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CONVEX SETS AND THE BOUNDED SLOPE CONDITION

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Let Ω be a bounded open convex set in R^n with boundary Γ. This paper concerns the class $B(\Gamma)$ of functions $\phi(x)$, defined on Γ, satisfying a bounded slope condition and its closure $\bar{B}(\Gamma)$ in $C^0(\Gamma)$. The class $\bar{B}(\Gamma)$ is of interest because of its occurrence in the theory of nonlinear, nonuniformly elliptic, boundary value problems. It is shown that $\bar{B}(\Gamma)$ is the set of continuous functions on Γ which, on flat pieces of Γ, are restrictions of linear functions of x. Thus $\bar{B}(\Gamma) = C^0(\Gamma)$ if and only if there are no line segments on Γ.

1. The set $B(\Gamma)$. Let Ω be a bounded, open subset of R^n and $\Gamma = \partial\Omega$ its boundary. A function $\phi(x)$ defined for $x \in \Gamma$ is said to satisfy a *bounded slope condition* (BSC) with a constant $K(\geqq 0)$ if, for every point $x_0 \in \Gamma$, there exists a pair of linear functions $\pi^{\pm}(x)$ of $x \in R^n$ satisfying

$$\begin{aligned} &\pi^{\pm}(x) = \sum_{k=1}^{n} a_k^{\pm}(x^k - x_0^k) + \phi(x_0) = a^{\pm}\cdot(x - x_0) + \phi(y_0) , \\ (1.0) \qquad &\pi^{-}(x) \leqq \phi(x) \leqq \pi^{+}(x) \text{ for } x \in \Gamma, \sum_{k=1}^{n} | a_k^{\pm} |^2 \equiv | a^{\pm} |^2 \leqq K^2 . \end{aligned}$$

For example, any linear function $\phi(x) = a.x + b$, restricted to Γ, satisfies a BSC with $K^2 = | a |^2$. On the other hand, if some function $\phi(x)$, $x \in \Gamma$, is not the restriction of a linear function and satisfies a BSC, then

$$(1.1) \qquad \Omega \text{ is convex .}$$

Below, we shall always assume (1.1).

The bounded slope (or an equivalent) condition occurs in the calculus of variations and the theory of nonlinear elliptic boundary value problems in papers of Hilbert, Lebesgue, Bernstein, Haar, Rado, von Neumann, etc., for recent references (e.g., to Nirenberg, Gilbarg, Stampacchia, and others), see [1], [2], [4, pp. 98–105], [5]. Since there are existence theorems for certain nonlinear, (nonuniformly) elliptic, Dirichlet boundary value problems on Ω with an arbitrary given boundary function ϕ in $B(\Gamma)$, where

$$(1.2) \qquad B(\Gamma) = \{\phi(x), x \in \Gamma : \phi \text{ satisfies a BSC}\} ,$$

it seems worthwhile to examine the set of functions $B(\Gamma)$.

Two results along these lines are the following, given in [1, pp.

504–505]:

(i) $\Gamma \in C^1 \Rightarrow B(\Gamma) \subset C^1(\Gamma)$;

(ii) $\Gamma \in C^{1,\lambda}, 0 < \lambda \leqq 1 \Rightarrow B(\Gamma) \subset C^{1,\lambda}(\Gamma)$.

The convex set Ω or its boundary Γ is called *uniformly convex* if there exists a constant $c > 0$ such that through every $x_0 \in \Gamma$, there passes a hyperplane $\pi \subset R^n$ supporting Γ and having the property that

$$\text{dist}\,(x, \pi) \geqq c\,|\,x - x_0\,|^2 \text{ for } x \in \Gamma\,. \tag{1.3}$$

The class $C^{k,\lambda}(\Gamma)$ will be defined as the set of functions $\phi(x), x \in \Gamma$, which are restrictions to Γ of functions of class $C^{k,\lambda}(R^n)$. (This generalizes the usual definition of class $C^{k,\lambda}(\Gamma)$ which requires that $\Gamma \in C^{k,\lambda}$.) Several authors have used the fact that

(iii′) Γ is uniformly convex $\Rightarrow B(\Gamma) \supset C^{1,1}(\Gamma)$;

for a detailed proof, see [3, p. 242]. Actually, the converse of this statement is also correct:

(iii) Γ is uniformly convex $\Leftrightarrow B(\Gamma) \supset C^{1,1}(\Gamma)$.

