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FIXED POINTS FOR ITERATES

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Let $f\colon X \to X$ be a continuous map of a compact polyhedron X into itself and H a homology theory with rational coefficients. In the first section a variety of theorems are proved connecting the existence or nonexistence of fixed points for certain iterates of f with a variety of other information such as: conditions on the Betti numbers of X, f being a homomorphism, certain induced homomorphisms $f_i^*\colon H_i(X) \to H_i(X)$ being isomorphisms, factors of the Lefschetz numbers $\Lambda(f^n)$, the gross behavior of f with respect to the components of X, and certain other iterates of f being fixed point free. One of the theorems proven is that if $H_i(X) = 0$ for odd i then there exists an $x \in X$ and an n, $1 \leqq n \leqq \Sigma_i \dim H_i(X)$, such that $f^n(x) = x$. Another theorem is that if f^n is fixed point free for $1 \leqq n \leqq p/2$ and $f^{p*} =$ identity then p divides the Euler characteristic of X.

The resuls of § 1 are applied to the problem of coincidence in § 2. In § 3 most of the results of the first two sections are shown to carry over to set-valued functions. Next it is shown that the behavior of f in any neighborhood of the fixed points of $f, f^2, \cdots, f^q$, where $q = \Sigma_i \dim H_i(X)$, determines all the Lefschetz numbers $\Lambda(f^n), n > 0$, and sheds a certain amount of light on f^*. The Euler characteristic of a compact polyhedron X is determined by the Lefschetz numbers $\{\Lambda(f^n)\}_{n=1}^\infty$ whenever $f\colon X \to X$ induces an isomorphism f^*. This fact is proven in the last section.

All the analysis centers around an index $K(f)$ discovered by J. L. Kelley and E. Spanier to whom I am indebted for invaluable help.[1] I am also grateful to V. Singh for several discussions.

1. Point-valued maps. In the following we will consider a compact polyhedron X and a continuous map f from X into X, $f\colon X \to X$. Let H be a homology theory with rational coefficients. β_n will stand for the n-Betti number of $X, \beta_n = \dim H_n X$. If L is a finite dimensional vector space over the rationals $\mathbf{R}a$ and $g\colon L \to L$ is a linear transformation, then $P(g)$ will denote the characteristic polynomial of g. If $L = 0$ we set $P(g) = 1$. The fundamental tool in all of what follows is the characteristic $K(f)$, [4], defined by

$$K(f) = \prod_i P(f_{2i}^*)/\prod_i P(f_{2i+1}^*) .$$

$K(f)$ can be expanded uniquely into a canonical formal Laurent series

[1] Actually, $K(f)$ is the zeta function well known to algebraic geometors.

$$K(f) = \lambda^{\chi}(1 + a_1\lambda^{-1} + a_2\lambda^{-2} + \cdots)$$

where $\chi = \Sigma(-1)^n\beta_n =$ the Euler characteristic of X. The coefficients $a_i = a_i(f)$ are the *canonical coefficients* of $K(f)$, [4]. Similarly $(K(f))^{-1}$ can be expanded uniquely into a canonical formal Laurent series.

$$(K(f))^{-1} = \lambda^{-\chi}(1 + b_1\lambda^{-1} + b_2\lambda^{-2} + \cdots) .$$

The coefficients $b_i = b_i(f)$ are then the canonical coefficients of $(K(f))^{-1}$. It is easily seen that $a_1 = a_2 = \cdots = a_n = 0$ if and only if $b_1 = b_2 = \cdots b_n = 0$.

Kelley and Spanier have shown in [4] that $\{a_n(f)\}_{n=1}^{\infty}$ and the Lefschetz numbers $\{\Lambda(f^n)\}_{n=1}^{\infty}$ are closely connected $(\Lambda(f) = \Sigma_i(-1)^i \operatorname{tr} f_i^*)$. In fact the n tuplet $\{a_1, a_2, \cdots, a_n\}$ determines and is determined by $\{\Lambda(f^1), \Lambda(f^2), \cdots, \Lambda(f^n)\}$ and furthermore $a_1 = a_2 = \cdots = a_n = 0$ if and only if $\Lambda(f^1) = \Lambda(f^2) = \cdots = \Lambda(f^n) = 0$. This gives immediately the following theorem.

THEOREM 1. *Let X be a compact polyhedron and $f: X \to X$ a continuous map from X into itself. If $f^1, f^2, \cdots, f^n$ are fixed point free then $a_1 = a_2 = \cdots a_n = 0$.*

We will now outline a more direct proof of Theorem 1 which resembles the proof of the Lefschetz fixed point theorem and possibly sheds more light on how $K(f)$ comes to have the geometrical significance given by Theorem 1.

Outline of second proof. If $Q = \lambda^n \sum_{i=0}^{\infty} c_i\lambda^{-i}$, $c_0 \neq 0$, is a formal Laurent series then we define $\mathscr{D}Q$ to be the smallest $i > 0$ such that $c_{i+1} \neq 0$ if such i exists and if not we define $\mathscr{D}Q = \infty$. It is easy to prove that if Q, R are two formal Laurent series then

$$\text{(i)} \qquad \mathscr{D}(Q \cdot R) \geqq \min(\mathscr{D}Q, \mathscr{D}R)$$

$$\text{(ii)} \qquad \mathscr{D}(Q/R) \geqq \min(\mathscr{D}Q, \mathscr{D}R) .$$

Next we note that if M is a square matrix and $P(M) = \det(\lambda I - M) = \lambda^n + c\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n$ then c_m is made up of sums and differences of "symmetrical" $m \times m$ sub-determinants of M. By a "symmetrical" $m \times m$ subdeterminant M' of M we mean a subdeterminant of M such that the set of indices corresponding to the rows of M used in M' is identical to the set of indices corresponding to the columns of M used in M'.

Now we choose a triangulation T of X sufficiently fine so that the following argument will hold good. We let $f^{\#}$ denote a chain map $f^{\#}: C(X, T) \to C(X, T)$ induced by a simplicial approximation to f. Let $B = (\sigma_1, \cdots, \sigma_q)$ be the canonical basis for $C_i(X, T)$ made of the i-

simplices of T, and let M be the matrix of $f_i^\#(\sigma_j) = \Sigma_k M_{jk}\sigma_k$. Assume that $f^1 \cdots f^n$ are fixed point free. We claim that if M' is a $m \times m$, $m \leqq n$, symmetrical subdeterminant of M then $M' = 0$. We may take M' to be formed from the first m rows and columns of M. Since T is very fine and f has no fixed points, $M_{11} = 0$. If $M_{1n} = 0$ for all $n = 1, 2, \cdots m$, we already have $M' = 0$. Supposing not, we may assume $M_{12} \neq 0$. Since T is very very fine and f and f^2 are fixed point free we must have $M_{21} = M_{22} = 0$. If $M_{2n} = 0$, all $n = 1, \cdots m$, we are done and if not we may assume that $M_{23} \neq 0$. Continuing in this way we either get $M' = 0$ or $M_{ij} = 0$ for $i \leqq j \leqq m$. But in the latter case $M' = 0$ also. This establishes the claim.

From the above we see that $\mathscr{D}P(f_i) \geqq n$ for all i. Thus by the elementary properties of $\mathscr{D}$ we have $\mathscr{D}[\prod_i P(f_{2i}^\#)/\prod_i P(f_{2i+1}^\#)] \geqq n$. But according to Kelley-Spanier [4] the quantity in brackets is equal to $K(f)$. Thus $\mathscr{D}K(f) \geqq n$ and $a_1, a_2, \cdots, a_n = 0$ as we wished to show.

Next we will find conditions which insure that $a_n \neq 0$ for some n.

LEMMA 2. *If $M = \{M_{ij}\}$ is an $n \times n$ matrix with real coefficients such that for each i, $\sum_j M_{ij} = 1$ then $\lambda - 1$ is a factor of the characteristic polynomial of M.*

Proof. We must show that 1 is an eigenvalue of M. But this follows from the observation that $MX = X$ where $X = (1, 1, \cdots, 1)$.

COROLLARY 3. *If $f: X \to X$ is a continuous map of a compact polyhedron X into itself then $(\lambda - 1) \mid P(f_0^*)$.*

THEOREM 4. *Let f be a continuous map of a compact polyhedron X into itself, $f: X \to X$. If $H_n(X) = 0$ for odd n then at least one of the functions $f^1, f^2, \cdots, f^\chi$ has a fixed point where χ is the Euler characteristic of X.*

Proof. In this case

$$\lambda^\chi + a_1\lambda^{\chi-1} + \cdots + a_\chi = K(f) = \prod_i P(f_{2i}) .$$

Since $\lambda - 1 \mid P(f_0^*)$ we have $\lambda - 1 \mid K(f)$. Consequently not all the a_i can vanish and the present theorem follows from Theorem 1.

