MAPPINGS AND DIMENSION IN GENERAL METRIC SPACES

JAMES KEESLING

In this paper necessary and sufficient conditions are developed for certain classes of continuous functions $f(X) = Y$, where X and Y are arbitrary metric spaces, to have the property that $\dim K = \dim f(K)$ for all closed $K \subset X$. In particular it is shown that if f is closed and $\dim f(K) > \dim K$ for some closed $K \subset X$, then there exists a closed $K' \subset X$ so that $\dim K' = 0$ and $\dim f(K') > 0$. These results are then used to show that if f is closed and finite to one so that the multiplicity function of f takes on at most $k + 1$ distinct values, then $\dim K \leq \dim f(K) \leq \dim K + k$ for all closed $K \subset X$.

The purpose of this paper is to investigate the relation of the dimension of a closed subset of the domain with the dimension of its image in the range under various classes of continuous functions. In the first part of the paper we investigate this relation for closed subsets of the domain which have dimension zero. Using these results we characterize the property of being dimension preserving on closed subsets for a large class of mappings. In the second part of the paper we then show several important types of mappings to be dimension preserving on closed subsets. In the last section we generalize a result of Hurewicz [3]. The results of this paper are related to those of a number of investigators among whom are: Alexandroff [1], R. Hodel [2], K. Nagami [7], J. Nagata [8, pp. 68–73], J. H. Roberts [9], J. Suzuki [10], and R. F. Williams [11]. As indicated in the title, the setting for our study is the class of metric spaces.

Notation: Throughout the paper X and Y denote metric spaces and f a continuous function from X onto Y. By $\dim X$ is meant the Lebesgue covering dimension of X. This is, of course, equal to the large inductive dimension of X, denoted $\operatorname{Ind} X$, in metric spaces. We let $\operatorname{ind} X$ denote the small inductive dimension of X. The relation $\operatorname{ind} X = \operatorname{Ind} X$ holds when X is locally separable but not in general otherwise. The necessary background in dimension theory for general metric spaces (resp. separable metric spaces) will be found in J. Nagata [8] (resp. Hurewicz and Wallman [4]). Any additional hypotheses on f, X, or Y will be explicitly stated in each theorem.

I. **Closed $K \subset X$ with $\dim K = 0$.** We will need the following lemma several times throughout the paper.

I.1. **Lemma.** *Let* $\dim X = n, n$ *finite, and let* $0 \leqq i \leqq n$. *Then* $\exists A, B$ *so that* $X = A \cup B$ *with* $\dim A = i$, $\dim B = n - i - 1$, *and with* A *an* F_σ *in* X.

Proof. By the decomposition theorem ([8], p. 19) it is possible to find A_1 and B_1 so that $X = A_1 \cup B_1$ with $\dim A_1 = i$ and $\dim B_1 = n - i - 1$. By Theorem II. 10, p. 32 [8] there exists $B \supset B_1$ with B a G_δ in X and $\dim B = n - i - 1$. Let $A = X - B$. Then A is an F_σ and $\dim A = i$.

I.2. **Definition.** Suppose for f we have $X = \bigcup_{i=1}^{\infty} A_i$, A_i closed in X, $f(A_i)$ closed in Y, and $f \,|\, A_i : A_i \to f(A_i)$ closed for all i. We say then that f is σ-*closed*. These mappings arise naturally as (1) continuous mappings of σ-compact spaces, (2) finite to one open mappings, and (3) open mappings with discrete point inverses on separable metric spaces.

I.3. **Lemma.** f *is* σ-*closed if and only if* $X = \bigcup_{i=1}^{\infty} A_i$ *so that for each* i, A_i *is an* F_σ *in* X, $f(A_i)$ *is an* F_σ *in* Y *and* $f \,|\, A_i : A_i \to f(A_i)$ *is a closed mapping.*

Proof. Suppose the latter condition holds. Then for each i let $A_i = \bigcup_{k=1}^{\infty} F_{k,i}$ with each $F_{k,i}$ closed in X and $F_{k,i} \subset F_{k+1,i}$ for all k. Likewise let $f(A_i) = \bigcup_{k=1}^{\infty} B_{k,i}$ with $B_{k,i}$ closed in Y and $B_{k,i} \subset B_{k+1,i}$. Define $C_{k,i} = F_{k,i} \cap f^{-1}(B_{k,i})$. Then one can easily verify that $X = \bigcup_{k,i} C_{k,i}$ satisfies the properties for f to be σ-closed.