As noted in [1], (ii) and (iii′) give the following assertion:

(iv) $\Gamma \in C^{1,1}$ and Γ uniformly convex $\Rightarrow B(\Gamma) = C^{1,1}(\Gamma)$.

Proof of (iii). In view of (iii′), it is sufficient to verify the following converse of (iii′):

(iii″) Γ is uniformly convex $\Leftarrow B(\Gamma) \ni \phi(x) = |\,x\,|^2, x \in \Gamma$.

Suppose, therefore, that $\phi(x) = |\,x\,|^2, x \in \Gamma$, satisfies a BSC, so that there exists a constant K and, for every $x_0 \in \Gamma$, a linear function $\pi^+(x)$ satisfying (1.0). Note that $\phi(x) = |\,x\,|^2 = |\,x - x_0 + x_0\,|^2$ satisfies

$$\phi(x) = |\,x - x_0\,|^2 + 2x_0\cdot(x - x_0) + \phi(x_0)\,.$$

Thus, by (1.0),

$$(a^+ - 2x_0)\cdot(x - x_0) \geqq |\,x - x_0\,|^2 \geqq 0 \text{ for } x \in \Gamma\,. \tag{1.4}$$

Hence $a^+ - 2x_0 \neq 0$, and the hyperplane $\pi\colon (a^+ - 2x_0)\cdot(x - x_0) = 0$ passes through x_0 and supports Ω. By (1.0), $|\,a^+ - 2x_0\,| \leqq K + 2R$, if Ω is contained in the sphere $|\,x\,| \leqq R$. Since $x \in \Omega$ implies that

$$0 \leqq (a^+ - 2x_0)\cdot(x - x_0) = |\,a^+ - 2x_0\,|\,\text{dist}\,(x, \pi)\;,$$

(1.4) shows that the inequality (1.3) holds with $c = 1/(K + 2R)$. This proves (iii″).

2. The set $\bar{B}(\Gamma)$. In [3], M. Miranda obtains generalized solutions for the Dirichlet boundary value problem associated with the minimal surface equation and a boundary function $\phi(x)$ in the set

$$\bar{B}(\Gamma) = \text{the closure of } B(\Gamma) \text{ in } C^0(\Gamma)\,. \tag{2.1}$$

Actually, Miranda assumes that Γ is uniformly convex [so that $\bar{B}(\Gamma) = C^0(\Gamma)$ by (iii′)] and deals with an arbitrary $\phi(x) \in C^0(\Gamma)$. His procedure is valid, however, if it is not assumed that Γ is uniformly convex but merely that $\phi(x) \in \bar{B}(\Gamma)$. Since his arguments apply equally well for other boundary value problems (cf., e.g., [2]), it is of interest to investigate the set of functions $\bar{B}(\Gamma)$ and, in particular, to see when $\bar{B}(\Gamma) = C^0(\Gamma)$.

The following terminology will be used below: A subset $\varLambda$ of Γ, which is neither empty nor a point, is called a *flat piece of* Γ if there exists a hyperplane π supporting Ω and $\varLambda = \pi \cap \Gamma$. A point $x_0 \in \Gamma$ is called an *extreme point of* Γ if it is not an interior point of a line segment on Γ. Ω or Γ is called *strictly convex* if every point $x_0 \in \Gamma$ is an extreme point (i.e., if there are no line segments on Γ). As usual, $\phi \mid \Sigma$ denotes the restriction of the function $\phi(x)$ to the set $x \in \Sigma$. Let