DEFINITION 5. Let $Q(\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n$, $c_0 \neq 0$, be a polynomial with rational coefficients. Set $\mathscr{S}Q =$ the greatest i such that $c_i \neq 0$.

We collect a few useful properties of $\mathscr{S}$ in the following lemma.

LEMMA 6. *Let Q and R be two polynomials with rational coefficients, L a finite dimensional vector space over the rationals, and $g\colon L \to L$ a linear transformation. Then*

(i) $\mathscr{S}(QR) = \mathscr{S}(Q) + \mathscr{S}(R)$.

(ii) $\mathscr{S}P(g) \leqq \operatorname{rank} g$.

(iii) *If g is an isomorphism then* $\mathscr{S}P(g) = \dim L$.

Proof. (i) is obvious. To prove (ii) consider a matrix for g relative to some basis. Then $r = \operatorname{rank} g =$ the largest nonzero minor of M. If $P(g) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n$ where $n = \dim L$ then c_i is a linear combination of certain $i \times i$ minors of M. Thus $c_{r+1} = c_{r+2} = \cdots = c_n = 0$ and consequently $\mathscr{S}P(g) \leqq r = \operatorname{rank} g$.

Now assume that g is an isomorphism. Then $c_n = (-1)^n \det M \neq 0$. Thus $\mathscr{S}P(g) = n = \dim L$, which proves (iii).

THEOREM 7. *Let f be a continuous map of a compact polyhedron X into itself, $f\colon X \to X$. Set $A = \{i \mid i$ is even and f_i^* is an isomorphism$\}$. If*

$$\Sigma_{i \in A} \dim H_i(X) > \Sigma_{i\,\mathrm{odd}} \operatorname{rank} f_i^*$$

then there exists an $x \in X$ and an $n \leqq \mathscr{S}(\prod_{i\,\mathrm{even}} P(f_i^*))$ such that $f^n(x) = x$. (The theorem is also true with "odd" and "even" interchanged throughout.)

Proof. Set $Q(\lambda) = \prod_i P(f_{2i}^*) = \lambda^q + c_1\lambda^{q-1} + \cdots + c_q$.

and

$$R(\lambda) = \prod_i P(f_{2i+1}^*) = \lambda^r + d_1\lambda^{r-1} + \cdots + d_r\ .$$

We make the following estimates on $\mathscr{S}(Q)$ and $\mathscr{S}(R)$.

$$\begin{aligned}\mathscr{S}(Q) = \mathscr{S}(\textstyle\prod_i P(f_{2i}^*)) = \Sigma_i \mathscr{S}P(f_{2i}^*) &\geqq \Sigma_{i \in A}\mathscr{S}P(f_i^*)\\ &= \Sigma_{i \in A} \dim H_i(X)\ .\end{aligned}$$

$$\mathscr{S}(R) = \mathscr{S}(\textstyle\prod_i P(f_{2i+1}^*)) = \Sigma_i \mathscr{S}P(f_{2i+1}^*) \leqq \Sigma_{i\,\mathrm{odd}} \operatorname{rank} f_i^*\ .$$

Set $N = \mathscr{S}(Q)$ and $M = \mathscr{S}(R)$. It follows from the hypothesis that $N > M$. Now Q, R and $K(f)$ must satisfy

$$R \cdot K(f) = Q\ .$$

Thus

$$0 \neq c_N = a_N + d_1 a_{N-1} + d_2 a_{N-2} + \cdots + d_M a_{N-M}\ .$$

Consequently not all of the numbers $a_N, a_{N-1}, \cdots, a_{N-M}$ can vanish. Therefore by Theorem 1 there exists an $n \leqq N = \mathscr{S}(Q) = \mathscr{S}(\pi_{i\,\mathrm{even}} P(f_i^*))$

and an $x \in X$ such that $f^n(x) = x$. (A dual proof with K^{-1} replacing K works for "odd" and "even" interchanged.)

COROLLARY 8. *Under the same hypothesis as the above theorem there exists an $x \in X$ and an $n \leqq \Sigma_{i\,\text{even}} \dim H_i(X)$ such that $f^n(x) = x$. (The dual result with "odd" and "even" interchanged also holds.)*

Proof. Simply note

$$\mathscr{S}(\textstyle\prod_{i\,\text{even}} P(f_i^*)) = \Sigma_{i\,\text{even}} \mathscr{S}P(f_i^*) \leqq \Sigma_{i\,\text{even}} \deg P(f_i^*) = \Sigma_{i\,\text{even}} \dim H_i(X)\ .$$

COROLLARY 9. *Let f be a homeomorphism of a compact polyhedron X onto itself, $f\colon X \to X$. If the Euler characteristic χ of X does not vanish then some iterate of f has a fixed point. In fact there exists an $x \in X$ and an*

$$n \leqq \max\,(\Sigma_{i\,\text{even}} \dim H_i(X), \Sigma_{i\,\text{odd}} \dim H_i(X))$$

such that $f^n(x) = x$.

Proof. Since f is a homeomorphism, each f_i^* is an isomorphism. Thus A contains all even or all odd i. Since $\chi \neq 0$ we have either

$$\text{☆} \qquad \begin{aligned}\Sigma_{i \in A} \dim H_i(X) = \Sigma_{i\,\text{even}} \dim H_i(X) &> \Sigma_{i\,\text{odd}} \dim H_i(X) \\ &\geqq \Sigma_{i\,\text{odd}} \operatorname{rank} f_i^*\end{aligned}$$

or ☆ with "odd" and "even" interchanged. In either case Corollary 8 implies that there exists an $x \in X$ and an

$$n \leqq \max(\Sigma_{i\,\text{even}} \dim H_i(X), \Sigma_{i\,\text{odd}} \dim H_i(X))$$

such that $f^n(x) = x$.

Comment. The hypothesis "f is a homeomorphism" in the above corollary could be weakened to "f^* is an isomorphism".

Corollary 9 has already been proven by F. B. Fuller [3].

THEOREM 10. *Let f be a continuous map of a compact polyhedron X into itself, $f\colon X \to X$. If $H_i(X) = 0$ for odd i and $f, f^2, \cdots, f^n$ are fixed point free then*

$$\Sigma_i \operatorname{rank} f_{2i}^* \geqq n + 1\ .$$

Proof. Note that if P is not a monomial then $\mathscr{D}(P) + 1 \leqq \mathscr{S}(P)$. (See second proof of Theorem 1.) $H_i(X) = 0$ for odd i implies that

$K(f)$ is a polynomial and since $\lambda - 1 \mid K(f)$, $K(f)$ is not a monomial. Because $f, f^2, \cdots, f^n$ are fixed point free $\mathscr{D}K(f) \geqq n$. Thus

$$n + 1 \leqq \mathscr{D}K(f) + 1 \leqq \mathscr{S}K(f) = \mathscr{S}\Pi_i P(f_{2i}^*)$$
$$\Sigma_i \mathscr{S}P(f_{2i}^*) \leqq \Sigma_i \operatorname{rank} f_{2i}^* \ .$$

If we take Σ_i rank f_i^* to be a measure of how nontrivial f is then Theorem 10 can be paraphrased very roughly as "If $H_i(X) = 0$ for odd i then f has to have a certain degree of nontriviality in order that $f, f^2, \cdots, f^n$ be fixed point free". For example, in order for f, $f^2, \cdots, f^{\chi-j-1}$ to be fixed point free, where $H_i(X) = 0$ for odd i, not more than j of the f_i^* can fail to be isomorphisms.

EXAMPLE A. Consider $\mathbf{R}^3$ to be $\mathbf{R} \times \mathbf{R}^2$ with elements (x, y) where $x \in \mathbf{R}$ and $y \in \mathbf{R}^2$. Let $r: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be a rotation about the origin through an angle π/n where n is a fixed positive integer. Define $X \subset \mathbf{R}^3$ and $f: X \rightarrow X$ by

$$X = \{(x, y) \mid \text{either } (0 \leqq x \leqq n \text{ and } \|y\| = 1) \text{ or}$$
$$x = 0, 1, \cdots, \text{ or } n \text{ and } \|y\| \leqq 1)\}$$
$$f(x, y) = \begin{cases} (x + 1, ry) \text{ if } 0 \leqq x \leqq n - 1 \\ ((n - x)n, ry) \text{ if } n - 1 < x \leqq n \end{cases}$$

for all $(x, y) \in X$.

It is easily seen that $\beta_0 = 1$, $\beta_2 = n$ and $\beta_i = 0$ for $i \neq 0$ or 2, and that f is continuous. Due to the small rotation r, any fixed point for $f, f^2, \cdots,$ of f^n must be of the form $(x, 0)$. There are exactly $n + 1$ such points in X and f permutes these points cyclicly. Thus $f, f^2, \cdots$, f^n are fixed point free.

