\ &\|H^{\nu+1} - H^\nu\|_1 \leqq 2\{\|B\|_1 + \|P^\nu\|_1 + \|Q^{\nu-1}\|_1\}\{\|P^\nu - P^{\nu-1}\|_1\\ &\qquad\qquad + \|Q^\nu - Q^{\nu-1}\|_1\}_1 \qquad (\nu = 1, 2, \cdots)\,. \end{aligned}$$

From (9) for the case $j < k$

$$\begin{aligned} &|P_{jk}^{\nu+1} - P_{jk}^\nu|\\ &= \left|\int_{-\infty}^{0} e^{-A_{jj}\sigma}\varepsilon\{H_{jk}^{\nu+1}(\psi, \varepsilon) - H_{jk}^\nu(\psi, \varepsilon)\}e^{A_{kk}\sigma}d\sigma\right|\\ &\leqq \int_{-\infty}^{0} K_2 e^{-\tilde{\mu}_j\sigma}|\varepsilon|\,\|H^{\nu+1} - H^\nu\|_0 K_2 e^{\mu_k\sigma}d\sigma\\ &\leqq |\varepsilon| K_4 \|H^{\nu+1} - H^\nu\|_0 \qquad (\nu = 1, 2, \cdots)\,. \end{aligned}$$

A similar inequality holds for the case $j > k$ so that

$$\text{(23)}\quad \|P^{\nu+1} - P^\nu\|_0 \leqq \delta_1 K_5 \|H^{\nu+1} - H^\nu\|_0 \quad (\nu = 1, 2, \cdots)\,.$$

Combining (21), (22), (23) we obtain

$$\begin{aligned}(24)\qquad &\| P^{\nu+1} - P^{\nu} \|_0 \\ &\leqq \delta_1 K_5 \{\| B \|_0 + \| P^{\nu} \|_0 + \| Q^{\nu-1} \|_0\}\{1 + \delta_1 K_3\} \| P^{\nu} - P^{\nu-1} \|_0 \\ &\qquad\qquad (\nu = 1, 2, \cdots) .\end{aligned}$$

From (8), (9) we observe that Q^0, P^1 are defined on N_δ and that on N_δ the inequality

$$(25)\qquad | Q^0(\theta, \varepsilon) | \leqq | B(\theta, \varepsilon) |$$

holds. Define

$$(26)\qquad K_6 = 2 \sup_{(\theta,\varepsilon) \in N_\delta} | B(\theta, \varepsilon) | + 3 \sup_{(\theta,\varepsilon) \in N_\delta} | P^1(\theta, \varepsilon) | .$$

Define V_{δ_1} as a subset of the positive integers,

$$(27)\qquad V_{\delta_1} = \{ \nu \mid \| B \|_0 + \| P^{\nu} \|_0 + \| Q^{\nu-1} \|_0 \leqq K_6 \} .$$

From (24), (26), (27)

$$(28)\qquad \| P^{\nu+1} - P^{\nu} \|_0 \leqq \delta_1 K_5 K_6 \{1 + \delta_1 K_3\} \| P^{\nu} - P^{\nu-1} \|_0, \quad \nu \varepsilon V_{\delta_1} .$$

Now restrict δ_1 further, if necessary, so that

$$(29)\qquad \begin{aligned} &\delta_1 K_3 < \frac{1}{2} \\ &\delta_1 K_5 K_6 \{1 + \delta_1 K_3\} < \frac{1}{2} ; \end{aligned}$$

then from (21), (25), (26), (28), (29) it follows that

$$\begin{aligned} &\| P^{\nu} \|_0 \leqq 2 \| P^1 \|_0 && (\nu = 1, 2, \cdots) \\ &\| Q^{\nu-1} \|_0 \leqq \| B \|_0 + \| P^1 \|_0 && (\nu = 1, 2, \cdots) \\ &\| B \|_0 + \| P^{\nu} \|_0 + \| Q^{\nu-1} \|_0 \leqq K_6 && (\nu = 1, 2, \cdots) \end{aligned}$$

so that by (27)

$$(30)\qquad V_{\delta_1} = \{\text{all positive integers}\}$$

and

$$(Q^{\nu}, H^{\nu}, P^{\nu}) \longrightarrow (Q, H, P)$$

in the C^0 topology on N_{δ_1} as $\nu \longrightarrow +\infty$.