$$\text{(2.2)} \qquad \begin{aligned} A(\Gamma) = \{\phi(x), x \in \Gamma : \phi \in C^0(\Omega);\ &\text{on every flat piece} \\ \varLambda \text{ of } \Gamma, \phi \mid \varLambda\ &\text{is the restriction of a linear function}\}\,, \end{aligned}$$

or, equivalently,

$$\begin{aligned} A(\Gamma) = \{\phi(x), x \in \Gamma : \phi \in C^0(\Gamma);\ &\text{on every line segment} \\ l \subset \Gamma, \phi \mid l\ &\text{is the restriction of a linear function}\}\,. \end{aligned}$$

It is understood that $A(\Gamma) = C^0(\Gamma)$ if Γ is strictly convex. The main result to be proved in this paper is

$$\text{(I)} \qquad \bar{B}(\Gamma) = A(\Gamma)\,;$$

in particular,

$$\text{(II)} \qquad \Gamma \text{ is strictly convex} \Leftrightarrow \bar{B}(\Gamma) = C^0(\Gamma)\,.$$

The proof will be given in § 5.

3. The functions ϕ^r and ϕ_r. It will be assumed that

$$\text{(3.1)} \qquad x = 0 \in \Omega$$

Let $(x, u) = (x^1, \cdots, x^n, u)$ denote coordinates in R^{n+1}. With a function $\phi \in C^0(\Gamma)$ and a number r satisfying

$$\text{(3.2)} \qquad |\phi(x)| < r \text{ for } x \in \Gamma\,,$$

associate the following sets in R^{n+1}:

$$\text{(3.3)} \qquad Z(r, \phi) = \{x = tx_0, u \leqq r + t[\phi(x_0) - r] \text{ for } x_0 \in \Gamma, t \geqq 0\}\,,$$

$$\text{(3.4)} \qquad W(r, \phi) = \{x = tx_0, u \geqq -r + t[\phi(x_0) + r] \text{ for } x_0 \in \Gamma, t \geqq 0\}\,.$$

The boundary $G^+(r, \phi)$ of $Z(r, \phi)[G^-(r, \phi)$ of $W(r, \phi)]$ is a cone with vertex at $(x, u) = (0, r)[(x, u) = (0, -r)]$ which opens downwards [upwards]. These cones were introduced in [1]. The convex hull of $Z(r, \phi)$ is the set of points (x, u) satisfying

$$(3.5) \qquad x = \sum_{i=1}^{m} \lambda_i t_i x_i,\ u \leqq r + \sum_{i=1}^{m} \lambda_i t_i [\phi(x_i) - r]\ ,$$

where $\lambda_i \geqq 0, \Sigma\lambda_i = 1, t_i \geqq 0, x_i \in \Gamma, m > 0$ arbitrary. If $T = \Sigma\lambda_i t_i > 0$ and $\mu_i = \lambda_i t_i / T$, then (3.5) can also be written as

$$(3.6) \qquad x = T\sum_{i=1}^{m} \mu_i x_i,\ u \leqq r + T\sum_{i=1}^{m} \mu_i[\phi(x_i) - r]\ ,$$

where $\mu_i \geqq 0, \Sigma\mu_i = 1, T \geqq 0, x_i \in \Gamma, m > 0$ arbitrary.

On R^n, define the function

$$(3.7) \qquad \phi^r(x) = \sup_{S(x)} \left\{r + T\sum_{i=1}^{m} \mu_i[\phi(x_i) - r]\right\}, \qquad x \in R^n\ ,$$

where the supremum is taken over the set

$$(3.8) \qquad S(x) = \left\{(T, \mu_1, \cdots, \mu_m, x_1, \cdots, x_m) : T\sum_{i=1}^{m} \mu_i x_i = x\right\}\ ,$$

and, as in (3.6), $T \geqq 0, \mu_i \geqq 0, \Sigma\mu_i = 1, x_i \in \Gamma, m > 0$ arbitrary. It is clear that

$$\phi^r(x) = \sup\{u\colon (x, u) \text{ in } (3.6),\ x \text{ fixed}\}\ ,$$

so that the closed convex hull of $Z(r, \phi)$ is the set

$$(3.9) \qquad \operatorname{co} Z(r, \phi) = \{(x, u)\colon u \leqq \phi^r(x), x \in R^n\}\ .$$