EXAMPLE B. Let n be and odd integer and X consist of n copies of a $2m (m > 0)$ dimensional sphere S. Let $\mathscr{A}: S \rightarrow S$ be the antipodal map and $g: X \rightarrow X$ be simply a cyclic permutation of the copies of S. Now set $f = g \circ \widetilde{\mathscr{A}}$ where $\widetilde{\mathscr{A}}: X \rightarrow X$ is the natural map induced on X by $\mathscr{A}$. Then f is a homeomorphism of X onto X and $\beta_0(X) = n$, $\beta_{2m}(X) = n$ and $\beta_i(X) = 0$ for $i \neq 0$ or $2m$. The action of g rules out any fixed points for $f, f^2, \cdots, f^{2n-1}$ except in the case of f^n. But $f^n = \widetilde{\mathscr{A}}$ since n is odd. Thus $f, f^2, \cdots, f^{2n-1}$ are fixed point free.

The above examples show that the bounds on n in Theorem 4 and Corollary 9 cannot, in general, be reduced. In special cases the number of iterates needed can be drastically reduced. One sort of case is where X contains a distinguished point x such as when x is the only cut point of X. Then for any homeomorphism f, we must have

$f(x) = x$. Another example is where $\beta_i = 0$ for odd i and $\beta_i = 0$ or 1 for even i. Then the Lefschetz number $\Lambda(f^2) = \Sigma_i(-1)^i$ trace $f_i^{*2} = \Sigma_i(\operatorname{tr} f_{2i}^*)^2 \geqq (\operatorname{tr} f_0)^2 = 1$. Therefore in this case f^2 always has a fixed point. Less trivial cases are covered in Theorems 18, 20 and 21. First we need to make a definition and prove some lemmas.

DEFINITION 11. Let $\{d_i\}$ be a sequence of integers and Q a polynomial over the rationals. $\{d_i\}$ is said to *factor* Q if there exists a sequence of polynomials $\{Q_i\}$ such that $Q = \Pi_i Q_i$ and $d_i = \deg Q_i$. In considering whether a given sequence $\{d_i\}$ factors a polynomial Q, one may disregard the d_i which vanish since they can always be made to correspond to the constant polynomial 1. If $Q = \Pi_i R_i$ is a prime factorization of Q then $\{d_i\}$ factors Q if and only if the numbers $\{\deg R_i\}$ can be partitioned into subsets whose sums correspond to the nonzero d_i. For example consider $Q = \lambda^n - 1$, $n \geqq 1$. Then a prime factorization of Q is $Q = \Pi_{d|n}\phi_d$ where ϕ_d is the d^{th} cyclotomic polynomial. $\deg \phi_d = \varphi(d)$ where φ is Euler's number-theoretic function, [9]. (φ is most easily calculated by the formula: If $d = \Pi_i p_i^{l_i}$ is the prime decomposition of d then $\varphi(d) = \Pi_i l_i p^{l_1-1}(p_i - 1.)$ Consider $n = 10$ and (a) $\{d_i\} = \{1, 3, 3, 3\}$ and (b)$\{\bar{d}_i\} = \{2, 8\}$. Then the factors d of 10 are $1, 2, 5$ and 10 and the corresponding $\varphi(d)$ are $1, 1, 4$ and 4. Thus $\{d_i\}$ does not factor $\lambda^{10} - 1$ but $\{\bar{d}_i\}$ does since $2 = 1 + 1$ and $8 = 4 + 4$.

LEMMA 12. *Let f be a continuous map of a compact polyhedron X into itself, $f\colon X \to X$. If $\beta_i = 0$ for odd i then $(K(f))(\lambda)$ is a polynomial, $\{\beta_i\}$ factors $K(f)$, and $(K(f))(f^*) = 0$.*

Proof. Since $\beta_i = 0$ for odd i we have

$$K(f) = \Pi_i P(f_{2i}^*) .$$

Thus $K(f)$ is a polynomial and because $\deg P(f_i^*) = \dim H_i(X) = \beta_i$ we see that $\{\beta_i\}$ factors $K(f)$. We know by the Hamilton-Cayley theorem that $(P(f_i^*))(f_i^*) = 0$. Thus $(K(f))(f_i^*) = 0$ for all i and consequently $(K(f))(f^*) = 0$.

LEMMA 13. *The polynomials $P(f_i^*)$ have integer coefficients.*

Proof. From the universal coefficient theorem, Spanier [8], we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H_i(X, \mathbf{Z}) \otimes \mathbf{R}a \arrow[r] \arrow[d, "{}^{\mathbf{Z}}f_i^* \otimes id"] & H_i(X, \mathbf{R}a) \arrow[r] \arrow[d, "{}^{\mathbf{R}a}f_i^*"] & H_{i-1}(X, \mathbf{Z})^*\mathbf{R}a \arrow[r] & 0 & \star \\
0 \arrow[r] & H_i(X, \mathbf{Z}) \otimes \mathbf{R}a \arrow[r] & H_i(X, \mathbf{R}a) \arrow[r] & H_{i-1}(X, \mathbf{Z})^*\mathbf{R}a \arrow[r] & 0 &
\end{tikzcd}$$

where $\mathbf{Z}$ = the integers, $\mathbf{R}\alpha$ = the rationals, and $*$ the torsion product. Since $\mathbf{R}\alpha$ is torsion free, $H_{i-1}(X, \mathbf{Z})^*\mathbf{R}\alpha = 0$. Thus ☆ reduces to

$$\begin{array}{ccc} H_i(X, \mathbf{Z}) \otimes \mathbf{R}\alpha & \approx & H_i(X, \mathbf{R}\alpha) \\ \Big\downarrow {}^{\mathbf{Z}}f_i^* \otimes id & & \Big\downarrow {}^{\mathbf{R}\alpha}f_i^* \\ H_i(X, \mathbf{Z}) \otimes \mathbf{R}\alpha & \approx & H_i(X, \mathbf{R}\alpha)\ . \end{array}$$

It is now apparent that an appropriate choice of generators for $H_i(X, \mathbf{Z})$ gives rise to a basis for $H_i(X, \mathbf{R}\alpha)$ relative to which ${}^{\alpha\mathbf{R}}f_i^*$ has a matrix with only integer entries. Thus $P({}^{\mathbf{R}\alpha}f_i^*) = P(f_i^*)$ has integer coefficients.

LEMMA 14. *The canonical coefficients a_i and b_i are integers.*

Proof. Since all characteristic polynomials are monic, so is $Q = \Pi_i P(f_{2i+1}^*)$. Thus we may write $Q = \lambda^\beta(1 - Q_1)$ where $Q_1(\lambda) = c_1\lambda^{-1} + c_2\lambda^{-2} + \cdots + c_\beta\lambda^{-\beta}$. Now K may be expanded into its formal Laurent series by the formula $\lambda^\chi(1 + a_1\lambda^{-1} + \cdots) = K(f) = \Pi_i P(f_{2i}^*)\cdot\lambda^{-\beta}\sum_{i=0}^{\infty} Q_1^i$. It is now clear that the a_i are just sums of products of the coefficients of $\Pi_i(f_{2i}^*)$ and Q_1 and are consequently integers. A completely analogous argument shows that the b_i are also integers.

We now prove a slight extension of a result in Kelley-Spanier [2].

LEMMA 15. *If $\Lambda(f^n) = 0$ for $1 \leqq n \leqq N-1$ then $a_n = 0$ for $1 \leqq n \leqq N-1$ and $-na_n = \Lambda(f^n)$ for $N \leqq n \leqq 2N-1$.*

Proof. From Kelley-Spanier [4] we have

$$\sum_{n=1}^{\infty} na_n\lambda^{n-1} = (1 + \sum_{k=1}^{\infty} a_k\lambda^k)(-\sum_{j=1}^{\infty}\Lambda(f^j)\lambda^{j-1})$$

this means that

$$-na_n = l_n + a_1l_{n-1} + a_2l_{n-2} + \cdots + a_{n-1}l_1 \ ☆$$

where $l_i = \Lambda(f^i)$. Using ☆ we first see that $a_n = 0$ for $1 \leqq n \leqq N-1$ and then using this fact and ☆ again we get $-na_n = \Lambda(f^n)$ for $N \leqq n \leqq 2N-1$.

It is already known, [2], that if f is periodic of prime period p in the sense that f^p = identity and f is fixed point free then $p|\chi$. Lemmas 14 and 15 immediately imply the following generalization of this result.

THEOREM 16. *Let $f: X \to X$ be a continuous map of a compact polyhedron X into itself. Suppose f^n is fixed point free for*

$1 \leqq n \leqq p/2$ $(p \geqq 2)$ *and* $f^{p*} =$ *identity. The* p *divides the Euler characteristic of* X. (p need not be prime.)

