PACIFIC JOURNAL OF MATHEMATICS
Vol. 25, No. 2, 1968

ON THE TETRAHEDRAL GRAPH

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Generalizing the concept of the triangular association scheme, Bose and Laskar introduced the tetrahedral graph the vertices of which are the $\binom{n}{3}$ unordered triplets selected from n symbols with two points adjacent if and only if their corresponding triplets have two symbols in common. If we let $d(x, y)$ denote the distance between two vertices x, y and $\Delta(x, y)$ the number of vertices adjacent to both x and y, then the tetrahedral graph possesses the following 4 properties:

(B0) **the number of vertices is** $\binom{n}{3}$

(B1) **it is connected and regular of degree** $3(n-3)$

(B2) **if** $d(x, y) = 1$ **then** $\Delta(x, y) = n - 2$

(B3) **if** $d(x, y) = 2$ **then** $\Delta(x, y) = 4$.

The question whether these conditions characterize tetrahedral graphs (no loops or parallel edges permitted) was answered in the affirmative by Bose and Laskar for $n > 16$. In the present paper characterizations of tetrahedral graphs are derived by strengthening each one of (B1), (B2), (B3) and these results are utilized to prove the sufficiency of (B0)-(B3) for $n=6$. (For $n < 4$ the problem is void, $n = 4, 5$ are trivial cases.)

All graphs considered in this paper are finite undirected without loops or parallel edges. As is readily seen the line-graph G of the complete graph with n vertices may be defined as a graph whose vertices are the $\binom{n}{2}$ unordered pairs taken from n symbols so that two pairs are adjacent if and only if they have a symbol in common. Letting $d(x, y)$ denote the distance between x and y and $\Delta(x, y)$ the number of vertices that are adjacent to both x and y, then G has the following properties:

(A0) the number of vertices is $\binom{n}{2}$

(A1) G is connected and regular of degree $2(n-2)$.

(A2) $d(x, y) = 1$ implies $\Delta(x, y) = n - 2$

(A3) $d(x, y) = 2$ implies $\Delta(x, y) = 4$.

Conner [2], Shrikhande [7], Hoffman [3, 4] and Li-chien [5, 6] showed that (A0)-(A3) completely characterize linegraphs of complete graphs except for $n = 8$ where 3 nonisomorphic graphs satisfying (A0)-(A3) exist. Bose and Laskar [1] took up the similar problem concerning unordered triplets chosen from n symbols we mentioned above.

For $n > 16$ (B0)-(B3) characterize tetrahedral graphs as was shown by Bose and Laskar in [1].

For $n < 4$ the characterization problem is meaningless.

For $n = 4$ a graph G satisfying (B0)–(B3) is necessarily the complete graph with 4 vertices and the vertices may be identified with the 4 unordered triplets chosen from 4 symbols.

For $n = 5$ the conditions (B0)–(B3) are identical with (A0)–(A3) hence a graph G satisfying (B0)–(B3) may be assigned as vertices the 10 unordered pairs of symbols taken from a set of 5 symbols with two vertices adjacent if and only if their corresponding pairs share a symbol. Replacing each pair by its complement in the set of the 5 symbols we obtain a graph G with triplets assigned to its vertices and G is readily seen to be tetrahedral.

In the following we assume $n \geqq 6$. K_i will denote the complete graph with i vertices, $S(x)$ the set of the vertices adjacent to x, $T(x,y)$ the set of the vertices adjacent to both x and y.

II. Characterizations of tetrahedral graphs.

LEMMA 1. *For a graph G satisfying* (B0)–(B3) *the following properties are equivalent*:

(C1) *For all $x \in G$ the subgraph induced by $S(x)$*[1] *can be partitioned into $3K_{n-3}$'s*:

$$X = \{x_1, x_2, \cdots, x_{n-3}\},\ Y = \{y_1, y_2, \cdots, y_{n-3}\},\ Z = \{z_1, z_2, \cdots, z_{n-3}\}$$

such that $\{x_i, y_i, z_i\}$ induces a K_3 for $i = 1, \cdots, n-3$[2].

