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INVARIANT MEASURES AND CESÀRO SUMMABILITY

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It is known that if T is a one-to-one, measurable, invertible and nonsingular transformation on the unit interval with a σ-finite invariant measure, then its induced transformation T_1 on L_1 functions f is such that $\lim_{n\to\infty} 1/n \sum_{k=1}^{n} T_1^k f(x)$ exists. In this note, a counterexample is constructed which shows that the converse is false.

Ornstein [4] constructed a linear, piecewise affine transformation on the unit interval which has no σ-finite invariant measure. Chacon [1] accomplished the same objective by constructing a transformation T whose induced transformation on L_1 functions f, denoted here by T_1, was such that

$$
\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_1^k f(x) = 0 \text{ a.e., and} \\
(1) \qquad &\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_1^k f(x) = \infty \text{ a.e. ,}
\end{aligned}
$$

since it is clear that T cannot have a σ-finite invariant measure if the sequence $\{1/n \sum_{k=1}^{n} T_1^k f(x)\}$ does not have a limit. (See also Jacobs [3].) The question arises as to whether the converse holds: if $\lim_{n\to\infty} 1/n \sum_{k=1}^{n} T_1^k f(x)$ exists, then T has a σ-finite invariant measure. It is the purpose of this paper to show that this statement is false by constructing a linear, piecewise affine transformation T on the interval $I = (0, 101/100]$ such that its induced transformation T_1 on L_1 functions f satisfies

$$
(2) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_1^k f(x) = 0 \text{ a.e.}
$$

Section 2 gives the construction of T, § 3 contains the proof that T has no σ-finite invariant measure, and § 4 shows that the induced transformation, T_1, satisfies (2).

The author is indebted to D. Ornstein for suggesting the method of construction of T, which parallels his construction in [1]. (See also [3].)

2. Construction of T. The transformation of T will be defined inductively step-by-step, and completely constructed in a denumerable number of steps. At each step, the domain of T will be extended to a subinterval of (1, 101/100], and T will not be altered where once defined.

At the first step, let T take $(0, 1/2]$ onto $(1/2, 1]$ in an order-preserving, affine way. Break up the interval $(1, 1 + (100)^{-1}/2]$ into 10^6 disjoint subintervals each of equal length $10^{-8}/2$. Denote $(0, 1/2]$ by I_1, $(1/2, 1]$ by I_2, and number the 10^6 subintervals just defined left to right by $I_3, \cdots, I_4, \cdots, I_{10^6+2}$. Let T take I_2 onto I_3, I_3 onto $I_4, \cdots, I_{10^6+1}$ onto I_{10^6+2}, in an order-preserving, affine way.

The domain of T will now be extended to some part of I_{10^6+2} using the method of [1]: split I_1 into two subintervals of equal length $I_{11} = (0, 1/4]$ and $I_{12} = (1/4, 1/2]$: split $I_2 = (1/2, 1]$ into $I_{21} = (1/2, 3/4]$ and $I_{22} = (3/4, 1]$. Similarly define I_{j1} and I_{j2} for $3 \leqq j \leqq 10^6 + 2$. It is clear that T already takes I_{j1} onto $I_{j+1,1}$ for $1 \leqq j \leqq 10^6 + 1$. Now split up all intervals I_{j2}, $1 \leqq j \leqq 10^6 + 2$ into 10^3 subintervals of equal length. By an obvious left-to-right numbering scheme, I_{j2} will be the union of consecutive disjoint subintervals $I_{j,2,1}, I_{j,2,2}, \cdots, I_{j,2,10^3}$ called the right part of I_j. I_{j1} is called the left part of I_j. It is clear that T already takes $I_{j,2,l}$ onto $I_{j+1,2,l}$ for $1 \leqq j \leqq 10^6 + 1$, $1 \leqq l \leqq 10^3$ in an order-preserving, affine way.