The following lemma lists various conditions under which we may wholly or partially determine $K(f)$. Both Lemma 17 and Theorem 18 may be considered refinements of Theorem 4.

LEMMA 17. *Let* $\beta_i = 0$ *for odd* i. *Then*

(a) *Let* $1 \leqq j \leqq \chi$. *If* f^n *is fixed point free for* $1 \leqq n \leqq \chi, n \neq j$, *then* $K(f) = \lambda^\chi - \lambda^{\chi-j} = \lambda^{\chi-j}(\lambda^j - 1)$.

(b) *If* f^n *is fixed point free for* $1 \leqq n \leqq \chi - 2$ *and not all of the* f_i *are isomorphisms then* $K(f) = \lambda^\chi - \lambda = \lambda(\lambda^{\chi-1} - 1)$.

(c) *Suppose* f *is a homeomorophism and* $1 \leqq j \leqq \chi - 1$. *If* f^n *is fixed point free for* $n \neq j, 1 \leqq n \leqq \chi - 1$, *then either* $K(f) = \lambda^\chi - 1$ *or* $K(f) = \lambda^\chi - 2\lambda^j + 1$.

Proof. (a) Assume hypothesis (a). Since $f^j(x) = x$ implies $f^{2j}(x) = x$ it follows that $2j \geqq \chi + 1$. We have $\Lambda(f^n) = 0$ for $1 \leqq n \leqq j-1$ and for $j + 1 \leqq n \leqq \chi$. Thus by Lemma 15 $a_n = 0$ for $1 \leqq n \leqq j - 1$ and for $j + 1 \leqq n \leqq \chi$. This leaves a_j to be determined. Since $\lambda - 1$ is a factor of $K(f)(\lambda)$ we have $K(f)(1) = 1 + a_j = 0$. Thus $K(f) = \lambda^\chi - \lambda^{\chi-j} = \lambda^{\chi-j}(\lambda^j - 1)$.

(b) Assume hypothesis (b). f_i^* is trivially an isomorphism for odd i. Thus there is an even i such that f_i^* is not an isomorphism. Then $\det f_i^* = 0$ and consequently $a_\chi = \pm \Pi_k \det f_{2k}^* = 0$. Since $\Lambda(f^n) = 0$ for $1 \leqq n \leqq \chi - 2$ we have $a_n = 0$ for $1 \leqq n \leqq \chi - 2$. This leaves only $a_{\chi-1}$ to be determined. Because $\lambda - 1$ is a factor of $K(f)(\lambda)$ we may write $K(f)(1) = 1 + a_{x-1} = 0$. Thus $K(f) = \lambda^\chi - \lambda = \lambda(\lambda^{\chi-1} - 1)$.

(c) Assume hypothesis (c). Since $f^j(x) = x$ implies $f^{2j}(x) = x$ it follows that $2j \geqq \chi$. Reasoning as before we see that $a_n = 0$ for $1 \leqq n \leqq \chi - 1, n \neq j$. This leaves a_j and a_χ to be determined. $a_\chi = \pm \Pi_i \det f_{2i}^*$. Using Lemma 13 we see that since f has an inverse f^{-1} the integer $\det f_{2i}^*$ has an integer inverse $\det f_{2i}^{-1*}$. Thus $\det f_{2i}^* = \pm 1$ for all i and therefore $a_\chi = \pm 1$. Because $K(f)(1) = 1 + a_j + a_\chi = 1 + a_j \pm 1 = 0$ we must have $K(f) = \lambda^\chi - 1$ or $K(f) = \lambda^\chi - 2\lambda^{\chi-j} + 1$.

Combining Lemmas 12 and 17 we obtain the following theorem.

THEOREM 18. *Let* X *be a compact polyhedron and* $f\colon X \to X$ *a continuous map from* X *into itself. Assume further that* $H_i(X) = 0$ *for odd* i. *Then under the additional hypothesis of* (a), (b) *or* (c) *of Lemma* 17 *we have respectively* (a) $\{\beta_i\}$ *factors* $\lambda^{\chi-j}(\lambda^j - 1)$ *and* (*see* *definition* 11) $f^{*\chi} - f^{*\chi-j} = 0$, (b) $\{\beta_i\}$ *factors* $\lambda(\lambda^{\chi-1} - 1)$ *and* $f^{*\chi} - f^{*} = 0$ *and* (c) $\{\beta_i\}$ *factors either* $\lambda^{\chi} - 1$ *or* $\lambda^{\chi} - 2\lambda^{\chi-j} + 1$ *and either* $f^{*\chi} - f^{*} = 0$ *or* $f^{\chi} - 2f^{*\chi-j} + I = 0$ *where* $I =$ *the identity transformation.*

Comment. Note that the condition "f is a homeomorphism" appearing in part (c) of Lemma 17 may be weakened to "f is a homotopy equivalence" or weakened even further to "there exists a continuous $g: X \to X$ such that $f^{*-1} = g^{*}$" for both Lemma 17 and Theorem 18.

COROLLARY 19. (a) *Under hypothesis* (a) *of Lemma* 17

$$\Lambda(f^{n}) = \begin{cases} j & \text{if } n = lj \text{ for some } l \geqq 1 \\ 0 & \text{otherwise } (n \geqq 1). \end{cases}$$

(b) *Under hypothesis* (b) *of Lemma* 17

$$\Lambda(f^{n}) = \begin{cases} \chi - 1 & \text{if } n = l(\chi - 1) \text{ for some } l \geqq 1 \\ 0 & \text{otherwise } (n \geqq 1). \end{cases}$$

Proof. Let $k = j$ for part (a) and $k = \chi - 1$ for part (b). Then by Lemma 15, $\Lambda(f^{k}) = -ka_k = k$ and $\Lambda(f^{m}) = 0$ for $1 \leqq m \leqq \chi - 1$, $m \neq k$. By Theorem 18, $f^{*\chi} = f^{*\chi-k}$. The Corollary now follows by reducing $\Lambda(f^{n})$ to $\Lambda(f^{n'})$ with $1 \leqq n' \leqq \chi - 1$ (or with $1 \leqq n' \leqq \chi$ if $k = \chi$).

Theorem 18, part (a), with $j = \chi = n + 1$ applies to Example A. Thus $f^{*n+1} =$ identity. A careful inspection of Example A will reveal that in fact f^{n+1} is homotopic to the identity.

Consider the polynomial $\lambda^{\chi} - 2\lambda + 1$ which is one of the possibilities in part (c) of Theorem 18 with $j = \chi - 1$. In light of the discussion following Definition 11, the prime factorization of $\lambda^{\chi} - 2\lambda + 1$ is of interest. $\lambda^{\chi} - 2\lambda + 1 = (\lambda - 1)P(\lambda)$ where $P(\lambda) = \lambda^{\chi-1} + \lambda^{\chi-2} + \cdots + \lambda - 1$. By applying Eisenstein's criteria to $P(\lambda + 1)$ it is found that $P(\lambda)$ is irreducible over the rationals for χ of the form $\chi = 2^{m}$. The author does not know of any value of χ for which $P(\lambda)$ is not irreducible.

The following two theorems sharpen the conclusions of Theorem 4 and Corollary 9 in certain special cases.

THEOREM 20. *Let* $f: X \to X$ *be a homeomorphism of a compact polyhedron* X *onto itself. Suppose* $H_i(X) = 0$ *for odd* i, $m =$ *the number of* i *such that* $\dim H_i(X) = 1$, *and* q *the greatest integer* j *such that* $j \leqq (m - 1)/2$. *Then there exists an* $x \in X$ *and an* n, $1 \leqq n \leqq \Sigma_i \dim H_{2i}(X) - q$ *such that* $f^{n}(x) = x$.

Proof. Suppose the conclusion were false. Then $a_1 = a_2 = \cdots = a_{\chi-q} = 0$. Therefore $(d^{q}/d\lambda^{q})K(\lambda) = (\chi!/(\chi - q)!)\lambda^{\chi-q}$. Because f is a homeomorphism, $P(f_i) = \lambda \pm 1$ for each i with $\dim H_i(X) = 1$. Thus, either 1 or -1 is a root of order $q+1$ of K. Consequently, either 1 or -1 is a root of $(d^q/d\lambda^q)/K(\lambda) = (\chi!/(\chi - q)!)\lambda^{\chi-q}$ which is impossible. Therefore the conclusion holds.

THEOREM 21. *Let $f\colon X \to X$ be a homeomorphism of a compact polyhedron X onto itself. Suppose* $\dim H_i(X) = 1$ *for at least m odd i and m even i, and $\chi \neq 0$. Then there exists an $x \in X$ and an n,* $1 \leqq n \leqq \max(\Sigma_i \dim H_{2i}(X),\ \Sigma_i \dim H_{2i+1}(X)) - m = q$ *such that* $f^n(x) = x$.