It is also clear that

$$(3.10) \qquad \phi^r(tx) = r + t[\phi^r(x) - r] \text{ for } x \in R^n, t \geqq 0\ ;$$

hence

$$(3.11) \quad \operatorname{co} Z(r, \phi) = \{x = tx_0, u \leqq r + t[\phi^r(x_0) - r] \text{ for } x_0 \in \Gamma, t \geqq 0\}\ .$$

Similarly, the closed convex hull of $W(r, \phi)$ is

$$(3.12) \quad \operatorname{co} W(r, \phi) = \{x = tx_0, u \geqq -r + t[\phi_r(x_0) + r] \text{ for } x_0 \in \Gamma, t \geqq 0\}\ ,$$

where

$$(3.13) \qquad \phi_r(x) = \inf_{S(x)} \left\{-r + T\sum_{i=1}^{m} \mu_i[\phi(x_i) + r]\right\}, x \in R^n\ ,$$

and $S(x)$ is given in (3.8). From (3.9) and its analogue, it follows that $\phi^r(x)$ is a concave and $\phi_r(x)$ is a convex function of x. In particular, $\phi^r, \phi_r \in C^0(R^n)$.

PROPOSITION 3.1. Necessary and sufficient in order that $\phi \in B(\Gamma)$ is that $\phi \equiv \phi^r | \Gamma \equiv \phi_r | \Gamma$ for large r.

This is merely a restatement of [1, Th. 2.1, p. 496].

PROPOSITION 3.2. (a) The functions ϕ, ϕ^r, ϕ_r satisfy

$$\phi_r(x) \leqq \phi(x) \leqq \phi^r(x) \text{ for } x \in \Gamma . \tag{3.14}$$

(b) If $\phi, \psi \in C^0(\Gamma)$ and $\phi \leqq \psi$ on Γ, then

$$\phi_r(x) \leqq \psi_r(x), \phi^r(x) \leqq \psi^r(x) \text{ on } \Gamma . \tag{3.15}$$

(c) Finally

$$(\phi^r \mid \Gamma)^r = \phi^r \text{ and } (\phi_r \mid \Gamma)_r = \phi_r . \tag{3.16}$$

Proof. The choice $(T, \mu_1, x_1) = (1, 1, x) \in S(x)$ for $x \in \Gamma$ implies (3.14) by (3.7), (3.13). The other assertions are trivial.

PROPOSITION 3.3. Let $|\phi(x)| \leqq M$ on Γ and $r > 2M + 1$ be fixed. Let $x \in \Gamma$. Then there exists a T and a Borel probability measure μ on Γ, depending on x and r, such that

$$x = T\int_\Gamma y d\mu, \phi^r(x) = r + T\int_\Gamma [\phi(y) - r] d\mu , \tag{3.17}$$

$$1 \leqq T \leqq 1/[1 - (2M + 1)/r] . \tag{3.18}$$

The arguments in the proof of this statement will also be used in other proofs below. Of course, one can obtain analogously

$$x = T\int_\Gamma y d\mu, \phi_r(x) = -r + T\int_\Gamma [\phi(y) + r] d\mu \tag{3.19}$$

for different T and μ.