The domain of T will now be extended to the subinterval

$$I_{10^6+2} - I_{10^6+2,2,10^3} = I_{10^6+2,1} \cup \left(\bigcup_{l=1}^{10^3-1} I_{10^6+2,2,l} \right),$$

as follows. Let T take $I_{10^6+2,1}$ onto $I_{1,2,1}$ and $I_{10^6+2,2,l}$ onto $I_{1,2,l+1}$ for $1 \leqq l \leqq 10^3 - 1$ in an order-preserving, affine way. Now relabel all intervals from left to right $I_1, \cdots, I_{M_1}$. This completes step one.

At the end of step $n - 1$, relabelling the intervals in an obvious way, T takes interval I_j onto I_{j+1} for $1 \leqq j \leqq M_n$ in an order-preserving, affine way. T is not yet defined on I_{M_n} and T will now be defined on part of I_{M_n}. Split I_{M_n} into 10^{6n} subintervals of equal length, and order them from left to right as $I_{M_n} + 1, \cdots, I_{N_n}$, where $N_n = M_n + 10^{6n}$. Now let T take I_j onto I_{j+1}, $M_n \leqq j \leqq N_n$ in an order-preserving, affine way. The domain of T will now be extended to some part of I_{N_n} using the method of [1].

For $1 \leqq j \leqq N_n$, split I_j into two disjoint intervals of equal length, written I_{j1} and I_{j2}, numbering from left to right. Divide the right interval I_{j2} into 10^{3n} disjoint subintervals of equal length, and denote them, from left to right, by $I_{j,2,l}$, $1 \leqq j \leqq N_n$ and $1 \leqq l \leqq 10^{3n}$. It is clear that T already takes $I_{j,1}$ onto $I_{j+1,1}$ and $I_{j,2,l}$ onto $I_{j+1,2,l}$ for $1 \leqq j \leqq N_n - 1$ and $1 \leqq l \leqq 10^{3n}$. The domain of T will now be extended to $I_{N_n} - I_{N_n,2,10^{3n}} = I_{N_n,1} \cup \left(\bigcup_{l=1}^{10^{3n}-1} I_{N_n,2,l} \right)$. Let T take $I_{N_n,1}$ onto $I_{1,2,1}$ and $I_{N_n,2,l}$ onto $I_{1,2,l+1}$ in an order-preserving affine way for $1 \leqq l \leqq 10^{3n} - 1$. This completes the definition of T at the n^{th} step. Now relabel all intervals from left to right as $I_1, I_2, \cdots, I_{M_{n+1}}$ to prepare for the $n + 1^{\text{st}}$ step.

3. Invariance properties of T.

DEFINITION. ([1], [3]). Two sets, E, F, are said to be finitely T-equivalent if they allow finite disjoint decompositions $E = \sum_{k=1}^{n} E_k$ and $F = \sum_{k=1}^{n} F_k$, such that for appropriate r_k, $T^{r_k}E_k = F_k$.

THEOREM. *T has no σ-finite invariant measure.*

Proof. We let m_0 denote Lebesgue measure. It suffices to show that T has the following property (See [1], [3] pp. 58–60, which this treatment follows):

For any integer n and any set $M \subset I$, such that $m_0(M) > 9/10$ there is a set of n mutually disjoint and T-equivalent subsets $M_1, \cdots, M_n$ contained in M such that $m_0(M_1) > 1/8$.

To show that this property holds, it suffices to choose $M \subset (0, 1]$ such that $m_0(M) > 9/10$. At step r, suppose $\bigcup I \subset (0, 1]$ where the union is taken only over those subintervals containing a subset of M. Renumber the subintervals $J_1, J_2, \cdots, J_p$, where T or its positive powers takes J_l onto J_{l+1}, $l = 1, 2, \cdots, P - 1$. Suppose $E = \{l: J_l \subset \bigcup_{j=1}^{r-1} I_j\}$. By the construction, $m_0(\bigcup_{l \in E} J_l) = 1/2$.