In order to show that our sequence converges in the C^1 topology we will need C^0 estimates on the rate of convergence of $(H^{\nu+1} - H^{\nu})$, and we will restrict δ_1 further, if necessary. From (9), (14)

$$\frac{\partial}{\partial\theta_l}P_{jk}^{\nu+1} - \frac{\partial}{\partial\theta_l}P_{jk}^{\nu}$$
$$= \int_{\pm\infty}^{0} e^{-A_{jj}\sigma}\Big[\varepsilon\Big\{\Big(\frac{\partial}{\partial\theta_1}H_{jk}^{\nu+1}\Big)(\psi,\varepsilon) - \Big(\frac{\partial}{\partial\theta_1}H_{jk}^{\nu}\Big)(\psi,\varepsilon)\Big\}\frac{\partial}{\partial\theta_l}\psi_1$$
$$+ \cdots + \varepsilon\Big\{\Big(\frac{\partial}{\partial\theta_m}H_{jk}^{\nu+1}\Big)(\psi,\varepsilon) - \Big(\frac{\partial}{\partial\theta_m}H_{jk}^{\nu}\Big)(\psi,\varepsilon)\Big\}\frac{\partial}{\partial\theta_l}\psi_m\Big]e^{A_{kk}\sigma}d\sigma$$

$$\frac{\partial}{\partial\varepsilon}P_{jk}^{\nu+1} - \frac{\partial}{\partial\varepsilon}P_{jk}^{\nu}$$
$$= \int_{\pm\infty}^{0} e^{-A_{jj}\sigma}\Big[\Big\{H_{jk}^{\nu+1}(\psi,\varepsilon) - H_{jk}^{\nu}(\psi,\varepsilon)$$
$$+ \varepsilon\Big\{\Big(\frac{\partial}{\partial\theta_1}H_{jk}^{\nu+1}\Big)(\psi,\varepsilon) - \Big(\frac{\partial}{\partial\theta_1}H_{jk}^{\nu}\Big)(\psi,\varepsilon)\Big\}\frac{\partial}{\partial\varepsilon}\psi_1$$
$$+ \cdots + \varepsilon\Big\{\Big(\frac{\partial}{\partial\theta_m}H_{jk}^{\nu+1}\Big)(\psi,\varepsilon) - \Big(\frac{\partial}{\partial\theta_m}H_{jk}^{\nu}\Big)(\psi,\varepsilon)\Big\}\frac{\partial}{\partial\varepsilon}\psi_m$$
$$+ \varepsilon\Big\{\Big(\frac{\partial}{\partial\varepsilon}H_{jk}^{\nu+1}\Big)(\psi,\varepsilon) - \Big(\frac{\partial}{\partial\varepsilon}H_{jk}^{\nu}\Big)(\psi,\varepsilon)\Big\}\Big]e^{A_{kk}\sigma}d\sigma$$

$$\|P^{\nu+1} - P^{\nu}\|_1 \leqq \delta_1 K_7 \|H^{\nu+1} - H^{\nu}\|_1 + K_7\|H^{\nu+1} - H^{\nu}\|_0 \tag{31}$$
$$(\nu = 1, 2, \cdots).$$

Combining (21), (22), (31)

$$\begin{aligned}\|P^{\nu+1} - P^{\nu}\|_1 \leqq{}& \delta_1 K_7 2\{\|B\|_1 + \|P^{\nu}\|_1 + \|Q^{\nu-1}\|_1\}\\ &\cdot\{1 + K_3\}\|P^{\nu} - P^{\nu-1}\|_1 + K_7\|H^{\nu+1} - H^{\nu}\|_0\end{aligned} \tag{32}$$
$$(\nu = 1, 2, \cdots).$$