This characterization of σ-closed will be useful in showing certain mappings to be σ-closed. The next theorem is the fundamental result of this section.

I.4. **Theorem.** *Let* f *be a closed mapping with* $\dim X = n$, $0 \leqq n < \infty$, *and* $\dim Y \geqq k(n + 1)$, $1 \leqq k < \infty$. *Then there is a closed set* $K \subset X$ *with* $\dim K = 0$ *and* $\dim f(K) \geqq k$.

Proof. This is clearly true for $\dim X = 0$. We proceed by induction. Suppose $\dim X = n > 0$ and that the theorem holds for all lesser values of n. By Lemma I.1 we let $X = A \cup B$ with A an F_σ, $\dim A = 0$ and $\dim B = n - 1$.

Case (i). $\dim f(A) \geqq k$.
In this case let $A = \bigcup_{i=1}^{\infty} F_i$ with F_i closed for all i. Since $f(F_i)$ is closed in Y for all i, by the sum theorem for closed sets ([8], p. 17) there is an i so that $\dim f(F_i) \geqq k$. Let $K = F_i$ in this case.

Case (ii). $\dim f(A) < k$.

In this case let $C \supset f(A)$ in Y with C a G_δ and $\dim C = \dim f(A) < k$. Then $Y - C$ is an F_σ and $\dim (Y - C) \geq \dim Y - \dim C - 1 \geq kn$. Let $Y - C = \bigcup_{i=1}^{\infty} F_i$ so that each F_i is closed. Then $\exists i$ so that $\dim F_i \geq kn$. Since $f^{-1}(F_i) \leq n - 1$, apply our induction assumption to $f \,|\, f^{-1}(F_i)$ to complete the proof for this case.

I.5. COROLLARY.   I.4 *remains true if we replace f closed with f σ-closed.*

I.6. COROLLARY.   *If f is σ-closed and $K \subset X$, K closed and $\dim K = n$, $0 \leq n < \infty$, with $\dim f(K) \geq k(n + 1)$, $1 \leq k < \infty$; then $\exists K'$ closed, $K' \subset K$ with $\dim K' = 0$ and $\dim f(K') \geq k$.*

1.7. COROLLARY.   *If f is σ-closed and f raises the dimension of any closed subset, then f raises the dimension of a closed subset of dimension zero.*

I.8. DEFINITION.   Let f be such that for every closed set $K \subset X$, $\dim K = \dim f(K)$. Such a function will be said to be *dimension preserving on closed subsets*. If $\dim f^{-1}(y) = 0$ for all $y \in Y$, then f is said to be 0-*dimensional*.

The next theorem is due to Hurewicz who proved it for separable metric spaces. It has been proved for more general spaces including arbitrary metric spaces by K. Morita and K. Nagami. For a proof see Nagata [8, pp. 63–68].

I.9. THEOREM.   *If f is closed and $\dim X - \dim Y \geq k \geq 0$, then $\exists y \in Y$ so that $\dim f^{-1}(y) \geq k$.*

Clearly the theorem is true if we replace closed by σ-closed. We can therefore state the following characterization of dimension preserving on closed subsets for σ-closed mappings.

I.10. THEOREM.   *If f is σ-closed, then f is dimension preserving on closed subsets if and only if f is 0-dimensional and has the property that $K \subset X$, K closed, and $\dim K = 0$ implies that*

$$\dim f(K) = 0 \, .$$

Proof. If f lowered the dimension of some closed subset, by I.9 f would not be 0-dimensional. If f raised the dimension of some closed set, then it would raise the dimension of some closed set of dimension zero.

Returning to I.4 we now prove that if X is Euclidean space, then we can improve the estimate given in I.4.

I.11. THEOREM. *Let* $X = E^n$ *and* $\dim Y \geqq kn + 1$ *where* E^n *denotes Euclidean* *n-space and* $1 \leqq k < \infty$. *Then* $\exists K$ *closed in* E^n, $\dim K = 0$ *with* $\dim f(K) \geqq k$.