Let $L = \max\{l: l \in E\}$. Assume $r > n$. Then for $L < s \leqq P, J$ is in the right part of the scheme and hence

(3) $m_0(J_s) \leqq 10^{-6r-3r} < 1/100nL$, since $L \sim 10^{3r}$. From this point on the proof is formally identical with that in [3], p. 60. This observation completes the proof.

4. Convergence of Cesàro sums.

DEFINITION. The transformation on L_1 functions f induced by T, denoted by T_1, is defined for $x_0 \in (0, 101/100]$ as

$$T_1 f(x) = f(T(x_1))R(T, x_0, x_1) ,$$

where $T(x_1) = x_0$ and $R(T, x_0, x_1)$ denotes the suitable Radon-Nikodym derivative of T defined almost everywhere which insures that

$$\int_0^{101/100} T_1 f(x)dx = \int_0^{101/100} f(x)dx .$$

T_1 is well defined. It is clear how to define powers of T_1. This may be expressed as $T_1^n f(x_0) = f(T^n(x_1))R(T^n, x_0, x_1)$, where $T^n(x_1) = x_0$ and $R(T^n, x_0, x_1)$ denotes the Radon-Nikodym derivative which insures that

$$\int_0^{101/100} T_1^n f(x)dx = \int_0^{101/100} f(x)dx .$$

Note that $R(T^n, x_0, x_1)$ is easy to compute. If $T^n(x_0) = x_1$ and $x_0 \in I_l$ and $x_1 \in I_m$, where I_l and I_m are intervals defined together in the same step in the definition of T such that $m \neq l$, then

$$R(T^n, x_0, x_1) = m_0(I_l)/m_0(I_m) = \text{length}\,(I_l)/\text{length}\,(I_m) \tag{4}$$

due to the piecewise affine character of T.

In order to show that for $f \in L_1$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_1^k f(x) = 0 \text{ a.e. } x \in I \tag{5}$$

is suffices to show (5) only for $f = 1$. This is so because if (5) holds for $f = 1$, by the Chacon-Ornstein theorem [2], for any $g \in L_1$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} T_1^k g(x) \Big/ \sum_{k=1}^{n} T_1^k f(x) \text{ exists a.e. } x \in I\,,$$

and hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_1^k g(x) = 0 \text{ a.e. } x \in I\,.$$

Thus it suffices to prove the following.

THEOREM. *For* $f = 1$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_1^k f(x) = 0 \text{ a.e. } x \in I\,.$$

Proof. The proof is divided into two cases; (a) $x \in (0, 1]$ and (b) $x \in (1, 101/100]$.

Case (a). Recall that M_n is the number of subintervals on which T or its range was defined at step n. Note that the point $x = 1$ is in the R_n-th interval at step n, where $R_n = M_n - \sum_{k=1}^{n} 10^{6k}$.

Define $f_n(1) = 1/R_n \sum_{l=1}^{R_n} T_1^l f(1)$. Then $f_n(1)$ is clearly the Cesàro sum of highest index (R_n) which can be defined at step n at the point $x = 1$ among the sums $1/p \sum_{l=1}^{p} T_1^l f(1)$. Also, for $x \in (0, 1]$, R_n is the maximum index p such that $1/p \sum_{l=1}^{p} T_1^l f(x)$ may be defined at step n.

Claim 1. $f_n(1) \leqq 10^{6n} \times 0(10^{3n})$.