(C2) *For all $x, y \in G$ with $d(x, y) = 1$ the subgraph induced by $T(x, y)$ consists of a K_{n-4} and a K_2 such that no vertex in K_{n-4} is adjacent to either vertex in K_2.*

(C3) *For all $x \in G$ the subgraph induced by $S(x)$ can be partitioned into $3K_{n-3}$'s such that for any pair $y, z \in S(x)$ with $d(y, z) = 2$ there are exactly 2 other vertices $v, w \in S(x)$ which are adjacent to both y, z.*

Proof. It is evident that property (C1) implies both (C2) and (C3). On the other hand assume (C2) and let $x_1 \in S(x)$. In $T(x, x_1)$ let $x_2, \cdots, x_{n-3}$ be the vertices in K_{n-4} and let y_1, z_1 be those in K_2. It follows from (C2) that in $T(x, y_1)$ the K_2-part is constituted by x_1, z_1 and further that the $n-4$ remaining vertices $y_2, \cdots, y_{n-3}$ form the K_{n-4}-part and are distinct from $x_2, \cdots, x_{n-3}$. Similarly for the pair x, z_1 the set $T(x, z_1)$ is made up by x_1, y_1 as K_2-part and by $n-4$ vertices $z_2, \cdots, z_{n-3}$ different from $x_i, y_i (i = 1, \cdots, n-3)$. Hence $S(x)$ has the form displaved in Fig. 1. (B2) implies that each of $x_i, y_i, z_i (i = 2, \cdots, n-3)$ is adjacent to exactly 2 vertices in $S(x)$ outside its own K_{n-3}.

[1] By the subgraph induced by a set S of vertices in G we mean the subgraph which has S as vertex-set and includes all edges between any two points in S.

[2] By (B2) it is clear that there exist no edges joining vertices of one K_{n-3} to another other than those of the specified K_3's.

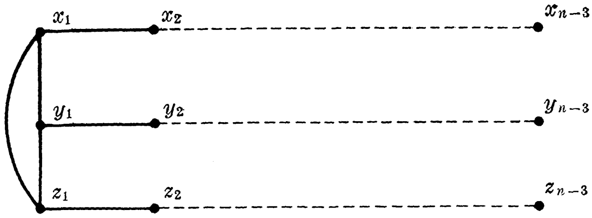

FIG. 1

If e.g. x_i were adjacent to $y_j, y_k (2 \leqq j, k)$ then the subgraph induced by $T(x, y_j)$ would consist of a K_{n-4} and a K_2 with $y_k \in K_{n-4}$, $x_i \in K_2$ and $d(x_i, y_k) = 1$, thus violating (C2). Hence each of x_i, y_i, z_i is adjacent to exactly one vertex of the two K_{n-3}'s not containing it—hence (C1) holds.

Next let us assume (C3). Let X, Y, Z be the $3\ K_{n-3}$'s of $S(x)$ as in (C1). Given $x_1 \in X$: In order to prove (C1) we have to exclude the following two possibilities:

(A) x_1 is adjacent to two vertices, say $y_1, y_2 \in Y$, lying in the same K_{n-3}.

(B) x_1 is adjacent to say $y_1 \in Y, z_1 \in Z$ in different K_{n-3}'s but $d(y_1, z_1) = 2$.

Suppose (A): The set $Y - \{y_1, y_2\}$ is nonempty (since $n \geqq 6$) and by (C3) no $y \in Y - \{y_1, y_2\}$ can be adjacent to any $x \in X$. Hence let us assume $y_3 \in Y - \{y_1, y_2\}$ is adjacent to $z_1, z_2 \in Z$. Since z_1 is adjacent to exactly two points in $S(x)$ outside Z (one of them being y_3) we conclude there is at most one vertex in $S(x)$ adjacent to both x_1 and z_1, thus contradicting (C3).

Suppose (B): By (C3) either y_1 is adjacent to some $z \in Z$ in which case z_1 must be adjacent to some $x_2 \in X$, or y_1 is adjacent to some $x_2 \in X$ with z_1 adjacent to some $y_2 \in Y$ or to x_2 also. Either possibility brings us back to case (A) with z_1 respectively y_1 playing the role of x_1 in (A).

REMARK 1. In a graph G satisfying (B0)-(B3) condition (C1) implies (C3'): For any pair of vertices x_1, y_1 with $d(x_1, y_1) = 2$ the subgraph induced by the 4 vertices x_2, x_3, y_2, y_3 adjacent to both x_1, y_1 is a cycle.