From (21), (22), (26), (27), (28), (30)

$$\|H^{\nu+1} - H^{\nu}\|_0 \leqq K_8\|P^{\nu} - P^{\nu-1}\|_0 \leqq 2^{-\nu+1}K_8\|P^1\|_0 \tag{33}$$
$$(\nu = 1, 2, \cdots),$$

and combining (32), (33)

$$\begin{aligned}\|P^{\nu+1} - P^{\nu}\|_1 \leqq{}& \delta_1 K_7 2\{\|B\|_1 + \|P^{\nu}\|_1 + \|Q^{\nu-1}\|_1\}\\ &\cdot\{1 + K_3\}\|P^{\nu} - P^{\nu-1}\|_1 + 2^{-\nu+1}K_7K_8\|P^1\|_0\end{aligned} \tag{34}$$
$$(\nu = 1, 2, \cdots).$$

Define

$$\begin{aligned}K_9 &= 2\|B\|_1 + 2(K_3 + 1)\|P^1\|_1 + (2K_3 + 3)K_{10}\\ K_{10} &= 2K_7K_8\|P^1\|_0.\end{aligned} \tag{35}$$

Although below it may be necessary to restrict δ_1 further, K_9 and K_{10} are computed using δ_1 which satisfies (29) and therefore K_9 and K_{10} are fixed constants for the remainder of the proof.

Define $V^1_{\delta_1}$ as a subset of the positive integers

$$V^1_{\delta_1} = \{\nu \mid \| B \|_1 + \| P^\nu \|_1 + \| Q^{\nu-1} \|_1 \leqq K_9\} . \tag{36}$$

Now restrict δ_1 further, if necessary, so that

$$2\delta_1 K_7 K_9 \{1 + K_3\} < \frac{1}{4} . \tag{37}$$

From (34), (35), (36), (37)

$$\| P^{\nu+1} - P^\nu \|_1 \leqq \frac{1}{4} \| P^\nu - P^{\nu-1} \|_1 + \left(\frac{1}{2}\right)^\nu K_{10} , \qquad \nu \in V^1_{\delta_1} . \tag{38}$$

From (38) by induction (as long as $1, 2, \cdots, \nu \in V^1_{\delta 1}$)

$$\begin{aligned} \| P^{\nu+1} - P^\nu \|_1 &\leqq \left(\frac{1}{4}\right)^\nu \| P^1 \|_1 + \left\{\left(\frac{1}{2}\right)^{2\nu-1} + \left(\frac{1}{2}\right)^{2\nu-2} \right. \\ &\quad \left. + \cdots + \left(\frac{1}{2}\right)^\nu\right\} K_{10} \leqq \left(\frac{1}{4}\right)^\nu \| P^1 \|_1 + \left(\frac{1}{2}\right)^{\nu-1} K_{10} \end{aligned} \tag{39}$$

$$\begin{aligned} \| P^\nu \|_1 &\leqq \left\{\left(\frac{1}{4}\right)^{\nu-1} + \left(\frac{1}{4}\right)^{\nu-2} + \cdots + \frac{1}{4} + 1\right\} \| P \|_1 \\ &\quad + \left\{\left(\frac{1}{2}\right)^{\nu-2} + \left(\frac{1}{2}\right)^{\nu-3} + \cdots + \frac{1}{2} + 1\right\} K_{10} + K_{10} \end{aligned} \tag{40}$$

and from (21), (25), (39)

$$\begin{aligned} \| Q^{\nu-1} \|_1 &\leqq \| Q^{\nu-1} - Q^{\nu-2} \|_1 + \cdots + \| Q^1 - Q^0 \|_1 + \| Q^0 \|_1 \\ &\leqq K_3 [\| P^{\nu-1} - P^{\nu-2} \|_1 + \cdots + \| P^1 - P^0 \|_1] + \| B \|_1 \\ &\leqq K_3 \left[\left\{\left(\frac{1}{4}\right)^{\nu-2} + \cdots + \frac{1}{4} + 1\right\} \| P^1 \|_1 \right. \\ &\quad \left. + \left\{\left(\frac{1}{2}\right)^{\nu-3} + \cdots + \frac{1}{2} + 1\right\} K_{10}\right] + \| B \|_1 . \end{aligned} \tag{41}$$