Proof. Suppose $\chi > 0$. The case $\chi < 0$ is handled similarly. Set $P = \Pi_i P(f^*_{2i})$ and $Q = \Pi_i P(f^*_{2i+1})$. Assume that $f, f^2, \cdots, f^q$ are fixed point free. Then from $QK = P$ it follows that

$$P(\lambda) = \lambda^\chi Q(\lambda) + c_{m-1}\lambda^{m-1} + c_{m-2}\lambda^{m-2} + \cdots + c_0 \quad ☆ .$$

Because f is a homeomorphism and $\chi > 0$ we have $c_0 = P(0) = \pm 1$ and $P(f_i^*) = \lambda \pm 1$ for each i with $\dim H_i(X) = 1$. Consider ☆ mod 2. Then $(\lambda - 1)^m$ is a factor of $P(\lambda)$ and $\lambda^\chi Q(\lambda)$ and therefore also of $R(\lambda) = c_{m-1}\lambda^{m-1} + \cdots + c_0 (\text{mod } 2)$. Noting that degree $R \leqq m-1$ we see that there is a contradiction unless $R = 0 \bmod 2$. But this is also impossible because $c_0 = 1 \bmod 2$. We must therefore admit that not all of the maps $f, f^2, \cdots, f^q$ are fixed point free.

Suppose now that in addition to the hypothesis of Theorem 21 we have $\dim H_i(X) = 0$ or 1 for all odd i, $m' =$ the number of odd i for which $\dim H_i(x) = 1$, and $f, f^2, \cdots, f^{q-1}$ are fixed point free. Then in place of ☆ we have

$$P(\lambda) = \bar{P}(\lambda)(\lambda - 1)^r(\lambda + 1)^{m-r} = \lambda^\chi(\lambda - 1)^s(\lambda + 1)^{m'-s} + R(\lambda) \quad ☆☆$$
$$m \leqq m',\ 0 \leqq r \leqq m,\ 0 \leqq s \leqq m'$$
$$\text{degree } R(\lambda) \leqq m .$$

Also $R(0) = P(0) = \pm 1$. For fixed values of r, s and $P(0)$, ☆☆ along with $R(0) = P(0)$ determines $R(\lambda) = c_m\lambda^m + \cdots + c_0$ because they give the value of $(d^n/d\lambda^n)R(a)$ for $a = 1$ with $0 \leqq n \leqq r-1$, and for $a = -1$ with $0 \leqq n \leqq m - r - 1$, and for $a = 0$ with $n = 0$. Thus P is determined up to $2(m+1)$ possibilities.

Similar considerations can be made with respect to Theorem 20.

The last theorem of this section is a generalization of Theorem 18a with $j = \chi - 1$.

THEOREM 22. *Let X be a compact polyhedron and $f\colon X \to X$ a continuous map from X into itself. Let $X_1, X_2, \cdots, X_q$ be the com-*

ponents of $X, X = X_1 U \cdots U X_q$. *Suppose further that* $H_i(X) = 0$ *for odd* i *and that* f^n *is fixed point free for* $1 \leqq n \leqq \chi(X) - 1$. *Then*

(i) $f^{\chi(X)*} =$ *identity*.

(ii) $\{\beta_i(X)\}$ *factors* $\lambda^{\chi}(X) - 1$.

(iii) $\beta_i(X_j) = \beta_i(X_l)$ *for* $j, l = 1, \cdots, q$ *and all* i.

(iv) $\{\beta_i(X_j)\}$ *factors* $\lambda^{\chi(X_j)} - 1 = \lambda^{\chi(X)/q} - 1$.

(v) *for a particular numbering of the components*

$$f(X_i) \subset X_{i+1} \text{ for } 1 \leqq i \leqq q \text{ where } X_{q+1} = X_1$$

(vi) *under the same numbering as in* (v)

$$(f \mid X_i)^*: H(X_i) \longrightarrow H(X_{i+1}) \text{ is an isomorphism for } 1 \leqq i \leqq q .$$

Proof. Since f is continuous for each $i, f(X_i) \subset X_j$ for some j. Thus f induces a map $g: \{1, \cdots, q\} \rightarrow \{1, \cdots, q\}$ by setting $g(i) = j$. Because g is a function from a finite set into itself it must have a cycle of length $k \leqq q$. Renumbering if necessary we may assume $f(X_i) \subset X_{i+1}$ for $1 \leqq i \leqq k - 1$ and $f(X_k) \subset X_1$. We may also assume that $\Sigma_i \beta_{2i}(X_1) \leqq \Sigma_i \beta_{2i}(X_j)$ for $1 \leqq j \leqq k$. Since $\beta_i(X) = 0$ for odd i and $\beta_i(X) = \Sigma_j \beta_i(X_j)$ we have $\beta_i(X_j) = 0$ for odd i and all j. Note that

$$\begin{aligned}
\chi(X) &= \Sigma_i \beta_{2i}(X) = \Sigma_i \Sigma_{j=1}^q \beta_{2i}(X_j) = \Sigma_{j=1}^q \Sigma_i \beta_{2i}(X_j) \\
&= \Sigma_{j=1}^k \Sigma_i \beta_{2i}(X_j) + \Sigma_{j=k+1}^q \Sigma_i \beta_{2i}(X_j) \\
&\geqq \Sigma_{j=1}^k \Sigma_i \beta_{2i}(X_1) + \Sigma_{j=k+1}^q \beta_0(X_j) \geqq k\chi(X_1) + q - k .
\end{aligned}$$

Thus $\chi(X) \geqq k\chi(X_1) + q - k$ with $\chi(X) > k\chi(X_1)$ unless $k = q$ and $\Sigma_i \beta_{2i}(X_j) = \Sigma_i \beta_{2i}(X_1)$ for $j = 1, 2, \cdots, q$.

Now set $h = f^k \mid X_1$. Then $h(X_1) \subset X_1$ and because f^n is fixed point free for $1 \leqq n \leqq \chi(X) - 1$, h^n is fixed point free for $1 \leqq n \leqq \chi(X_1) - 1$. Therefore by Theorem 18a, $h^{*\chi(X_i)} =$ identity and $\{\beta_i(X_1)\}$ factors $\lambda^{\chi(X_1)} - 1$. By Theorem 4, $h^{\chi(X_1)}$ has a fixed point. Thus $f^{k\chi(X_1)}$ has a fixed point and consequently $\chi(X) \leqq k\chi(X_1) \leqq \chi(X)$. Therefore $\chi(X) = k\chi(X_1)$ and so $k = q$ and $\Sigma_i \beta_{2i}(X_j) = \Sigma_i \beta_{2i}(X_1)$ for $j = 1, \cdots, q$. This shows that X_1 could have been any of the X_j's. We can write

$$\text{identity} = h^{\chi(X_1)*} = (f \mid X_q)^* \cdots (f \mid X_1)^* \cdots (f \mid X_q)^* \cdots (f \mid X_1)^* .$$

Thus $(f \mid X_1)^*: H(X_1) \rightarrow H(X_2)$ is one-to-one. Therefore $\beta_i(X_1) \leqq \beta_i(X_2)$. But since $\Sigma_i \beta_{2i}(X_1) = \Sigma_i \beta_{2i}(X_2)$ we must have $\beta_i(X_1) = \beta_i(X_2)$ for all i. Since X_1 could have been chosen to be any of the X_j we conclude that $\beta_i(X_j) = \beta_i(X_l)$ for all $j, l = 1, 2, \cdots, q$ and all i, and that $(f \mid X_i)^*: H(X_i) \rightarrow H(X_{i+1})$ is an isomorphism for $1 \leqq i \leqq q$ where $X_{q+1} = X_1$.

Theorem 18a applies to $f: X \rightarrow X$ and thus we have $\{\beta_i(X)\}$ factors

$\lambda^{\chi(X)} - 1$ and $f^{\chi(X)*} =$ identity. We have now established all the conclusions of the theorem.

Observe that $f^m(x) = x$ implies $f^{nm}(x) = x$ for all $n \geqq 1$. Armed with this fact we can play funny games with most of the above theorems. For example the conclusion of Theorem 4 could be replaced by "$f^{\chi!}$ has a fixed point" or "either f^α or f^β has a fixed point where $\alpha = \Pi_{\substack{i \text{ odd} \\ 1 \leqq i \leqq \chi}} i$ and $\beta = \Pi_{\substack{i \text{ even} \\ 1 \leqq i \leqq \chi}} i$".

GENERALIZATIONS. Note that in most of the results the hypothesis and/or the conclusion can be stated in terms of the Lefschetz numbers $\Lambda(f^n)$. Consequently they can be used in conjuction with the results of Atiyah and Bott [1] and O'Neill [6] concerning the degrees of fixed points, and essential fixed points and fixed point sets.