Proof. In the subgraph induced by $S(x_3)$, x_1 and y_1 are in different K_{n-3}'s with x_1 adjacent to a vertex, say x_2, in the K_{n-3} containing y_1, and y_1 in turn adjacent to y_2 in the K_{n-3} containing x_1. We have $d(x_2, y_2) = 2$ and no other vertices in $S(x_3)$ are adjacent to both x_1, y_1. Now let us consider $S(x_2)$. There x_3, y_1 are in the same K_{n-3}, x_1 is in another and y_1 is adjacent to a vertex y_3 in the K_{n-3} which contains x_1. Since y_3 evidently is different from y_2, it must be the fourth point adjacent to x_1, y_1; furthermore $d(x_3, y_3) = 2$, $d(x_2, y_3) = 1$. Similarly one gets $d(y_2, y_3) = 1$; hence x_2, x_3, y_2, y_3 induce a cycle.

REMARK 2. It can be shown with only a little difficulty that for $n = 6$ the converse of Remark 1 holds i.e. (C3′) implies (C1) in a graph satisfying (B0)–(B3). As we will not make use of this fact subsequently the proof is omitted.

LEMMA 2. *Let a graph G satisfy* (B0)–(B3) *and* (C1). *Let X, Y, Z be as in Lemma* 1, *i.e.* $S(x) = X \cup Y \cup Z$. *Then*

$$S(z_i) = (T(x, z_i) - \{x_i, y_i\}) \cup \{x\} \cup (T(x_i, z_i) - \{x, y_i\}) \cup \{x_i\} \cup (T(y_i, z_i) - \{x, x_i\}) \cup \{y_i\} \quad \text{for all } i$$

with each one of the sets on the right hand side inducing a K_{n-3}. For $S(x_i)$, $S(y_i)$ the analogous statements hold.

Proof. Follows instantly from (C1).

Note that Lemma 2 implies that $S(z_i)$ is completely determined by $S(x)$, $S(x_i)$, $S(y_i)$.

THEOREM 1. *A graph G is tetrahedral if and only if G satisfies* (B0)–(B3) *and any* (*and hence all*) *of the conditions* (C1)–(C3).

Proof. Necessity follows readily from the definition. To prove the sufficiency let us first interpret tetrahedral graphs geometrically.

For $n \geqq 6$ let $C_n = \{(i, j, k) \mid 1 \leqq i, j, k \leqq n; i, j, k \text{ integral}\}$ i.e., the set of all integral lattice points of the 3-cube with sides extending from 1 to n. Let $C'_n = \{(i, j, k) \mid 1 \leqq i, j, k \leqq n; i \neq j \neq k \neq i; i, j, k \text{ integral}\}$ then $|C'_n| = n(n-1)(n-2)$. Now it is evident that G is tetrahedral if and only if its vertices can be identified with the lattice points in C'_n (each vertex appears exactly 6 times in C'_n) such that two vertices are adjacent if and only if they lie on a straight line parallel to a coordinate-axis.

Thus in order to prove the theorem it suffices to show that the vertices of a graph G satisfying (B0)–(B3) and say (C1) can be arranged in C'_n in the above fashion.

For simplicity let us denote the vertices of G by the natural numbers from 1 to $\binom{n}{3}$.

Let

$$\left\{\begin{matrix} 2, \cdots, n-2 \\ n-1, \cdots, 2n-5 \\ 2n-4, \cdots, 3n-8 \end{matrix}\right\}$$

be the 3 K_{n-3}'s constituting $S(1)$ with numbers in the same colum inducing K_3's. Place 1 at the spots $(3, 2, 1), (2, 3, 1), (1, 3, 2)$ in C'_n and $\{2, \cdots, n-2\}$ on the lattice points $\{(i, 2, 1) \mid 4 \leqq i \leqq n\}$ in this order, $\{n-1, \cdots, 2n-5\}$ on $\{(i, 3, 1) \mid 4 \leqq i \leqq n\}$, and $\{2n-4, \cdots, 3n-8\}$ on $\{(i, 3, 2) \mid 4 \leqq i \leqq n\}$.