Hence from (35), (36), (40), (41)

$$V^1_{\delta_1} = \{\text{all positive integers}\}$$

and from (21), (22), (39)

$$(Q^\nu, H^\nu, P^\nu) \longrightarrow (Q, H, P)$$

in the C^1 topology on N_{δ_1} as $\nu \longrightarrow +\infty$. The assertion concerning multiple periodicity follows from standard arguments. This completes the proof of Theorem 1.

Using the proof method developed for Theorem 1, we can prove

THEOREM 2. *For system* (1) *where* $b(\theta, \varepsilon)$ *and* $B(\theta, \varepsilon)$ *are*

$C^k(1 \leqq k < \infty)$ with uniformly bounded derivatives on some N_δ, the change of variables (3) *and the transformed system* (4) *are also C^k with uniformly bounded derivatives on some N_{δ_1} provided $0 < \delta_1 \leqq \delta$ is sufficiently small.*

4. The analytic case. In this section we first construct a formal power series change of variables and then give a counterexample to show that in general the power series does not converge. The procedure is then modified in Theorem 3.

Consider the real analytic system of ordinary differential equations

$$\begin{aligned}\dot{\theta} &= a + \varepsilon b(\theta, \varepsilon)\\ \dot{x} &= (A + \varepsilon B(\theta, \varepsilon))x\end{aligned} \tag{42}$$

which is the same as (1) except that b, B are defined and analytic on

$$N_\delta^0 = \{(\theta, \varepsilon) \mid |\mathscr{I}(\theta)| < \delta, |\varepsilon| < \delta\}$$

and $b_{(\theta,\varepsilon)}, B_{(\theta,\varepsilon)}$ are uniformly bounded on N_δ^0, where $\mathscr{I} \equiv$ imaginary part, $\mathscr{I}(\theta) \equiv (\mathscr{I}(\theta_1), \cdots, \mathscr{I}(\theta_m))$. If we look for an analytic change of variables

$$x = (I + \varepsilon P(\theta, \varepsilon))y$$

such that

$$\dot{y} = (A + \varepsilon Q(\theta, \varepsilon))y ,$$

where P and Q are defined and analytic on some region $N_{\delta_1}^0$, P and Q satisfying (5); then we must solve (7a, b). Because (42) is analytic, it is natural to write (see the left side of 7b)

$$\dot{P}_{jk} = \sum_{l=1}^{m} \{a_l + \varepsilon b_l(\theta, \varepsilon)\} \frac{\partial}{\partial \theta_l} P_{jk}$$

where $\theta = (\theta_1, \cdots, \theta_m)$, $a = (a_1, \cdots, a_m)$, $b = (b_1, \cdots, b_m)$. Then, rather than (7a, b) we try to solve

$$Q_{jj} = B_{jj} + \varepsilon \sum_{l=1}^{n} B_{jl} P_{lj} \qquad (j = 1, \cdots, n) \tag{43a}$$

$$\begin{aligned}\sum_{l=1}^{m} a_l \frac{\partial}{\partial \theta_l} P_{jk} &= A_{jj} P_{jk} - P_{jk} A_{kk} + B_{jk}\\ &\quad + \varepsilon \sum_{l=1}^{m} B_{jl} P_{lk} - \varepsilon P_{jk} Q_{kk} - \varepsilon \sum_{l=1}^{m} b_l \frac{\partial}{\partial \theta_l} P_{jk}\\ &\qquad (j, k = 1, \cdots, n;\ j \neq k) .\end{aligned} \tag{43b}$$