Proof. Let A be a countable dense set in E^n. Then since $f(A)$ is countable, it must have dimension zero. Let $B \supset f(A)$ be a G_δ so that $\dim B = 0$. Then $\dim (Y - B) \geqq kn$ and $Y - B$ is an F_σ. Let $Y - B = \bigcup_{i=1}^\infty F_i$ be a countable union of closed sets. Let $\dim F_i \geqq kn$. Since $f^{-1}(F_i) \subset E^n - A$, $f^{-1}(F_i)$ has no interior points and $\dim f^{-1}(F_i) \leqq n - 1$. Now $f \mid f^{-1}(F_i)$ is σ-closed since E^n is σ-compact. The proof is now completed by applying I.5 to $f \mid f^{-1}(F_i)$.

I.12. COROLLARY. *If* $X = E^1$ *and* $\dim Y = m > 1$, *then* $\exists K$ *closed in* X, $\dim K = 0$ *with* $\dim f(K) \geqq m - 1$.

The concluding theorem of this section is for an arbitrary continuous function which lowers dimension.

I.13. THEOREM. *Let* $\dim Y = n$ *and* $\dim X \geqq k(n + 1)$ *where* $0 \leqq n < \infty$ *and* $1 \leqq k < \infty$; *then* $\exists K \subset X$, K *closed,* $\dim K \geqq k$ *with* $\dim f(K) = 0$.

Proof. By induction on n. In case $n = 0$, let $K = X$. Let $\dim Y = n > 0$ and suppose the theorem is true for all lesser values of n. Let $Y = A \cup B$ with A an F_σ, $\dim A = 0$ and $\dim B = n - 1$. We then have two cases.

Case (i). $\dim f^{-1}(A) \geqq k$.
In this case let $A = \bigcup_{i=1}^\infty F_i$ closed and get an i so that

$$\dim f^{-1}(F_i) \geqq k .$$

Case (ii). $\dim f^{-1}(A) < k$.
Then let $C \supset f^{-1}(A)$ be a G_δ so that $\dim C = \dim f^{-1}(A) < k$. Then $\dim (X - C) \geqq kn$. Let $X - C = \bigcup_{i=1}^\infty F_i$ where each F_i is closed in X. Then $\exists i$ so that $\dim F_i \geqq kn$. Since $f(F_i) \subset B$, $\dim f(F_i) \leqq n - 1$. We now apply our induction assumption to $f \mid F_i$ and theorem follows.

I.14. COROLLARY. *If f lowers the dimension of any closed set of the domain, then f takes a closed set of positive dimension onto a set of dimension zero.*

II. **Functions dimension preserving on closed subsets.** In this section we will show the following types of functions to be dimension preserving on closed subsets and σ-closed: (1) f open with $f^{-1}(y)$ discrete with X separable; (2) f finite to one and open; (3) f σ-closed, open, $f^{-1}(y)$ discrete for all $y \in Y$; (4) f σ-closed and exactly k to one. We will first prove (3) and then show that (1) and (2) follow. We will then show that (4) is a result of (2). Some examples and other theorems will be inserted at appropriate points in this chain of argument.

II.1. THEOREM. *Let f be open with $f^{-1}(y)$ discrete for all $y \in Y$. Then for all $K \subset X$, dim $K \leqq$ dim $f(K)$.*

Proof. Let $\{A_{\alpha,n} : \alpha \in \Gamma_n\}$ be a locally finite open cover of X so that diam $(A_{\alpha,n}) \leqq 1/n$ for all $\alpha \in \Gamma_n$.   Let $B_{\alpha,n} = \{x \in A_{\alpha,n} : f^{-1}f(x) \cap A_{\alpha,n} = x\}$. By the openness of f, $B_{\alpha,n}$ is closed in $A_{\alpha,n}$ as is $f(B_{\alpha,n})$ in $f(A_{\alpha,n})$. Therefore $B_{\alpha,n}$ and $f(B_{\alpha,n})$ are $F'_\sigma s$. It is easily seen that

$$f \mid B_{\alpha,n} : B_{\alpha.n} \to f(B_{\alpha,n})$$

is open. Since it is also one to one and continuous, it is a homeomorphism. Therefore dim $K \cap B_{\alpha,n} = $ dim $f(K \cap B_{\alpha,n})$ for all n and all $\alpha \in \Gamma_n$. By the sum theorem, $\exists B_{\alpha,n}$ so that dim $K \cap B_{\alpha,n} \geqq k$ if dim $K \geqq k$, noting that $\bigcup_{\alpha,n} B_{\alpha,n} = X$ is σ-locally finite. But then dim $f(K) \geqq k$.