Proof. For a given $x \in \Gamma$, choose $(T, \mu_1, \cdots, \mu_m, x_1, \cdots, x_m) \in S(x)$ such that the error η defined by

$$\phi^r(x) = (1 - T)r + T\sum_{i=1}^m \mu_i \phi(x_i) + \eta \tag{3.20}$$

satisfies

$$0 \leqq \eta \leqq 1 ; \tag{3.21}$$

cf. (3.7), (3.8). It is clear from (3.8) that $T \geqq 1$ if $x \in \Gamma$. Since $|\phi(x)| \leqq M$, (3.14) shows that $\phi^r(x) \geqq -M$ for $x \in \Gamma$. Hence, by (3.20),

$-M \leqq (1 - T)r + TM + 1$ and, since $T \geqq 1$, we have $0 \leqq 1 - 1/T \leqq (1 + 2M)/r$. Consequencely, (3.18) holds.

For the choice $(\mu_1, \cdots, \mu_m, x_1, \cdots, x_m)$, write

$$x = T\sum_{i=1}^{m} \mu_i x_i = T\int_\Gamma y d\mu, \ \sum_{i=1}^{m} \mu_i \phi(x_i) = \int_\Gamma \phi(x) d\mu \ , \tag{3.22}$$

where μ is a probability (Borel) measure on Γ with support on the finite set $\{x_1, \cdots, x_m\}$. For each $k = 1, 2, \cdots$, choose $T^k, \mu_1^k, \cdots, \mu_{m(k)}^k, x_1^k, \cdots, x_{m(k)}^k \in S(x)$ so that $\eta = \eta^k$ in (3.19) satisfies $\eta^k \to 0$. If $r > 1 + 2M$, it can be supposed that $T = \lim T^k$ exists and satisfies (3.18), and that $\mu = \lim \mu^k$ exists weakly. Writing $T = T^k$ and $\mu = \mu^k$ and letting $k \to \infty$, we obtain (3.17) from (3.20) and (3.22).

PROPOSITION 3.4. We have

$$\phi_r(x) \leqq \phi_s(x) \leqq \phi^s(x) \leqq \phi^r(x) \ \text{ for } s \geqq r, x \in \Omega \ . \tag{3.23}$$

Furthermore

$$\phi^\infty(x) = \lim_{r\to\infty} \phi^r(x), \ \phi_\infty(x) = \lim_{r\to\infty} \phi_r(x) \tag{3.24}$$

exist for $x \in \Gamma$ and

$$\phi_\infty(x) \leqq \phi(x) \leqq \phi^\infty(x) \ \text{ for } x \in \Gamma \ . \tag{3.25}$$

Note that $\phi^r(x), \phi^r(x)$ are defined on R^n, while $\phi^\infty(x), \phi_\infty(x)$ are defined only on Γ.

Proof. Consider only ϕ^r, ϕ^∞. The expression $\{\cdots\}$ in (3.7) can be written

$$\{\cdots\} = r(1 - T) + T\sum_{i=1}^{m} \mu_i \phi(x_i) \ .$$

The convexity of Ω implies that $T \geqq 1$ in (3.8) if $x \in \Omega$. Thus the term $r(1 - T)$ is a nonincreasing function of r. This gives the last inequality in (3.23) which, together with (3.14), implies the existence of $\phi^\infty(x)$ and the inequality $\phi^\infty(x) \geqq \phi(x)$ for $x \in \Gamma$.

PROPOSITION 3.5. The relation

$$\phi^\infty(x) = \phi(x) \ [\text{or } \phi_\infty(x) = \phi(x)] \tag{3.26}$$

holds for all $x \in \Gamma$ if and only if $\phi(x)$ is a concave [or convex] function on every flat piece of Γ; in which case

$$\phi^r(x) \to \phi(x) \ [\text{or } \phi_r(x) \to \phi(x)] \ \text{ uniformly on } \Gamma \ , \tag{3.27}$$

as $r \to \infty$. In particular,

$$\phi^\infty = \phi = \phi_\infty \text{ on } \Gamma \Leftrightarrow \phi \in \Lambda(\Gamma) .$$

The proof shows that both parts of (3.26) hold at an extreme point x of Γ for every $\phi \in C^0(\Gamma)$.