Since B is analytic, it has a power series expansion

$$B(\theta, \varepsilon) = \sum_{g=0}^{\infty} \varepsilon^g B^g(\theta) .$$

(The super-scripts on B designate matrix coefficients in the expansion, not powers of the matrix B; this remark will hold below relative to P, Q, etc.) If we assume P and Q have power series expansions

$$P(\theta, \varepsilon) = \sum_{g=0}^{\infty} \varepsilon^g P^g(\theta)$$
$$Q(\theta, \varepsilon) = \sum_{g=0}^{\infty} \varepsilon^g Q^g(\theta) \ ,$$

then in (43) we can solve for the coefficients of P and Q recursively. Equating corresponding powers of ε in (43b) we have

$$\sum_{l=1}^{m} a_l \frac{\partial}{\partial \theta_l} P^g_{jk} = A_{jj} P^g_{jk} - P^g_{jk} A_{kk} + Z^g_{jk} \tag{44}$$

where $Z^g_j = Z^g_{jk}(\theta)$ is some known matrix function of θ; Z^g depends only on the coefficients of P and Q of degree less than g. The unique bounded solution of (44) in given by

$$P^g_{jk}(\theta) = \int_{\pm\infty}^{0} e^{-A_{jj}\sigma} Z^g_{jk}(\theta + a\sigma) e^{A_{kk}\sigma} d\sigma \tag{45}$$

where the lower limit of integration is chosen $+\infty$ for $A_{kk} < A_{jj}$ and $-\infty$ for $A_{jj} < A_{kk}$ with $<$ the order relation in (2).

Thus from (43a) and (45) we can construct the formal power series for P and Q recursively. If $\dim \theta > 1$, the following counter example shows that in general the formal power series for P and Q need not converge.

Consider the real analytic, four dimensional system of differential equations

$$\begin{aligned} \dot{\theta} &= a + \varepsilon b \\ \dot{x} &= (A + \varepsilon B(\theta))x \end{aligned} \tag{46}$$

where $\theta = (\theta_1, \theta_2)$, $a = (a_1, a_2)$, $b = (1, 0)$, $x = (x_1, x_2)$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & B_{12}(\theta) \\ 0 & 0 \end{pmatrix},$$

with $\lambda_1 \neq \lambda_2$. From (43a, b)

$$Q_{11} = B_{11} + \varepsilon\{B_{11}P_{11} + B_{12}P_{21}\} = \varepsilon P_{21} \tag{47a}$$

$$Q_{22} = B_{22} + \varepsilon\{B_{21}P_{12} + B_{22}P_{22}\} = 0 \tag{47b}$$

$$\begin{aligned} \sum_{l=1}^{2} a_l \frac{\partial}{\partial \theta_l} P_{12} &= (\lambda_1 - \lambda_2)P_{12} + B_{12} \\ &\quad + \varepsilon\{B_{11}P_{12} + B_{12}P_{22}\} - \varepsilon P_{12}Q_{22} - \varepsilon \frac{\partial}{\partial \theta_1} P_{12} \\ &= (\lambda_1 - \lambda_2)P_{12} + B_{12} - \varepsilon \frac{\partial}{\partial \theta_1} P_{12} \end{aligned} \tag{47c}$$

$$
\begin{aligned}
\sum_{l=1}^{2} a_l \frac{\partial}{\partial \theta_l} P_{21} &= (\lambda_2 - \lambda_1) P_{21} + B_{21} \\
&\quad + \varepsilon\{B_{21}P_{11} + B_{22}P_{21}\} - \varepsilon P_{21}Q_{11} - \varepsilon \frac{\partial}{\partial \theta_1} P_{21} \\
&= (\lambda_2 - \lambda_1) P_{21} - (\varepsilon P_{21})^2 - \varepsilon \frac{\partial}{\partial \theta_1} P_{21} \,.
\end{aligned} \tag{47d}
$$

From (47d), (45) we have $P_{21} \equiv 0$; from (47a, b) $Q \equiv 0$. Thus we have only P_{12} to compute in (47c).