II.2. EXAMPLE. Let I be the unit interval $[0, 1]$ in E^1 with the ordinary topology and let I^* be the same set with the discrete topology. Define $f : I \times I^* \to I$ so that $f(a, b) = a$. Then f is open and $f^{-1}(y)$ is discrete for all y. Let $\Delta = \{(a, a) : a \in I\} \subset I \times I^*$. Then Δ is discrete and closed, hence dim $\Delta = 0$. But $f(\Delta) = I$. Therefore we cannot prove equality in II.1 even for closed K in X without additional assumptions on f.

II.3. THEOREM. *If f is open, σ-closed, and $f^{-1}(y)$ is discrete for all $y \in Y$, then f is dimension preserving on closed subsets.*

Proof. Since f is 0-dimensional we need only show that the second condition of I.10 holds. Let $K \subset X$ with dim $K = 0$ and K closed and suppose dim $f(K) > 0$. We may assume $K \subset A_i$ for some i where $X = \bigcup_{j=1}^{\infty} A_j$ as in the definition of σ-closed. Let $\{A_{\alpha,n} : \alpha \in \Gamma_n\}$ be a locally finite open cover of X with diam $(A_{\alpha,n}) \leqq 1/n$. Let $F_{\alpha,n} \subset A_{\alpha,n}$ so that $\{F_{\alpha,n} : \alpha \in \Gamma_n\}$ forms a closed cover of X. Let

$$Q_{\alpha,n} = \{x \in F_{\alpha,n} \cap K: f^{-1}f(x) \cap A_{\alpha,n} = x\} .$$

Then $Q_{\alpha,n}$ is closed in X by the openness of f. Now $\{Q_{\alpha,n}: \alpha \in \Gamma_n\}$ is locally finite, hence closure preserving. Since $Q_{\alpha,n} \subset A_i$ and $f \mid A_i$ is closed we get that $\{f(Q_{\alpha,n}): \alpha \in \Gamma_n\}$ is closure preserving in Y. By a theorem due to Nagami ([8], p. 18) we have that if $\dim f(Q_{\alpha,n}) \leq k$ for all $\alpha \in \Gamma_n$, then $\dim \bigcup_\alpha f(Q_{\alpha,n}) \leq k$. Now $f \mid Q_{\alpha,n}$ is a homeomorphism onto $f(Q_{\alpha,n})$, hence $\dim f(Q_{\alpha,n}) \leq 0$. Now $B_n = \bigcup_{\alpha \in \Gamma_n} f(Q_{\alpha,n})$ is closed with $\dim B_n = 0$. We can easily verify that $\bigcup_{n=1}^\infty B_n = f(K)$. Therefore $\dim f(K) = 0$, a contradiction of our assumption that $\dim f(K) > 0$. That is, f must be dimension preserving on closed subsets.

II.4. THEOREM. *If f is open with discrete point inverses and X is separable, then f is dimension preserving on closed subsets.*

Proof. Let $\{A_i\}$ be a countable basis for X. Let

$$B_i = \{x \in A_i: f^{-1}f(x) \cap A_i = x\}.$$

Then B_i is an F_σ as is $f(B_i)$. Now $f \mid B_i$ is a homeomorphism onto $f(B_i)$. Applying I.3 we get that f is σ-closed. Applying II.3 we get that f is dimension preserving on closed subsets.

II.5. *Note.* If we actually apply the proof of I.3 to this case we get $X = \bigcup_{i=1}^\infty A_i$, A_i closed, $f(A_i)$ closed, with $f \mid A_i$ a homeomorphism onto $f(A_i)$.