All the results have been stated for compact polyhedra. But since the analysis basically concerns the Lefschetz numbers $\Lambda(f^n)$ it is easily seen that all the results of this section and the next hold whenever the rational homology groups $H_n(X)$ are finitely generated and trivial for n sufficiently large, and the Lefschetz fixed point theorem holds. In particular the results hold for compact metric ANR's [5].

If the homology is taken with coefficients in a field $\mathscr{F}$ with nonzero characteristic then $\Lambda(f^1) = \cdots = \Lambda(f^n) = 0$ need not imply $a_1 = \cdots = a_n = 0$ (Kelley and Spanier [4]). Thus Theorem 1 no longer follows from the Lefschetz fixed point theorem. But the alternate proof is still good for polyhedra. Consequently most of the results of this section hold for polyhedra and homology with coefficients in an arbitrary field.

2. Applications to the problem of coincidence. Let $h, g: X \to X$ be two continuous maps from the compact polyhedron X into itself. A point x is a coincident point for f and g if $f(x) = g(x)$. In order to apply the preceding fixed point theorems we will assume that g is a homeomorphism. Then $h(x) = g(x)$ is equivalent to $f(x) = x$ where $f = g^{-1} \circ h$. If g^{-1} and h commute then $f^n(x) = x$ would give $g^n(x) = h^n(x)$. We can still obtain such a result even when g^{-1} and h may not commute.

THEOREM 23. *Let X be a compact polyhedron, $h: X \to X$ a continuous map and $g: X \to X$ a homeomorphism. Assume that h^* and g^* commute, $h^*g^* = g^*h^*$. If in addition either conditions* (a) *or* (b) *hold then there exists an $x \in X$ and an n, $1 \leqq n \leqq q$, such that $h^n(x) = g^n(x)$.*

(a) *$H_i(X) = 0$ for odd i and $q = \chi = \Sigma_i \dim H_{2i}(X)$.*

(b) *h is a homeomorphism (or just h^* is an isomorphism), the Euler index $\chi \neq 0$, and $q = \max(\Sigma_i \dim H_{2i}(X), \Sigma_i \dim H_{2i+1}(X))$.*

Proof. Consider case (a). As in the proof of Theorem 4 we can conclude that the a_i's associated with $K(g^{-1}h)$ cannot all vanish for $1 \leqq i \leqq q = \chi$. This implies (Kelley-Spanier [4]) that $\Lambda((g^{-1}h)^n) \neq 0$ for some $n, 1 \leqq n \leqq q$. But

$$\begin{aligned} 0 \neq \Lambda((g^{-1}h)^n) &= \Lambda(((g^{-1}h)^n)^*) = \Lambda((g^{*-1}h^*)^n) \\ &= \Lambda(g^{*-n}h^{*n}) = \Lambda((g^{-n}h^n)^*) = \Lambda(g^{-n}h^n) \, . \end{aligned}$$

Thus by the Lefschetz fixed point theorem there exists an $x \in X$ such that $g^{-n}h^n(x) = x$ or equivalently $h^n(x) = g^n(x)$ for some n, $1 \leqq n \leqq q$. This proves the theorem for case (a). Case (b) is handled in a completely analogous way refering to Corollary 9 in place of Theorem 4.

We will consider here one more interesting case.

THEOREM 24. *If h, g and X are as in Theorem 23 with the additional hypothesis that $g_i = \alpha_i I$ where each $\alpha_i = \pm 1$ then either there exists an even $n, 1 \leqq n \leqq q$, such that $h^n(x) = x$ for some $x \in X$ or there exists an odd $n, 1 \leqq n \leqq q$, such that $h^n(x) = g(x)$ for some $x \in X$.*

Proof. Nothing that $g^{*n} = g^*$ or I depending on whether n is even or odd the conclusion follows much the same as in the proof of Theorem 25.

All the results of § 1 may be similarly applied to give coincidence theorems like Theorems 23 and 24. Note that the condition $g_i^* = \alpha_i I$ implies that g^* commutes with any homomorphism φ: $H(X) \to H(X)$. For an example of this last theorem consider the two dimensional surface X in Figure 1. Define g by $g(x, y, z) = (x, -y, -z)$ for all $(x, y, z) \in X$. Intuitively it seems clear that $\beta_0 = 1, \beta_1 = 4, \beta_2 = 1, \beta_i = 0$ for $i > 2$, and that the circles a, b, c and d form a basis for $H_1(X)$. Still reasoning intuitively we conclude that $g_0^* = I$, $g_1^* = -I$ and $g_2^* = I$.

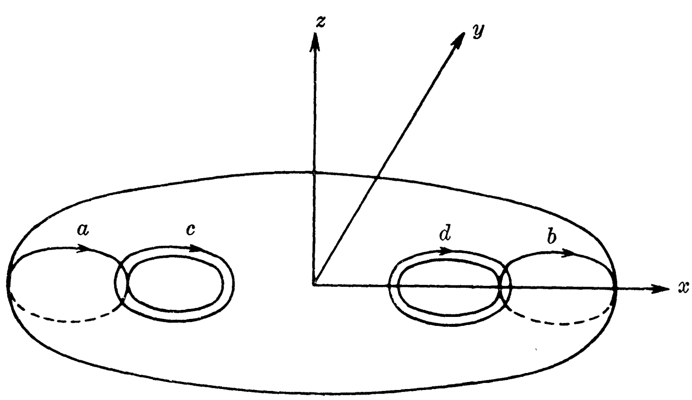

FIGURE 1.

Providing that all of this is indeed so we can conclude from Theorem 24 that if h is any homeomorphism $h: X \to X$ then either $h(x) = g(x)$, $h^3(x) = g(x)$, $h^2(x) = x$, or $h^4(x) = x$ has a solution $x \in X$.

3. **Set-valued maps.** We follow the exposition of the theory of set-valued maps in [7]. Let X and Y be topological spaces. A set-valued function $F: X \to Y$ assigns to each point $x \in X$ a closed nonempty subset $F(x)$ of Y. If $F: X \to Y$ is a set-valued function, let $F^{-1}: Y \to X$ be the function such that $x \in F^{-1}(y)$ if and only if $y \in F(x)$. Then F is upper (lower) semicontinuous provided F^{-1} is closed (open). If both conditions hold, F is continuous.

All spaces we deal with will be assumed to be compact polyhedra with a metric denoted by d. If $\varepsilon > 0$ is a real number, we shall also denote by $\varepsilon: X \to X$ the set-valued function such that $\varepsilon(x) = \{x' \mid \mathrm{d}(x, x') \leqq \varepsilon\}$ for each $x \in X$. Let A and B be chain groups with supports in X and Y respectively, and let $\varepsilon > 0$ be a number. A chain map $\varphi: A \to B$ is accurate with respect to a set-valued function $F: X \to Y$ provided $|\varphi(a)| \subset F(|a|)$ for each $a \in A$. Further, φ is ε-accurate with respect to F provided φ is accurate with respect to the composite function $\varepsilon F \varepsilon$.

Let H denote Čech homology theory with rational coefficients.

Definition. A homomorphism $h: H(X) \to H(Y)$ is an induced homomorphism of a set-valued function $F: X \to Y$ provided that given $\varepsilon > 0$ there is a chain map $\varphi: C(X) \to C(Y)$ such that φ is ε-accurate with respect to F and $\varphi_* = h$.

If h_F and h_G are induced homomorphisms of upper semicontinuous functions $F: X \to Y$ and $G: Y \to Z$, then $h_G h_F$ is an induced homomorphism of GF. If $F: X \to Y$ is a continuous point-valued map then the Čech homology homomorphism F_* is an induced homomorphism of F.

O'Neill [7] proves the following lemma.