Proof. Proceeding by induction, we first obtain an upper bound for $f_1(1)$. The point $x = 1$ is in interval $I_{M_1} - 10^6$ which is of length $1/4 \times 10^{-3}$ and the intervals that map into I_1 at step 1 by T or its positive powers are each of one of the following types:

Type 1. I_1, I_2 each of length 1/4, and hence each contributing 10^3

to the sum $\sum_{l=1}^{R_n} T_1^l f(1)$ by (4);

Type 2. $I_3, \cdots, I_{10^6+2}$, each of length $1/4 \times 10^{-8}$, and so by (4) each contributing 10^{-5} to the above sum;

Type 3. $I_{10^6+3}, I_{10^6+4}, I_{2\times 10^6+5}, I_{2\times 10^6+6}, I_{3\times 10^6+7}, I_{3\times 10^6+8}, \cdots, I_{(10^3-1)10^6+2(10^3-1)+1}$, $I_{(10^3-1)10^6+2(10^3-1)+2}$, each of length $1/4 \times 10^{-3}$, and hence each contributing 1 to the sum; and

Type 4. $I_{10_6+5}, \cdots, I_{2\times 10^6+4}, I_{2\times 10^6+7}, I_{3\times 10^6+6}, \cdots, I_{(10^3-1)\times 10^6+2(10^3-1)+3}, \cdots$, $I_{10^3\times(10^6)+2(10^3-1)+3}$, each of length $1/4 \times 10^{-11}$, and hence contributing 10^{-8} to the sum.

Multiplying the contribution of each type of interval by a number at least as large as the number of each such interval, adding these four terms, and dividing by a number smaller than the total number of summands R_n yields the following upper bound

$$(6) \quad f_1(1) < \frac{2 \times 10^3 + 10^6 \times 10^{-5} + 10^6 \times 10^3 \times 10^{-8} + 2 \times 10^3 \times 1}{10^9}$$

or $f_1(1) < 6 \times 10^{-6}$

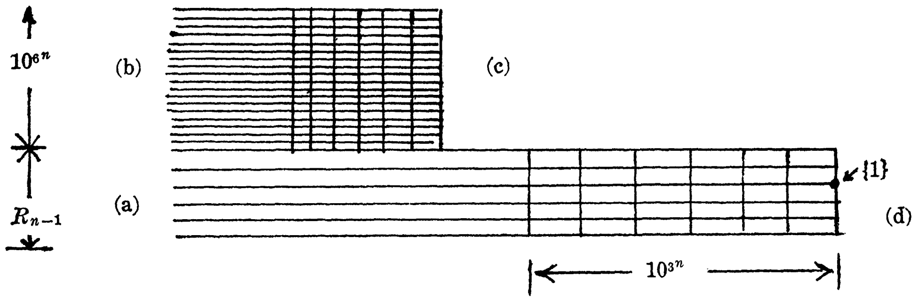

FIGURE 1.

Consider the above diagram representing the four types of domain of definition on which T and its positive powers are defined at step n. The domain (a) is the set of left parts of (0, 1] together with the left parts of the subintervals of (1, 101/100] added to the domain before step n. Domain (b) is the set of left parts of the subinterval of (1, 101/100] added to the domain of definition of T at step n. Domain (c) is the right part of the subinterval added to the domain of T at step n. Domain (d) is the right part of (0, 1] together with the right part of the subintervals of (1, 101/100] added to the domain before step n. The numbers on the diagram refer to the respective number of subintervals into which the left parts right parts, of (0, 1] and appropriate subintervals of (1, 101/100] are divided at the n^{th} step.

Using an obvious notation,

$$(7) \qquad R_n f_n(1) = \sum_{l=1}^{R_n} T_1^l f(1) = \sum_{(a)} + \sum_{(b)} + \sum_{(c)} + \sum_{(d)} T_1^l f(1) ,$$

where

$$(8)\qquad \sum_{(a)} T_1^l f(1) = f_{n-1}(1) R_{n-1} \times 10^{3^n} ,$$

since the length ratio of left part intervals to the corresponding right part intervals is 10^{3^n};

$$(9)\qquad \sum_{(b)} T_1^l f(1) = 10^{-6^n} \times 2^{-n} \times \frac{1}{200} \times 2^n \times 10^{\Sigma_{j=1}^n 3^j} \times 10^{6^n} = \frac{1}{200} \times 10^{\Sigma_{j=1}^n 3^j} ,$$