II.6. LEMMA. *If f is open and exactly k to one, then f is closed.*

Proof. Let $y_n \to y$ be a sequence of distinct points in Y. Let $x_n \in f^{-1}(y_n)$ for all n and suppose that $\{x_n\}$ has no limit point in X. Then we may choose $\{V_i\}_{i=1}^k$ a disjoint collection of open sets, each containing no point of the set $\{x_n\}$ where $z_i \in V_i$ and $f^{-1}(y) = \{z_1, \cdots, z_k\}$. By the openness of f, $\exists N_i$ such that $n \geq N_i$ implies that $f^{-1}(y_n) \cap V_i \neq \varnothing$. Let $N = \max \{N_i\}_{i=1}^k$. Then $f^{-1}(y_N) \cap V_i \neq \varnothing$ for all i. But $f^{-1}(y_N)$ consists of exactly k points. Therefore there is exactly one point of $f^{-1}(y_N)$ in each V_i and every point is in one of the V_i's. Therefore $x_N \in V_i$ for some i, a contradiction. Therefore the sequence $\{x_n\}$ must have some convergent subsequence and f must be closed.

II.7. THEOREM. *If f is open and finite to one, then f is dimension preserving on closed subsets.*

Proof. Let $Y_k = \{y \in Y: f^{-1}(y)$ has k or more points$\}$. Let $\mu(y)$ be the number of points in $f^{-1}(y)$. Since f is open, μ is lower semi-

continuous. Therefore Y_k is open for all k. Therefore

$$B_k = Y_k \cap (Y - Y_{k+1})$$

is an open set intersected with a closed set, hence is an F_σ. But $B_k = \{y: \mu(y) = k\}$. Therefore $f \mid f^{-1}(B_k)$ is closed onto B_k by II.6 since an open function restricted to an inverse set is open onto its image. Applying I.3 and II.3 we have that f is dimension preserving on closed subsets.

II.8. THEOREM. *If f is closed and exactly k to one, then f is dimension preserving on closed subsets.*

Proof. Let $y \in Y$ and let $f^{-1}(y) = \{x_1, \cdots, x_k\}$. If $k = 1$, then f is a homeomorphism. Suppose $k > 1$ and define

$$\Sigma_y = \min \{d(x_i, x_j): x_i \neq x_j\}.$$

Then $\Sigma_y > 0$ for all $y \in Y$. Using the closedness of f it is possible to show that if $A_n = \{y \in Y: \Sigma_y \geq 1/n\}$, then A_n is closed in Y and $f \mid f^{-1}(A_n)$ onto A_n is open. Thus f is dimension preserving on closed subsets by II.7.

II.9. THEOREM. *If f is σ-closed and exactly k to one, then f is dimension preserving on closed subsets.*

Proof. Let $X = \bigcup_{i=1}^{\infty} A_i$ as in the definition of σ-closed. We may assume $A_i \subset A_{i+1}$ for all i. Let $B_i = \{x \in A_i: f^{-1}f(x) \subset A_i\}$. Then $f(B_i) = \{y \in f(A_i): \mu_i(y) = \max \mu_i\}$ where $\mu_i(y)$ is the number of points in $f^{-1}(y) \cap A_i$. If we define $\Sigma_y = \min \{d(x_i, x_j); x_i \neq x_j\}$ where $f^{-1}(y) = \{x_1, \cdots, x_k\}$ and then define $C_n = \{y \in f(A_i): y \in f(B_i)$ and $\Sigma_y \geq 1/n\}$. Then using the closedness of $f \mid A_i$, C_n will be a closed set in Y. But $f(B_i) = \bigcup_{n=1}^{\infty} C_n$. Therefore $f(B_i)$ is an F_σ and since $f^{-1}f(B_i) = B_i$, B_i is also an F_σ. Now if $x \in X$, then since $f^{-1}f(x)$ is finite, $\exists A_i$ so that $f^{-1}f(x) \subset A_i$. Therefore $x \in B_i$ and $\bigcup_{i=1}^{\infty} B_i = X$. Now $f \mid B_i$ is closed onto $f(B_i)$ since B_i is an inverse set in a set on which f is closed. Since $f \mid B_i$ is exactly k to one, $f \mid B_i$ is dimension preserving on closed subsets. Thus f is dimension preserving on closed subsets. Thus f is dimension preserving on closed subsets on all of X by the sum theorem.