Proof. Consider only ϕ^r and ϕ^∞. Suppose that the first part of (3.26) holds for all $x \in \Gamma$. Let Δ be a flat piece of Γ. Then Δ is a closed convex set and $\phi^r \mid \Delta$ is a concave function. Thus $\phi \mid \Delta$ is the limit of a nonincreasing sequence of concave functions and hence is concave.

Conversely, let $\phi \in C^0(\Gamma)$ and $\phi \mid \Delta$ be a concave function on every flat piece Δ of Γ. Let $\sigma(r) = \max[\phi^r(x) - \phi(x)]$ for $x \in \Gamma$, so that $\sigma(r) \geqq 0$ is a nonincreasing function of r for large r. Let

$$(0 \leqq) c = \lim \sigma(r) \text{ as } r \to \infty .$$

It will be shown that $c = 0$; i.e., that the first limit relation in (3.27) holds uniformly on Γ. Choose a sequence of points $x_j \in \Gamma$ such that, for large j, $\phi^j(x_j) \geqq \phi(x_j) + c$. In Proposition 3.3, let $r = j$, $x = x_j$ and, correspondingly $T = R_j$, $\mu = \mu_j$. Thus

$$x_j = T_j \int_\Gamma y d\mu_j,\ \phi^j(x_j) = j(1 - T_j) + T_j \int_\Gamma \phi(y) d\mu_j ,$$

where $T = T_j$ satisfies (3.18) with $r = j$. Consequently

$$\phi(x_j) + c \leqq \phi^j(x_j) \leqq T_j \int_\Gamma \phi(y) d\mu_j .$$

After a selection of a suitable subsequence (and a suitable renumbering), it can be supposed that $x_0 = \lim x_j$ exists and $\mu = \lim \mu_j$ exists weakly. In particular, $x_0 \in \Gamma$ and μ is a (Borel) probability measure on Γ. Letting $j \to \infty$ gives

$$x_0 = \int_\Gamma y d\mu,\ \phi(x_0) + c \leqq \int_\Gamma \phi(y) d\mu , \tag{3.28}$$

since $T_j \to 1$.

If x_0 is an extreme point of Γ, then the first part of (3.28) shows that the support of μ is the point x_0. The second part then gives $\phi(x_0) + c \leqq \phi(x_0)$. Thus $c = 0$.

If x_0 is not an extreme point of Γ, then x_0 is an interior point of the smallest flat piece Δ of Γ containing x_0. The set Δ is a closed convex subset of Γ and, by the first part of (3.28), the support of μ is contained in Δ. Thus the second part of (3.28) gives

$$\phi(x_0) + c \leqq \int_\Delta \phi(y) d\mu \leqq \phi(x_0) ,$$

where the last inequality is a consequence of the first part of (3.28) and the fact that $\phi \mid \Delta$ is a concave function. Again, we obtain $c = 0$. This completes the proof.

4. **$B(\Gamma)$-approximations.** As in the last section, let $x = 0 \in \Omega$ and $\phi(x) \in C^0(\Gamma)$. Let $r > 0$ be fixed (and sufficiently large) and, in terms of the operations $\phi \rightarrow \phi^r$, $\phi \rightarrow \phi_r$ define two functions $g, h \in C^0(R^n)$, which depend on ϕ and r:

$$g = (\phi^r)_r \text{ and } h = g^r = [(\phi^r)_r]^r . \tag{4.1}$$

In (4.1) and below, if $\psi \in C^0(R^n)$, the functions ψ^r, ψ_r mean $(\psi \mid \Gamma)^r$, $(\psi \mid \Gamma)_r$, respectively.

PROPOSITION 4.1. The functions (4.1) satisfy

$$\phi_r \leqq g \leqq h \leqq \phi^r \text{ on } \Gamma , \tag{4.2}$$

$$h = g^r \text{ and } g = h_r \text{ on } \Gamma . \tag{4.3}$$

Proof. By (3.14), $\phi^r \geqq \phi$ on Γ. By the analogue of (3.15), this implies that $g = (\phi^r)_r \geqq \phi_r$ on Γ. The analogue of the first part of (3.14) with ϕ replaced by ϕ_r gives $g = (\phi^r)_r \leqq \phi^r$ on Γ. Hence, by (3.15) and (3.16), $h = g^r \leqq (\phi^r)^r = \phi^r$ on Γ. Thus (4.2) is proved.