where $10^{-6^n} \times 2^{-n} \times 1/200$ is the length of a (b) interval, $2^{-n} \times 10^{-\Sigma_{j=1}^n 3^j}$ is the length of the (d) interval containing the point 1, and 10^{6^n} is the number of (b) intervals;

$$(10)\qquad \sum_{(c)} T_1^l f(1) = \frac{1}{200} \times 2^{-n} \times 10^{-6^n} \times 10^{-3^n} \times \left[10^{\Sigma_{j=1}^n 3^j} \times 2^n\right] \times 10^{6^n} \times 10^{3^n} = \frac{1}{200} \times 10^{\Sigma_{j=1}^n 3^j} ,$$

since each subinterval of (c) has length $((100) \times 2^{n+1} \times 10^{6^n+3^n})^{-1}$, the subinterval containing the point 1 has length $(10^{\Sigma_{j=1}^n 3^j} \times 2^n)^{-1}$ and there are a total of $10^{6^n+3^n}$ subintervals in (c);

$$(11)\qquad \sum_{(d)} T_1^l f(1) < R_{n-1} f_{n-1}(1) \times 10^{3^n}$$

since there are 10^{3^n} sets of intervals on which T and its positive powers were defined at the $n - 1^{st}$ step in (d).

Clearly

$$(12)\qquad R_n < 10^{6^n+3^n} .$$

Hence from (6) – (12) inclusive,

$$(13)\qquad f_n(1) < \frac{2f_{n-1}(1) \times R_{n-1} \times 10^{3^n} + 2 \times \left(\frac{1}{200}\right) \times 10^{\Sigma_{j=1}^n 3^j}}{10^{6^n+2^n}} .$$

By the induction hypothesis, $f_{n-1}(1) = 10^{-6^{n-1}} \times 0(10^{3^{n-1}})$.

Using this in (13), $f_n(1) < 10^{-6^n} \times 0(10^{3^n})$, completing the induction argument.

Now consider $x \in (0, 1]$ such that in addition, x is in the right part of the scheme. In the diagram below, at step n, the second subinterval in the right part of the scheme which is also a subinterval of $(0, 1]$ is denoted by Q. This interval is I_{r_0}, where $r_0 = M_{n-1} + 10^{6^n} + 1 > 10^{6^n}$. Let $x_0 \in Q$.

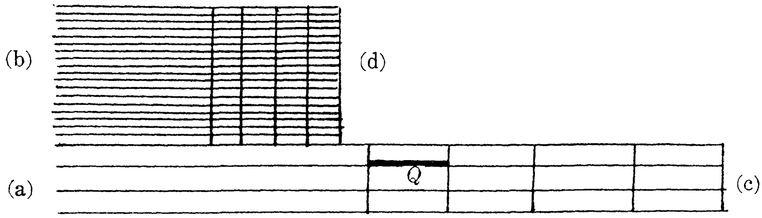

FIGURE 2.

Claim. Suppose r is such that $M_n > r > M_{n-1}$, and $x \in (0, 1]$ and also in the right part of the scheme at step n. Then under these conditions,

$$\max_{x,r} \frac{1}{r} \sum_{l=1}^{r} T_1^l f(x) = \frac{1}{r_0} \sum_{l=1}^{r_0} f(x_0) . \tag{14}$$

This is clear since the largest Radon-Nikodym derivatives in the above Cesàro sum come about as a result of T and its positive powers taking points from the left part of the scheme to its right part.