II.10. COROLLARY. *If f is σ-closed and exactly k to one except for a set $B \subset Y$ which is an F_σ with $\dim B = 0$ and $f^{-1}(y)$ has less than k points for all $y \in B$, then f is dimension preserving on closed subsets.*

Proof. Let $C_n = \{y \in Y - B : \Sigma_y \geq 1/n\}$. Then by the σ-closedness of f, C_n is an F_σ. Now $f \mid f^{-1}(C_n)$ is σ-closed onto C_n and exactly k to one, therefore dimension preserving on closed subsets. Now if $K \subset X$, K closed with $\dim K = 0$ and $\dim f(K) > 0$, then

$$\dim f(K) \cap B > 0 \,.$$

But by assumption $\dim B = 0$. Therefore $\dim f(K) = 0$ and f is dimension preserving on closed subsets.

II.11. EXAMPLE. Let I be the unit interval and C the Cantor ternary set. Let $f : C \to I$ be the cantor ternary function onto the interval. Then f is one to one except for a countable set $B \subset I$. However, for $y \in B$, $f^{-1}(y)$ has two points. Therefore our assumption that $f^{-1}(y)$ have fewer than k points in II.10 was not superfluous.

III. Closed $K \subset X$ with $\dim K < \dim f(K)$. The results which motivated the theorems of this section are due mainly to Hurewicz [3]. His basic result is the following.

III. 1. THEOREM. *Let f be closed and finite to one with X and Y separable metric spaces. Then if $\dim Y - \dim X \geq k \geq 0$, then μ takes on at least $k + 1$ distinct values, where $\mu(y)$ is the number of points in $f^{-1}(y)$.*

Apparently there has been some difficulty in generalizing this result to arbitrary metric spaces since the statement in J. Nagata [8, p. 68] leaves out a considerable amount of the force of III.1. Also his statement of Suzuki's result (footnote p. 73 [8]) is again only a partial result. Our results in this section will imply the full force of III.1 in arbitrary metric spaces and will be considerably more general. Our basic result is III.2.

III.2. THEOREM. *If f is σ-closed and finite to one and if $K \subset X$ is closed with $\dim f(K) - \dim K \geq k \geq 0$, then μ takes on at least $k + 1$ values on $f(K)$.*

We will need a sequence of lemmas before we can prove III.2.

III.3 LEMMA. *Let $\dim X = n$ and $A \subset X$ be an F_σ so that*

$$\dim A \leq n - 1 \,.$$

Then there is an $N \subset X - A$, N an F_σ, with $\dim N = 0$ and

$$\dim (X - N) = n - 1 \,.$$

Proof. Let $X = B \cup C$ with B an F_σ, $\dim B = n - 1$ and $\dim C = 0$. Then $A \cup B$ is the union of two F_σ's each of dimension less than n. Therefore $\dim (A \cup B) = n - 1$. Let $D \supset A \cup B$ be a G_δ with $\dim D = n - 1$. Let $N = X - D$. Then $\dim N = 0$ since $N \subset C$. Clearly N has the other required properties.

III.4. **Lemma.** *Let f be σ-closed and finite to one. Let $Y_m = \{y \in Y: \mu(y) \geq m\}$. Then Y_m is an F_σ in Y.*

Proof. Let $X = \bigcup_{i=1}^{\infty} A_i$ be as in the definition of σ-closed. We may suppose that $A_i \subset A_{i+1}$. It is sufficient to show that

$$B_i = \{y \in Y_m: f^{-1}(y) \subset A_i\}$$

is an F_σ for all i. Let $C_n = \{y \in Y_m \cap f(A^i): \exists \{x_1, \cdots, x_m\} \subset f^{-1}(y) \cap A_i$ so that $d(x_i, x_j) \geq 1/n$ for $i \neq j\}$. Then using the closedness of f on A_i, it is possible to show that C_n is closed in A_i for all n. But $\bigcup_{n=1}^{\infty} C_n = B_i$. Therefore Y_m is an F_σ as asserted.