In order to prove (4.3), note that $h \geqq g$ on Γ implies that $h_r \geqq g_r \quad g$ on Γ. Also, from $h \leqq \phi^r$, it follows that $h_r \leqq (\phi^r)_r = g$ on Γ. Consequently $h_r = g$ on Γ. In view of (4.1), this completes the proof.

REMARK. If we could verify that $h = g$ on Γ, then we could complete the proof of (I) at this point; cf. Proposition 4.6 and § 5. It will remain undecided whether "$h = g$ on Γ" always holds, but it will be shown that this relation is valid if, for example, the extreme points of Γ are dense on Γ. This fact will be sufficient for the proof of (I).

In the remainder of this section, we make the following assumption:

(A) Let $g, h \in C^0(R^n)$ satisfy (4.3), hence

$$g \leqq h \text{ on } \Gamma . \tag{4.4}$$

For the sake of brevity, some statements and their proofs will be given only for h. It will be clear that analogous statements hold for g. These analogous statements will be utilized below.

If μ is a Borel measure on Γ, supp μ will denote its support and co (supp μ) will denote the closed convex hull of supp μ.

PROPOSITION 4.2. Let $x_0 \in \Gamma$. Then there exists a $T \geqq 1$ and a Borel probability measure μ on Γ such that

$$x_0 = T\int_\Gamma y d\mu,\ h(x_0) = r + T\int_\Gamma [h(y) - r] d\mu\ , \tag{4.5}$$

$$g = h \text{ on supp } \mu\ ; \tag{4.6}$$

and $h(x)$ is the restriction of a linear function of x on the set

$$\{x = ty,\ t \geqq 0,\ y \in \text{co}\,(\text{supp } \mu)\}\ . \tag{4.7}$$

Proof. By Proposition 3.3 and $h = g^r$, there exists a $T \geqq 1$ and a Borel probability measure μ satisfying

$$x_0 = T\int_\Gamma y d\mu,\ h(x_0) = r + T\int_\Gamma [g(y) - r] d\mu\ . \tag{4.8}$$

Hence $g \leqq h$ on Γ gives

$$h(x_0) \leqq r + T\int_\Gamma [h(y) - r] d\mu \tag{4.9}$$

and inequality holds unless (4.6) is valid. Using the fact that

$$h(ty) = r + t[h(y) - r] \text{ for } t \geqq 0\ ,$$

we can write (4.9) as

$$h(x_0/T) - r \leqq 2\int_0^1\int_\Gamma [h(ty) - r] dt d\mu\ , \tag{4.10}$$

while

$$x_0/T = 2\int_0^1\int_\Gamma ty dt d\mu \text{ and } 2\int_0^1\int_\Gamma t dt d\mu = 1\ . \tag{4.11}$$

Since $r - h(x)$ is a convex function of x, (4.11) implies that

$$2\int_0^1\int_\Gamma [h(ty) - r] dt d\mu \leqq h(x_0/T) - r\ ; \tag{4.12}$$

the inequality holds unless $h(x)$ is the restriction of a linear function of x on the set (4.7).

The sign of equality must hold in (4.10) and (4.12), hence in (4.9). Thus we conclude that (4.6) is valid and that $h(x)$ is a linear function on the set (4.7).

PROPOSITION 4.3. Let $n = 2$, so that Γ is a curve. Then $h(x) =$

$g(x)$ on Γ. Furthermore, if $x_*, x^* \in \Gamma$, l is the line segment $[x_* x^*]$, $x = 0 \notin l$, and $h(x)$ is a linear function of x on the sector $S = \{x = ty, t \geqq 0, y \in l\}$, then $l \subset \Gamma$.