Suppose B_{12} has the Fourier series representation

$$B_{12}(\theta) = \sum_{\nu_1, \nu_2 = -\infty}^{+\infty} \beta_{\nu_1, \nu_2} e^{i\{\nu_1\theta_1 + \nu_2\theta_2\}} \,.$$

If $\psi = (\psi_1, \psi_2)$, $\psi = \psi(t, \theta, \varepsilon)$ is the unique solution of the θ-equation in (46) with initial condition θ at $t = 0$, then

$$
\begin{aligned}
\psi_1(t, \theta, \varepsilon) &= \theta_1 + (a_1 + \varepsilon)t \\
\psi_2(t, \theta, \varepsilon) &= \theta_2 + a_2 t \,.
\end{aligned}
$$

From (47c)

$$
\begin{aligned}
\frac{d}{dt} P_{12}(\psi(t, \theta, \varepsilon)) &= (\lambda_1 - \lambda_2) P_{12}(\psi(t, \theta, \varepsilon)) \\
&\quad + \sum \beta_{\nu_1, \nu_2} e^{i\{\nu_1(\theta_1 + (a_1+\varepsilon)t) + \nu_2(\theta_2 + a_2 t)\}}
\end{aligned}
$$

and therefore, when ε is real,

$$P_{12}(\theta, \varepsilon) = \sum [\lambda_2 - \lambda_1 + i\{\nu_1(a_1 + \varepsilon) + \nu_2 a_2\}]^{-1} \beta_{\nu_1, \nu_2} e^{i\{\nu_1\theta_1 + \nu_2\theta_2\}} \,.$$

But since

$$[\lambda_2 - \lambda_1 + i\{\nu_1(a_1 + \varepsilon) + \nu_2 a_2\}] = 0$$

when

$$\varepsilon = \nu_1^{-1}[-i(\lambda_2 - \lambda_1) - \nu_2 a_2] - a_1 \,,$$

we conclude that in general $P_{12}(\theta, \varepsilon)$ can not be analytic in ε.

Rather than (42) we consider the real system of ordinary differential equations

$$
\begin{aligned}
\dot{\theta} &= a + \tilde{\varepsilon} b(\theta, \tilde{\varepsilon}) \\
\dot{x} &= (A + \varepsilon B(\theta, \varepsilon))x
\end{aligned} \tag{48}
$$

where θ, a, b are real vectors; a is a constant vector; $\tilde{\varepsilon}$ is a real perturbation parameter; b is defined and $C^k (1 \leqq k < \infty)$ on

$$N_\delta^1 = \{(\theta, \tilde{\varepsilon}) \mid \theta \in R^m, \, -\delta < \tilde{\varepsilon} < \delta\}$$

with $m = \dim \theta$, $R^m \equiv m$-dimensional euclidean space; b and all its derivatives of order less than or equal to k with respect to the components of $(\theta, \tilde{\varepsilon})$ are uniformly bounded on N^1_δ; A is the real, constant matrix given in (1); ε is a complex perturbation parameter; B is real analytic in ε and C^k in (θ, ε) on

$$N^2_\delta = \{(\theta, \varepsilon) \mid \theta \in R^m, |\varepsilon| < \delta\}$$

and has a power series expansion on N^2_δ,

$$B(\theta, \varepsilon) = \sum_{g=0}^{\infty} \varepsilon^g B^g(\theta) ;$$

B and all its derivatives of order less than or equal to k with respect to the components of θ are uniformly bounded on N^2_δ, and these derivatives of B are equal to the term by term derivatives of the power series for B.