Claim 2. At step n,

$$\frac{1}{r_0} \sum_{l=1}^{r_0} T_1^l f(x_0) < f_{n-1}(1) . \tag{15}$$

Proof. From the above diagram,

$$\sum_{l=1}^{r_0} T_1^l f(x_0) = \sum_{(a)} + \sum_{(b)} + \sum_{(c)} T_1^l f(x_0) , \tag{16}$$

where

$$\sum_{(a)} T_1^l f(x_0) = 10^{3^n} \times M_{n-1} \times f_{n-1}(1) , \tag{17}$$

$$\sum_{(b)} T_1^l f(x_0) = (200)^{-1} \times 2^{-n} \times 10^{-6^n} \times \left(10^{\sum_{j=1}^{n} 3^j} \times 2^n\right) \times 10^{6^n} < 10^{n3^n} , \tag{18}$$

and,

$$\sum_{(c)} T_1^l f(x_0) = M_{n-1} \times f_{n-1}(1) . \tag{19}$$

Hence from (15) – (19),

$$\frac{1}{r_0} \sum_{l=1}^{r_0} T_1^l f(x_0) < \frac{(10^{3^n} + 1) \times M_{n-1} \times f_{n-1}(1) + 10^{n3^n}}{10^{6^n}}$$

$$\sim \frac{(10^{3^n}) \times 10^{3^{n-1}+6^{n-1}} \times f_{n-1}(1) + 10^{n3^n}}{10^{6^n}} < f_{n-1}(1) .$$

This establishes the claim.

Now a.e. $x \in (0,1]$ is in the right part of the scheme for infinitely many steps n since at each step, every subinterval is divided into two equal subintervals, one of which becomes a member of the left part of the scheme, and the other, the right part. Further, higher powers of $T_1^l f(x)$ can only be defined at a given stage n if x is in the right part of the scheme. These remarks plus Claims 1 and 2 above establish that

$$\lim_{r\to\infty} \frac{1}{r} \sum_{l=1}^{r} T_1^l f(1) \longrightarrow 0 \text{ for a.e. } x \in (0, 1] ,$$

which is case (a).

For case (b), let $x \in (1, 101/100]$. The procedure to be followed parallels that in case (a).

$$\text{Define } f_{1k}\Big(1 + (100)^{-1} \times \sum_{l=1}^{r} 2^{-j}\Big) = \frac{1}{M_{r_k}} \sum_{l=1}^{M_{rk}} T_1^l f\Big(1 + (100)^{-1} \times \sum_{j=1}^{r} 2^{-j}\Big) ,$$

where $k \geqq r + 1$ and $M_{r_k} = M_k - \sum_{j=r+1}^{k} 10^{6^j}$. That is, M_{r_k} is the highest power of T that may be defined at step k with domain on a part of the r^{th} subinterval $(1 + (100)^{-1} \times 2^{-r}, 1 + (100)^{-1} \times 2^{-r+1}]$ which is taken from $(1, 101/100]$.

Claim 3. $f_{1k}(1 + (100)^{-1} \times \sum_{j=1}^{r} 2^{-j}) = 10^{-6^k} \times 0(10^{3^{k-1}})^{\to 0} \longrightarrow 0$ for fixed r as $k \longrightarrow \infty$.

Claim 4. Let

$$x \in \Big(1 + (100)^{-1} \times \sum_{j=1}^{r-1} 2^{-j}\Big), 1 + (100)^{-1} \times \sum_{j=1}^{r} 2^{-j}\Big] .$$

Suppose that at step $k > r, x$ is in the right part of the scheme. Then for

$$M > N_{k-1}, \frac{1}{M} \sum_{l=1}^{M} T_1^l f(x) < f_{1,k-1}\Big(1 + (00)^{-1} \times \sum_{j=1}^{r} 2^{-j}\Big) .$$

The proof of Claim 3 follows as for Claim 1, and that for Claim 4 as for Claim 2. The proofs use the fact that $10^{6^n} \gg 10^{3^n}$ as n increases. The details are omitted.

Since a.e. $x \in (1, 101/100]$ is in the right part of the scheme for infinitely many steps n, and since higher powers of $T_1^l f(x)$, for fixed x, are defined when x is in the right part of the scheme at some step, Claims 3 and 4 yield the result for case (b).

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