Lemma. *Let X be a compact polyhedron, $F: X \to Y$ a set-valued function. Then $h: H(X) \to H(Y)$ is an induced homomorphism of F if and only if given $\varepsilon > 0$ there is an arbitrarily fine triangulation T of X and an ε-accurate chain map $\varphi: C(X, T) \to C(Y)$ such that $\varphi_* = h$.*

Now assume that X is a compact polyhedron, n is a fixed positive integer, and $F: X \to X$ is a continuous set-valued function such that if $x \in X$ then $F(x)$ is either homologically trivial or consists of n homologically trivial components. It is then shown in [7] that there exists a homomorphism $h: H(X) \to H(X)$ "induced by F" in the sense that for each $\varepsilon > 0$ there exists an arbitrarily fine triangulation T of X and a chain map $\varphi: C(X, T) \to C(X)$ such that φ is ε-accurate with respect to F and $\varphi_* = h$. Furthermore, from the proof it can be seen that φ can be chosen so that if v is a vertex of T then $\varphi(v) = \bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_n$ where $v_1, v_2, \cdots, v_n$ (repetitions allowed) are points of X and $\bar{v}_1, \cdots, \bar{v}_n$ are the associated 0-chains in $C(X)$. Suppose X has r components $X_1, \cdots, X_r$ where $r \geqq 1$. Let w_i be a point in X_i and $e_i \in H_0(X)$ the homology class of w_i. Then for each $i, h_0(e_i) = \varphi_{*0}(e_i) = \Sigma_j a_{ij} e_j$ where each a_{ij} is a nonnegative integer and $\Sigma_j a_{ij} = n$ for each i. Now set $F^* = n^{-1}h$. Then F^*, (which possibly is not uniquely determined by F) is also induced by F and $F^*(e_i) = \Sigma_j b_{ij} e_j$ where $b_{ij} \geqq 0$ for all i and j, and $\Sigma_j b_{ij} = 1$ for each i. Thus by Lemma 2, $\lambda - 1 \mid P(F_0^*)$.

DEFINITION 25. A homomorphism $h: H(X) \to H(X)$ is a nice induced homomorphism of a set-valued function $F: X \to X$ provided h is an induced homomorphism of F and $h_0(e_i) = \Sigma_j c_{ij} e_j$ where the e_i are as above, $c_{ij} \geqq 0$ for all i and j, and $\Sigma_j c_{ij} = 1$ for each i.

It is easily seen that the Čech homology homomorphism of a continuous point-valued map is nice and that the composition of nice induced homomorphisms is also nice.

We quote one more result from [7].

LEMMA. *Let X be a compact polyhedron, $F: X \to X$ an upper semi-continuous set-valued function. If h is an induced homology homomorphism of F and the Lefschetz number $\Lambda(h) = \Sigma(-1)^q$ trace h_q is not zero, then F has a fixed point.*

We are now in a position to carry over most of § 1 to certain set-valued functions. Let γ be a positive integer, $F: X \to X$ a continuous set-valued function of a compact polyhedron X such that if $x \in X$ then $F(x)$ is either homologically trivial or consists of γ homologically trivial components. From above we know that F has a nice induced homomorphism $F^*: H(X) \to H(X)$. Now if f is replaced by F and f^* is replaced by F^* and fixed point equations of the form $f^n(x) = x$ replaced by $x \in F^n(x)$ in all the definitions, lemmas, theorems, corollaries and proofs of § 1, then all the results remain valid and the proofs correct with the following few exceptions. Skip Lemmas 13 and 14, Theorem 16, part (c) of Lemma 17 and part (c) of Theorem 18, Theorems 20, 21, 22, and change "$f(F)$ is a homeomorphism" in Corollary 9 to "$f^*(F^*)$ is an isomorphism". The two theorems of § 2 also remain valid with h replaced by F, h^* replaced by F^*, $h^n(x) = g^n(x)$ replaced by $g^n(x) \in F^n(x)$, $g(x) = h^n(x)$ by $g(x) \in F^n(x)$ and $h^n(x) = x$ by

$x \in F^n(x)$. The proofs remain correct also if expressions such as F^{n*} are interpreted appropriately.

The following theorem is analogous to Theorem 22.

THEOREM 26. *Let n be a positive integer and X is a compact polyhedron. Suppose $F: X \to X$ is a continuous set-valued function such that for each $x \in X$, $F(x)$ is homologically trivial or consists of n homologically trivial components. Assume further that $H_i(X) = 0$ for odd i. If X has r components $X_1, X_2, \cdots, X_r$, and F^m is fixed point free for $1 \leqq m \leqq \chi(X) - 1$ where $\chi(X) = \Sigma_i \dim H_{2i}(X)$ then*

(i) *For an appropriate numbering or the components*

$$F(X_i) \subset X_{i+1}, 1 \leqq i \leqq r, \text{ where } X_{r+1} = X_1 .$$

(ii) *$\{\beta_i(X)\}$ factors $\lambda^{\chi(X)} - 1$.*

(iii) *$F^{*\chi(X)} =$ identity.*

(iv) *$\beta_i(X_j) = \beta_i(X_k)$ for all i, j and k.*

(v) *for each j, $\{\beta_i(X_j)\}$ factors $\lambda^{\chi(X_j)} - 1$.*

Outline of proof. Define $F_{ij}(\chi) = (F(x)) \cap X_j$ for all $x \in X_i$. For each (i, j) there are only two cases: (a) $F_{ij}(x) = \varnothing$ for all $x \in X_i$, (b) there exists an n', $1 \leqq n' \leqq n$, such that $F_{ij}: X_i \to X_j$ is a continuous set-valued function such that for each $x \in X_i$, $F_{ij}(x)$ is homologically trivial or consists of n' homologically trivial components. Define $g(i) = \{j \mid \text{case } b) \text{ applies to } F_{ij}\}$. Then g will have a "cycle" which we will assume is $1, 2, \cdots, p$, $1 \leqq p \leqq r$, in the sense that $i + 1 \in g(i)$ for $1 \leqq i \leqq p - 1$ and $1 \in g(p)$. We will also assume that this is the shortest cycle of g. Then reasoning as in Theorem 22 with $F_{i,i+1}$ replacing $f \mid X_i$ etc., we find that $\beta_i(X_1)$ factors $\lambda^{\chi(X_1)} - 1$ and $p = r$. $\beta_i(X_j) = \beta_i(X_k)$ all i, j and k. $1, 2, \cdots, p = r$ being the shortest cycle for g implies $g(i) = \{i + 1\}$ for $1 \leqq i \leqq r - 1$ and $g(r) = \{1\}$. Thus $F(X_i) \subset X_{i+1}$ for $1 \leqq i \leqq r$ where $X_{r+1} = X_1$. Because F^m is fixed point free for $1 \leqq m \leqq \chi(X) - 1$ the alalogous theorem for set-valued functions to Theorem 18 applies and we obtain $\{\beta_i(X)\}$ factors $\lambda^{\chi(X)} - 1$ and $F^{*\chi(X)} =$ identity.

4. The behavior of f near fixed points. We shall consider here how much information is given by the behavior of f near fixed points of f and certain of its iterates. It is apparent from the work of O'Neill [6] that th Lefschetz number $\Lambda(f)$ is determined by $f \mid V$ where V is any open set containing all the fixed points of f. Thus $f \mid U$ determines $\Lambda(f), \Lambda(f^2), \cdots, \Lambda(f^n)$ whenever U contains all the fixed points for $f, f^2, \cdots, f^n$ (for the U contains all the images of these fixed points under the maps $f, f^2, \cdots, f^n$). We know that $\Lambda(f), \cdots, \Lambda(f^n)$

determine $a_1, \cdots, a_n$. The next question is how many a_i are needed to determine $K(f)$. The answer is contained in the following lemma.

LEMMA 27. *Let*

$$P(\lambda) = c_0\lambda^a + c_1\lambda^{\alpha-1} + \cdots + c_\alpha,$$
$$Q(\lambda) = d_0\lambda^\beta + d_1\lambda^{\beta-1} + \cdots + d_\beta, d_0 \neq 0 ,$$

and $P(\lambda)/Q(\lambda) = K(\lambda) = e_0\lambda^{\alpha-\beta} + e_1\lambda^{\alpha-\beta-1} + \cdots$ *where* $c_i, d_i,$ *and* $e_i \in \mathbf{R}a$. *Then all* e_n *are determined by* $e_0, e_1, \cdots, e_{\alpha+\beta}$.

Proof. Since $P = KQ$ we have

$$\text{☆} \qquad c_n = \sum_{i=0}^{\beta} d_i e_{n-i} \qquad \text{where we set } e_l = 0 \text{ for } l < 0 \text{ and } c_n = 0 \text{ for } n > \alpha .$$

Set $e^n = (e_n, e_{n-1}, \cdots, e_{n-\beta+1})$. Then solving ☆ for e_n we see that there is a linear mapping L such that for $n > \alpha$, $e^n = Le^{n-1}$. Consider the set $\{e^\alpha, e^{\alpha+1}, \cdots, e^{\alpha+\beta}\}$ of $\beta + 1$ vectors Ra^β. There must exist $a, j, 0 \leqq j \leqq \beta$ and $c_i \in \mathbf{R}a$ such that $e^{\alpha+j} = \sum_{i=1}^{j} c_i e^{\alpha+j-i}$. (A set of such c_i can be found through solving the appropriate linear equations.) Now if $n \geqq \alpha + \beta$, then

$$e^n = L^{n-\alpha-j}e^{\alpha+j} = \sum_{i=1}^{j} c_i L^{n-\alpha-j}e^{\alpha+j-i}$$
$$= \sum_{i=1}^{j} c_i e^{n-i} . \quad \text{☆}$$

The conclusion of the lemma now follows easily from ☆.