Next we look at the set $S(2)$. Besides $\{1, 3, \cdots, n-2\}$ inducing a K_{n-3}, there are two more K_{n-3}'s one headed by $n-1$, the other by $2n-4$. ($n-1, 2n-4$ cannot be in the same K_{n-3} since both are adjacent to 1.)

Thus

$$S(2) = \begin{Bmatrix} 1, & 3, \cdots, n-2 \\ n-1, & 3n-7, \cdots, 4n-12 \\ 2n-4, & 4n-11, \cdots, 5n-16 \end{Bmatrix}$$

with numbers in the same column inducing K_3's.

Now place 2 at the spots $\{2, 4, 1\}$ and $(1, 4, 2)$ in C'_n, $\{n-1, 3n-7, \cdots, 4n-12\}$ on the line $\{(i, 4, 1) \mid 3 \leqq i \leqq n\}$, $\{2n-4, 4n-11, \cdots, 5n-16\}$ on the line $\{(i, 4, 2) \mid 3 \leqq i \leqq n\}$. Now the situation is as follows (Fig. 2): We claim:

$$\begin{aligned} d(n, 3n-7) &= d(2n-3, 4n-11) = 1 \\ d(n+1, 3n-6) &= d(2n-2, 4n-10) = 1 \\ &\vdots \\ d(n, 4n-11) &= d(2n-3, 3n-7) = 2 \\ d(n+1, 4n-10) &= d(2n-2, 3n-6) = 2 \\ &\vdots \end{aligned}$$

It now follows from $d(2, n) = 2$ and Remark 1 that the fourth point i beside $1, 3, n-1$ adjacent to both $2, n$ has to satisfy $d(i, 1) = 2$, $d(i, 3) = d(i, n-1) = 1$. Hence i must be $3n-7$ and furthermore $d(n, 4n-11) = 2, d(2n-3, 4n-11) = 1, d(2n-3, 3n-7) = 2$. A similar argument proves the other assertions. That no other points among the ones introduced thus far are adjacent beside those already mentioned also follows easily with the help of Remark 1.

Next we consider $S(3)$: $\{1, 2, 4, \cdots, n-2\}$ induce one K_{n-3}; we have already found that $n, 2n-3, 3n-7, 4n-11$ are also in $S(3)$. Since $n, 2n-3$ are both adjacent to 1 they must be in different K_{n-3}'s, and so must $3n-7, 4n-11$ be in different K_{n-3}'s. Hence

$$S(3) = \begin{Bmatrix} 1, & 2, & 4, \cdots, n-2 \\ n, & 3n-7, & 5n-15, \cdots, 6n-21 \\ 2n-3, & 4n-11, & 6n-20, \cdots, 7n-26 \end{Bmatrix}$$

with elements in the same column inducing K_3's.