III.5. **Theorem.** *Suppose f is σ-closed and finite to one. Let $Y_k = \{y \in Y: \mu(y) \geq k\}$. Let $m = \min \{\mu(y): y \in Y\}$. Then if $K \subset X$ is closed with $\dim K \leq n$ and $\dim [f(K) \cap Y_{m+1}] < n$, then $\dim f(K) \leq n$.*

Proof. Let K satisfy the hypotheses of the theorem. We may assume n to be finite. If $n = 0$, then $\dim [f(K) \cap Y_{m+1}] < 0$ means that $f(K) \cap Y_{m+1} = \varnothing$. Therefore $K \subset f^{-1}(Y_m - Y_{m+1})$. By II.9 $\dim f(K) \leq 0$. Now let $n > 0$ and suppose that the theorem holds for lesser values of n. By the hypothesis $\dim [f(K) \cap Y_{m+1}] \leq n - 1$. Since f is 0-dimensional we have $\dim [f^{-1}(f(K) \cap Y_{m+1}) \cap K] \leq n - 1$. Let $A = f^{-1}(f(K) \cap Y_{m+1}) \cap K$. Then A is an F_σ in K by III.4. By III.3 let $N \subset K - A$ be an F_σ so that $\dim N \leq 0$ and $\dim (K - N) \leq n - 1$. Then $N \subset f^{-1}(Y_m - Y_{m+1})$. Since f is dimension preserving on closed subsets on $f^{-1}(Y_m - Y_{m+1})$, $\dim f(N) \leq 0$. Since f is σ-closed, $f(N)$ is also an F_σ. Let $N' \subset Y_{m+1} \cap f(K)$ so that N' is an F_σ in $Y_{m+1} \cap f(K)$, $\dim N' \leq 0$ and so that $\dim [Y_{m+1} \cap f(K) - N'] \leq n - 2$. Note that N' is also an F_σ in Y, since $f(K)$ and Y_{m+1} are both F_σ's. Now let:

$$Y' = f(K) - N' \cup f(N)$$
$$K' = f^{-1}(Y') \cap K.$$

Then $f(K') = Y'$. We also have;

$$\dim [Y' \cap Y_{m+1}] = \dim [f(K) \cap Y_{m+1} - N'] < n - 1.$$

Now $K' \subset K - N$ and therefore $\dim K' \leq n - 1$. Observing that $f \mid f^{-1}(Y')$ onto Y' is σ-closed and K' is closed in $f^{-1}(Y')$ we can apply

our induction assumption to get $\dim f(K') = \dim Y' \leqq n - 1$. Let $M = f(K) - Y'$. Then $M = f(N) \cup N'$ and since $f(N)$ and N' are both F_σ's, $\dim M \leqq 0$. By the sum theorem, $\dim f(K) \leqq \dim Y' + \dim M + 1 \leqq n$.

III.6. COROLLARY. *If f is σ-closed and finite to one and $K \subset X$ is closed with $\dim f(K) > \dim K$, then if $m = \min \{\mu(y): y \in f(K)\}$, then $\dim [f(K) \cap Y_{m+1}] \geqq \dim f(K) - 1$.*

Proof. In III.5 let $n = \dim f(K) - 1$.

Proof of III.2. We simply apply III.6 inductively in the following manner. Let $m_1 = \min \{\mu(y): y \in f(K)\}$. Then if $k = 0$ we are finished. Suppose $k > 0$. Then by III.6 we get that

$$\dim [f(K) \cap Y_{m_1+1}] \geqq \dim f(K) - 1 \geqq 0 .$$

Let $m_2 = \min \{\mu(y): y \in f(K) \cap Y_{m_1+1}\}$. Then $m_2 > m_1$ and μ takes on the value m_2 on $f(K)$. Let $K' = K \cap f^{-1}(Y_{m_1+1})$ which is closed in $f^{-1}(f(K) \cap Y_{m_1+1})$. Since f restricted to this latter set is σ-closed we now can say that if $k - 1 > 0$, then $\dim f(K') - \dim K' \geqq k - 1 > 0$ and we can repeat the above process for K' to get an

$$m_3 = \min \{\mu(y): y \in f(K') \cap Y_{m_2+1}\} .$$

We can continue this process until $k - i = 0$, that is, at least k times to get a sequence $\{m_1 < \cdots < m_{k+1}\}$ with $\mu(y_i) = m_i$ for some $y_i \in f(K)$ for each $i \leqq k + 1$.