It should be noted that although P and Q may not be determined from K, a $\bar{P}$ and $\bar{Q}$ can be found (once K is known) such that $\bar{P}/\bar{Q} = K$. To see this note that if $\bar{e}^n = (e_n, e_{n-1}, \cdots, e_{n-\beta})$, $\bar{d} = (\bar{d}_0, \bar{d}_1, \cdots, \bar{d}_\beta)$, $\bar{d}$ is perpendicular to $\bar{e}^{\alpha+1}, \bar{e}^{\alpha+2}, \cdots, \bar{e}^{\alpha+\beta}$, $\bar{Q}(\lambda) = \bar{d}_0\lambda^\beta + \cdots + \bar{d}_\beta$ and $\bar{P} = \bar{Q}K$, then $\bar{P}$ is a polynomial and $\bar{P}/\bar{Q} = K$. If we now set $P' = \bar{P}/(\bar{P}, \bar{Q})$ and $Q' = \bar{Q}/(\bar{P}, \bar{Q})$, $((\bar{P}, \bar{Q})$ = the greatest common divisor of $\bar{P}$ and $\bar{Q})$ then P' and Q' are factors of original P and Q respectively.

We can now make the following conclusions. If $f \mid V$ is given where V is an open set containing all the fixed points of $f^1, f^2, \cdots, f^n$ then $\Lambda(f), \Lambda(f^2), \cdots, \Lambda(f^n)$ are determined. If $\Lambda(f), \Lambda(f^2), \cdots, \Lambda(f)^n$ are given with $n = \Sigma_i \beta_i$ then $a_1, a_2, \cdots, a_n$ are determined and these latter numbers determine $K(f)$. Knowning $K(f)$ means that we know a_i for all i and from these we can find $\Lambda(f^i)$ for all i. We have thus proven.

THEOREM 28. *$f \mid V$ determines $\Lambda(f^i)$ for all i.*

As above we may calculate for $K(f)$ a P' and Q' such that $K(f) = P'/Q', (P', Q') = 1$, and P' and Q' are factors of $\Pi_i P(f_{2i}^*)$ and $\Pi_i P(f_{2i+1}^*)$ respectively. Under certain ideal circumstances we may even be able to determine $P(f_i^*)$ for all i. For example, if it should be the case that $\deg P' = \deg \Pi_i P(f_{2i}^*)$, $\deg Q' = \deg \Pi_i P(f_{2i+1}^*)$, the nonzero β_{2i} are distinct from each other, the nonzero β_{2i+1} are also distinct from each other, and the β_{2i} and β_{2i+1} correspond to the degrees of the (irreducible) factors of P' and Q' respectively, then $P' = \Pi_i P(f_{2i}^*)$, $Q' = \Pi_i P(f_{2i+1}^*)$ and the $P(f_i)$'s can be identified from the factorizations of P' and Q' into irreducible factors. Other circumstances also lead to whole or partial determinations of the $P(f_i^*)$.

One need not always know all the a_i, $1 \leqq i \leqq \Sigma_j \beta_j$, in order to calculate $K(f)$. In the special case where $a_i = 0$ for $1 \leqq i \leqq \max(\Sigma_i \beta_{2i}, \Sigma_i \beta_{2i+1})$ it is not difficult to show that $K(f) = \lambda^\chi$. This implies the following theorem.

THEOREM 29. *Let f be a continuons map of a compact polyhedron X into itself, $f: X \rightarrow X$. If f^n is fixed point free for $1 \leqq n \leqq \max(\Sigma_i \dim H_{2i}(X), \Sigma_i \dim H_{2i+1}(X))$ then the Lefschetz indices $\Lambda(f^m)$ vanish for all $m \geqq 0$.*

5. Lefschetz numbers determine Euler characteristic.

THEOREM 30. *Let X_1 and X_2 be compact polyhedra, and $f_1: X_1 \rightarrow X_1$ and $f_2: X_2 \rightarrow X_2$ homeomorphisms (homotopy equivalences or just that f_1^* and f_2^* are isomorphisms will suffice). If the Lefschetz numbers agree, $\Lambda(f_1^n) = \Lambda(f_2^n)$ for all $n \geqq 1$, then the Euler characteristics of X_1 and X_2 are the same, $\chi(X_1) = \chi(X_2)$.*

Proof. We may assume $\chi(X_1) \geqq \chi(X_2)$. From the proof of Lemma 15 we have the formula

$$-na_n(f_i) = \Lambda(f_i^n) + a_1(f_i)\Lambda(f_i^{n-1}) + \cdots + a_{n-1}(f_i)\Lambda(f_i^1) \qquad n \geqq 1, i = 1, 2 .$$

Thus $a_n(f_1) = a_n(f_2)$ for all $n \geqq 1$. Since

$$K(f_i)(\lambda) = \lambda^{\chi(X_i)}(1 + a_1(f_i)\lambda^{-1} + \cdots) \qquad (i = 1, 2)$$

we see that

$$K(f_1)(\lambda) = \lambda^{\chi(X_1) - \chi(X_2)} K(f_2)(\lambda) \quad ☆☆$$

Set $P_i = \Pi_j P(f_{i,2j}^*)$ and $Q_i = \Pi_j P(f_{i,2j+1}^*)$ for $i = 1, 2$. Then $K(f_i) = P_i/Q_i$ for $i = 1, 2$. Since each $f_{i,j}^*$ is an isomorphism, $\det(f_{i,j}^*) \neq 0$ for $i = 1, 2$, and $j \geqq 0$. Thus $P_i(0) \neq 0$ and $Q_i(0) \neq 0$ for $i = 1, 2$. From ☆☆ we have

$$P_1(\lambda)Q_2(\lambda) = \lambda^{\chi(X_1)-\chi(X_2)}P_2(\lambda)Q_1(\lambda)$$

If $\chi(X_1) \neq \chi(X_2)$ then we would have the impossibility that either $P_1(0) = 0$ or $Q_2(0) = 0$. Thus $\chi(X_1) = \chi(X_2)$ as we wished to prove.

To see how one may actually calculate $\chi(X)$ from $\{\Lambda(f^n)\}_{n=1}^{\infty}$ where $f: X \to X$ is a homeomorphism we make the following observations. Note that if $K(f) = P/Q$ where P and Q are two polynomials with $P(0) \neq 0$ and $Q(0) \neq 0$ then $\chi(X) =$ degree $P -$ degree $Q = \mathscr{S}\mathscr{P} - \mathscr{S}\mathscr{Q}$. Next from the formula

$$K(f) = (\lambda^a + 1)\Pi_i P(f_{2i}^*)/(\lambda^a + 1)\Pi_i P(f_{2i+1}^*)$$

we see that for any integer $q \geqq$ degree $\Pi_i P(f_{2i+1}^*)$ there exists monic polynomials P and Q such that $P(0) \neq 0$, $Q(0) \neq 0$, degree $Q = q$ and $K(f) = P/Q$. If $Q(\lambda) = \lambda^q + c_1\lambda^{q-1} + \cdots + c_q$ and $P(\lambda) = \lambda^p + d_1\lambda^{p-1} + \cdots + d_p$ then $Q(0) = c_q$, $P(0) = d_p$ and $K = P/Q$ is equivalent to $\sum_{i=0}^{q} c_i a_{n-i} = d_n$ where we have set $c_0 = a_0 = d_0 = 1$.

There observations lead to the following procedure for calculating $\chi(X)$ from $\{\Lambda(f^n)\}$. First calculate the $a_n = a_n(f)$ from ☆ of Lemma 15. Then define ${}^q a^n \equiv (a_{n-q}, a_{n-q+1}, \cdots, a_n) \in \mathbf{R}^{q+1}$. Next let c^q be an $q + 1$ tuple $(c_q, \cdots, c_0) = c$ with $c_0 = 1$, $c_q \neq 0$ and $c^q \cdot a^n = 0$ for $n = 2q, 2q + 1, \cdots, 2q + q$ (where "$\cdot$" is the usual dot product in $\mathbf{R}^{q+1}$) if such a $q+1$ tuple exists; let $c^q = (0, 0, \cdots, 0)$ if no such $q + 1$ tuple exists.

Now set $p_q + 1 =$ the smallest nonnegative integer N such that $c^q \cdot a^n = 0$ for $N \leqq n \leqq 3q$. Then $\chi(X) = p_q - q$ for

$$q \geqq \max(\text{degree } \Pi_i P(f_{2i+1}^*), |\chi(X)|) .$$

Thus, if some *a priori* upper bound can be put on $\sum_i \dim H_i(X)$ then the above procedure becomes a finite procedure for calculating $\chi(X)$ from a finite number of the $\Lambda(f^n)$.

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