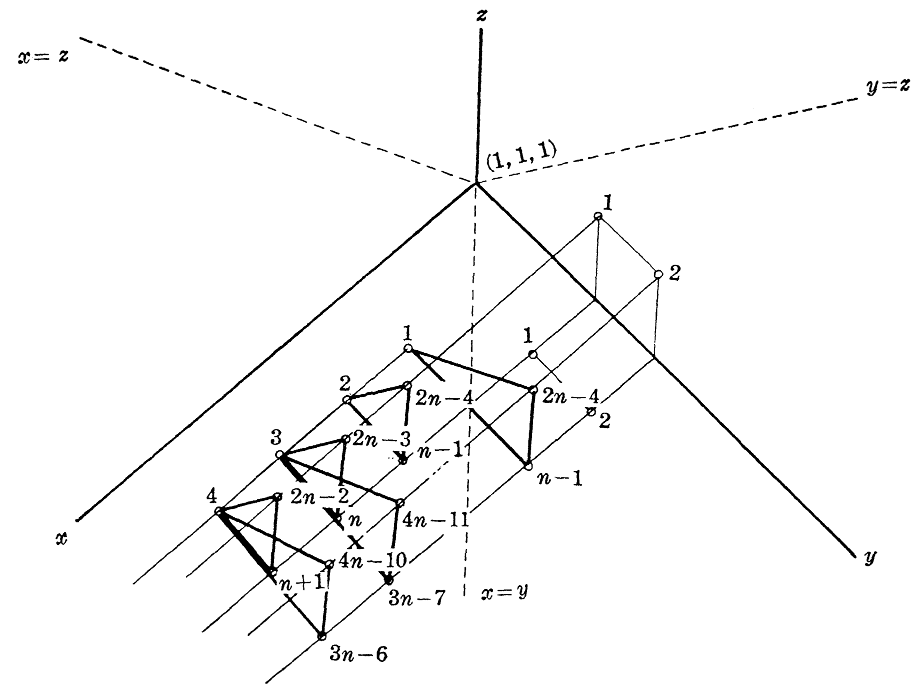

FIG. 2

Place 3 at $(2, 5, 1)$ and $(1, 5, 2)$ in C'_n, $\{n, 3n-7, \cdots, 6n-21\}$ on the line $\{(i, 5, 1) \mid 3 \leqq i \leqq n\}$, $\{2n-3, \cdots, 7n-26\}$ on the line $\{(i, 5, 2) \mid 3 \leqq i \leqq n\}$.

We claim:

$$(1)\qquad \begin{aligned} d(n+1, 5n-15) &= d(3n-6, 5n-15) = 1 \\ d(2n-2, 6n-20) &= d(4n-10, 6n-20) = 1 \end{aligned}$$

$$(2)\qquad \begin{aligned} d(n+1, 6n-20) &= d(3n-6, 6n-20) = 1 \\ d(2n-2, 5n-15) &= d(4n-10, 5n-15) = 1 \end{aligned}$$

and similarly for the lines parallel to the y-axis starting at $n+2$, $n+3, \cdots$ resp. $2n-1, 2n, \cdots$. (1) follows as before by considering $3, n+1$; $3, 3n-6$; $3, 2n-2$, $3, 4n-10$, and (2) is a consequence of (1). Again we note that no other edges beside those already mentioned exist between vertices 1 to $7n-26$.

In this way one considers all sets $S(a)$ for $1 \leqq a \leqq n-2$ and fills the lattice points $\{(i, j, k) \mid 1 \leqq i \leqq n, 1 \leqq j \leqq n, k = 1, 2\}$ in C'_n in the same fashion, thus obtaining $\binom{n-1}{2} + \binom{n-2}{2} = (n-2)^2$ vertices.

Now we turn to vertices corresponding to points $\{(i,j,k) \mid j=3, k=1\}$ in C'_n. In $S(n-1)$ two K_{n-3}'s are already known: $\{1, n, n+1, \cdots, 2n-5\}$, $\{2, 3n-7, \cdots, 4n-12\}$ with $1 = d(1, 2) = d(n, 3n-7) = \cdots$ and further $2n-4 \in S(n-1)$ Hence

$$S(n-1)\left\{\begin{matrix} 1, & n, & n+1, \cdots, 2n-5 \\ 2, & 3n-7, & 3n-6, \cdots, 4n-12 \\ 2n-4, & (n-2)^2+1, & (n-2)^2+2, \cdots, (n-2)^2+(n-4) \end{matrix}\right\}$$

with numbers in the same colum inducing K_3's.

Place $n-1$ at $(1,4,3)$ and $\{2n-4, (n-2)^2+1, \cdots, (n-2)^2+(n-4)\}$ on the line $\{(i,4,3) \mid 2 \leqq i \leqq n\}$ in C'_n.

We claim:

$$d(2n-3, (n-2)^2+1) = d(4n-11, (n-2)^2+1) = 1$$
$$d(2n-2, (n-2)^2+2) = d(4n-10, (n-2)^2+2) = 1$$
$$\vdots$$

This assertion is verified by considering the pairs $2n-3, n-1$; $4n-11, n-1$; $2n-2, n-1$; $4n-10, n-1$; $\cdots$ and applying Remark 1.

In this manner we fill up all the lattice points $\{(i,j,k) \mid 1 \leqq i \leqq n, 4 \leqq j \leqq n, k=3\}$ thus obtaining $\binom{n-3}{2}$ new vertices. By Lemma 2 the vertices adjacent to $2n-4, 2n-3, \cdots$ i.e. to points on the line $\{(i,j,k) \mid 4 \leqq i, 3=j, 2=k\}$ have already been taken care of, so we may turn to points on the line $\{(i,j,k) \mid j=4, k=1\}$. We place $3n-7$ at $(1,5,4)$ and proceed in the usual manner.