III.7. THEOREM. *If f is σ-closed and $K \subset X$ is closed and locally Euclidean with $\dim f(K) - \dim K \geq k > 0$ then μ takes on at least $k + 2$ distinct values in $f(K)$.*

Proof. Since K is locally Euclidean in the subspace topology it must be locally separable and hence $\text{ind } K = \dim K$. Therefore $\dim K = \sup \{n: K \text{ contains a Euclidean neighborhood of dimension } n\}$. Let us assume μ takes on a finite number of values on $f(K)$. Let $m = \max \{\mu(y): y \in f(K)\}$ and let $B = \text{int}_K (f^{-1}(Y_m) \cap K)$ be the interior in K of $f^{-1}(Y_m) \cap K$. Since B is open in K which is closed in X, K must be an F_σ in X. Since $B \subset f^{-1}(Y_m - Y_{m+1})$, $\dim f(B) = \dim B \leqq \dim K$. Let $K' = K - B$. Then K' is closed. Now $f(K) = f(K') \cup f(B)$ is the union of two F_σ's. Therefore

$$\dim f(K) = \max \{\dim f(K'), \dim f(B)\} = \dim f(K') .$$

We now have two cases:

Case (i). $f^{-1}(Y_m) \cap K' = \emptyset$.

In this case μ takes on at least $k + 1$ values on $f(K')$. It also takes on the additional value m which makes $k + 2$ values.

Case (ii). $f^{-1}(Y_m) \cap K' \neq \emptyset$.

In this case we repeat the construction of $\{m_1 < \cdots < m_{k+1}\}$ as in III.2 on $f(K')$. But simply note that for $m = \max \{\mu(y): y \in f(K')\}$, $f^{-1}(Y_m) \cap K'$ has no interior points in K. Therefore

$$\dim [f^{-1}(Y_m) \cap K'] < \dim K ,$$

so we can continue the construction one more step to get a sequence $\{m_1 < \cdots < m_{k+2}\}$.

The last result shows that it is possible to generalize these results to σ-closed functions with $Fr(f^{-1}(y))$ finite for all $y \in Y$.

III.8. THEOREM.   *Let f be σ-closed with $Fr(f^{-1}(y))$ finite for all $y \in Y$; then if $K \subset X$ is closed with $\dim f(K) - \dim K \geq k > 0$, then if we let $X_m = \bigcup \{Fr(f^{-1}(y)): Fr(f^{-1}(y))$ has exactly m points$\}$, then $K \cap X_m \neq \emptyset$ for at least $k + 1$ distinct m's.*

Proof. We note that $\bigcup_{m=1}^{\infty} X_m = X - \bigcup_{y \in Y} \text{int} (f^{-1}(y))$ and consequently $\bigcup_{m=1}^{\infty} X_m$ is closed. Therefore f restricted to this set is σ-closed and finite to one. If $A = f(K) - f(K \cap (\bigcup X_m))$ then by the σ-closedness of f, A is the countable union of closed discrete sets, therefore A is an F_σ of dimension zero. Since $\dim f(K) > 0$, we must have $\dim f(K) = \dim f(K \cap (\bigcup X_m))$. Applying III.2 to $K \cap (\cup X_m)$ we get that $K \cap X_m \neq \emptyset$ for at least $k + 1$ distinct m's.

III.9. THEOREM.   *If K is locally Euclidean in III.8, then we can replace $k + 1$ by $k + 2$.*

Proof. Combining the proofs of III.7 and III.8 with slight modification will yield III.9.

IV. Final remarks. One may have in mind to generalize the results contained in this paper to arbitrary sets without restricting them to be closed. The following example serves to indicate that without strong conditions on the function the prospects are not promising.

IV.1. EXAMPLE.   Let $X = \{(x, y) \in E^2; 0 \leq x \leq 1$ and $y = 0$ or $y = 1\}$. Let $f: X \to [0, 1]$ so that $f(x, y) = x$. Then f is open and closed and two to one. Let A be the rationals in $[0, 1]$ and B the irrationals in $[0, 1]$. Let $K = \{0\} \times A \cup \{1\} \times B$. Then $\dim K = 0$ and

$$\dim f(K) = 1 \, .$$

Once one has this example in mind he will likely be able to construct a variety of other examples. This indicates how important our assumption was that the sets be closed.

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