THEOREM 3. *For system* (48) *there exists a unique change of variables*

$$x = (I + \varepsilon P(\theta, \tilde{\varepsilon}, \varepsilon))y \tag{49}$$

such that

$$\dot{y} = (A + \varepsilon Q(\theta, \tilde{\varepsilon}, \varepsilon))y \tag{50}$$

where I is the identity matrix; P and Q are real analytic in ε and C^k in $(\theta, \tilde{\varepsilon}, \varepsilon)$ on some

$$N^3_{\delta_1} = \{(\theta, \tilde{\varepsilon}, \varepsilon) \mid \theta \in R^m, -\delta_1 < \tilde{\varepsilon} < \delta_1, |\varepsilon| < \delta_1\}$$

with $0 < \delta_1 \leqq \delta$ sufficiently small; P and Q have power series expansions on $N^3_{\delta_1}$,

$$\begin{aligned} P(\theta, \tilde{\varepsilon}, \varepsilon) &= \sum_{g=0}^{\infty} \varepsilon^g P^g(\theta, \tilde{\varepsilon}) \\ Q(\theta, \tilde{\varepsilon}, \varepsilon) &= \sum_{g=0}^{\infty} \varepsilon^g Q^g(\theta, \tilde{\varepsilon}) ; \end{aligned} \tag{51}$$

P and Q and all their derivatives of order less than or equal to k with respect to the components of $(\theta, \tilde{\varepsilon})$ are uniformly bounded on $N^3_{\delta_1}$, and these derivatives of P and Q are equal to the term by term derivatives of the power series for P and Q, respectfully; with P and Q partitioned into sub-matrices similar to A,

$$\begin{aligned} P_{jj}(\theta, \tilde{\varepsilon}, \varepsilon) &\equiv 0 \qquad && (j = 1, \cdots, n) \\ Q_{jk}(\theta, \tilde{\varepsilon}, \varepsilon) &\equiv 0 \qquad && (j, k = 1, \cdots, n; j \neq k) \end{aligned} \tag{52}$$

if system (48) *has multiple period ω in θ, then P and Q also have multiple period ω in θ.*

Proof. Following the proof method of Theorem 1, we obtain from (49), (50), (52)

$$\begin{aligned} Q_{jj} &= B_{jj} + \varepsilon \sum_{l=1}^{n} B_{jl} P_{lj} && (j = 1, \cdots, n) \\ \dot{P}_{jk} &= A_{jj} P_{jk} - P_{jk} A_{kk} + B_{jk} \\ &\quad + \varepsilon \sum_{l=1}^{n} B_{jl} P_{lk} - \varepsilon P_{jk} Q_{kk} && (j, k = 1, \cdots, n; j \neq k) . \end{aligned} \tag{53}$$

Assuming P and Q have power series representations (51) with (as of yet) undetermined coefficients, we obtain upon equating coefficients of ε^g in (53)

$$Q^g_{jj} = B^g_{jj} + W^g_{jj} \tag{54a}$$

$$\dot{P}^g_{jk} = A_{jj} P^g_{jk} - P^g_{jk} A_{kk} + B^g_{jk} + Z^g_{jk} \tag{54b}$$

where $W^g = W^g(\theta, \tilde{\varepsilon})$ and $Z^g = Z^g(\theta, \tilde{\varepsilon})$ are matrices which depend on the coefficients of P and Q of degree less then g. To solve (54b) define

$$\begin{aligned} P^g_{jk}(\theta, \tilde{\varepsilon}) = \int_{\pm\infty}^{0} e^{-A_{jj}\sigma} \{ & B^g_{jk}(\psi(\sigma, \theta, \tilde{\varepsilon})) \\ & + Z^g_{jk}(\psi(\sigma, \theta, \tilde{\varepsilon}), \tilde{\varepsilon})\} e^{A_{kk}\sigma} d\sigma \end{aligned} \tag{55}$$

where $\psi = \psi(t, \theta, \tilde{\varepsilon})$ is the unique solution to the θ-equation in (48) with initial condition θ at $t = 0$. Using (54), (55) we compute the coefficients of P and Q recursively. By restricting δ_1 sufficiently, one easily shows that P^g, Q^g $(g = 0, 1, 2, \cdots)$ have, with respect to the components of $(\theta, \tilde{\varepsilon})$, uniformly bounded derivatives of order less than or equal to k. To show that the power series for P and Q converge and that the term-by-term derivatives up to order k of the power series also converge, one uses the proof method of Theorem 1. The assertion concerning multiple periodicity follows from a standard argument. This completes the proof of Theorem 3.

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