Proceeding in the same fashion we gradually fill up all the lattice points $\{(i,j,k) \mid j>k\}$ in C'_n obtaining finally

$$\binom{n-1}{2} + \binom{n-2}{2} + \cdots + \binom{3}{2} + \binom{2}{2} = \binom{n}{3}$$

i.e., all vertices of G.

It is easily seen that reflection about the plane $y=z$ fills the other half of C'_n and that by means of this construction one actually arrives at a tetrahedral graph.

III. The case $n=6$.

LEMMA 3. *Given a graph G which satisfies* (B0)-(B3). *Let* $D_i(x) = \{y \in G \mid d(x,y) = i\}$ $i \geqq 1$. *Then*

$$|D_1(x)| = 3(n-3) \tag{3}$$

$$|D_2(x)| = 3\binom{n-3}{2} \tag{4}$$

$$\Big|\bigcup_{i>2} D_i(x)\Big| = \binom{n-3}{3} \tag{5}$$

for all vertice $x \in G$.

Proof. (3) is condition (B1) of the hypothesis. (B2) implies that there are exactly $3(n-3)-(n-1)=2(n-4)$ edges joining an arbitrary vertex in $D_1(x)$ with $D_2(x)$. Now since by (B3) any vertex in $D_2(x)$ is adjacent to exactly 4 vertices in $D_1(x)$ we have the equality $2(n-4)\cdot 3(n-3) = 4\cdot|D_2(x)|$ and hence (4). (5) is now a consequence of (3) and (4).

COROLLARY. *For* $n \leqq 8$ *the diameter of* $G \leqq 3$.

(5) and the fact that $\binom{n-3}{3} < 3(n-3)$ for $n \leqq 8$ immediately prove this assertion.

THEOREM 2. *For* $n = 6$ *a graph* G *satisfying* (B0)-(B3) *is tetrahedral.*

Proof. In the light of Theorem 1 we have to show that in this case (B0)-(B3) imply any (and hence all) of the conditions (C1)-(C3′). Let us prove that (C2) follows from (B0)-(B3).

Let x be any vertex, then by Lemma 3

$$|D_1(x)| = 9,\ |D_2(x)| = 9,\ |D_3(x)| = 1\ .$$

Furthermore let z be such that $d(x, z) = 3$ then for any

$$y \in D_1(x) \cup D_2(x) (= D_1(z) \cup D_2(z))$$

$$d(x, y) = 1 \Longleftrightarrow d(z, y) = 2\ . \tag{6}$$

For simplicity let us denote the vertices of G by $1, 2, \cdots, 20$ and suppose that

$$D_1(1) = \{2, 3, \cdots, 10\}$$
$$D_2(1) = \{11, 12, \cdots, 19\}$$
$$D_3(1) = \{20\}$$

with $d(2, 19) = d(3, 18) = \cdots = d(10, 11) = 3$.

By (B2) 2 is adjacent to 4 vertices in $D_1(1)$—say 3, 4, 5, 6. To prove (C2) we have to show each one of 3, 4, 5, 6 is joined to exactly one other vertex of this set.

(6) now implies

$$d(2, 11) = d(2, 12) = d(2, 13) = d(2, 14) = 1.$$

Case (A). 3 is adjacent to each one of $\{4, 5, 6\}$. Then by (B2) $d(3, 7) = d(3, 8) = d(3, 9) = d(3, 10) = 2$ and hence 1, 4, 5, 6, 11, 12, 13, 14 would all be adjacent to both 2 and 3, contradicting (B2).

Case (B). 3 is adjacent to two of $\{4, 5, 6\}$ say 4, 5. Then by a similar argument 1, 4, 5 and three vertices among $\{11, 12, 13, 14\}$ would be adjacent to 2 and 3, a contradiction.

Case (C). 3 is adjacent to none of $\{4, 5, 6\}$. Then say $d(3, 7) =$

$d(3, 8) = d(3, 9) = 1$ and hence by (6) $d(3, 14) = d(3, 13) = d(3, 12) = 2$ leaving only 1 and 11 as vertices adjacent to both 2 and 3—thus again contradicting (B2).

Hence the only possible case: 3 (and similarly 4, 5, 6) is adjacent to exactly one among $\{3, 4, 5, 6\}$, thus proving the theorem.

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Received May 19, 1967.

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