Proof. Suppose, if possible, that there is a point $x_0 \in \Gamma$ where $h > g$. Then, in Proposition 4.2, supp $\mu \neq \{x_0\}$. Thus, by Proposition 4.2 and its analogue for g, there is an arc on Γ, say, with endpoints x_*, x^*, containing x_0 in its interior, $g = h$ at $x = x_*, x^*$, and g, h are linear functions of x on the sector S.

Clearly, by linearity, $g = h$ on l, since $g = h$ at $x = x_*, x^*$. As g, h are linear on half-lines emanating from $x = 0$, it follows that $l \subset \Gamma$. For otherwise, there are points of the arc from x_* to x^* where $g(x) > h(x)$. Since this impossible $l \subset \Gamma$ and $g = h$ at $x = x_0 \in l$.

PROPOSITION 4.4. Let $n \geqq 2$ be arbitrary; $x_0, x_1 \in \Gamma$, and l the line segment $[x_0 x_1]$. Suppose that $x = 0 \notin l$, $g = h$ at $x = x_0$ and $x = x_1$, and $h(x)$ is a linear function of x on the plane sector $S = \{x = ty, t \geqq 0, y \in l\}$. Then $l \in \Gamma$.

Proof. Let π_2 be the 2-dimensional plane in x-space containing the segment l and the point $x = 0$. In the course of this proof, only points $x \in \pi_2$ are considered.

Starting with the function $\psi = h \mid \Gamma_0 (= \Gamma \cap \pi_2)$, apply the procedure at the beginning of this section to obtain functions

$$g_{(0)} = (\psi^{(r)})_{(r)} \text{ and } h_{(0)} = g_{(0)}^{(r)} \text{ on } \Gamma_0 = \Gamma \cap \pi_2 \, .$$

The superscripts and subscripts (r) indicate the operations $\psi \to \psi^r$ and $\psi \to \psi_r$, except that only points $x_i \in \pi_2$ are involved in the analogues of (3.7), (3.13).

It is clear that $\psi^{(r)} = h = h^r$ on Γ_0 and so, $h \geqq g_{(0)} \geqq g$ on Γ_0, but $g_{(0)} = g = h$ at $x = x_0, x_1$. The definition of the operation $\phi \to \phi^r$ implies that $h_{(0)} = g_{(0)}^{(r)} = h$ on S, $h_{(0)} = g = h$ at $x = x_0, x_1$. Consequently, $h_{(0)}$ is the linear function h on the sector S. By the last proposition, $h_{(0)} = g_{(0)}$ on Γ_0 and $l \subset \Gamma_0 \subset \Gamma$. This completes the proof.

COROLLARY. *In Proposition* 4.2,

$$\text{co}\,(\text{supp}\,\mu) \subset \Gamma \, . \tag{4.13}$$

In other words, either supp $\mu = \{x_0\}$ or supp μ is contained in a flat piece of Γ.

PROPOSITION 4.5. Let x_0 be an extreme point of Γ. Then $h(x_0) = g(x_0)$.

Proof. If the assertion is false, then, in the last Corollary, $\mathrm{co}\,(\mathrm{supp}\,\mu)$ is in a flat piece of Γ. Thus, in Proposition 4.2, $T = 1$ and $x_0 \in \mathrm{co}\,(\mathrm{supp}\,\mu)$, but $x_0 \notin \mathrm{supp}\,\mu$. This contradicts the assumption that x_0 is an extreme point of Γ.

COROLLARY. *Suppose that the extreme points of Γ are dense on Γ. Then $h(x) \equiv g(x)$ on Γ. In particular, $h \mid \Gamma = g \mid \Gamma \in B(\Gamma)$.*

The last assertion follows from Proposition 3.1.

PROPOSITION 4.6. Assume that the extreme points of Γ are dense on Γ. Then $\bar{B}(\Gamma) \supset \Lambda(\Gamma)$.

Proof. Let $\phi \in \Lambda(\Gamma)$ and $\varepsilon > 0$. By Propositions 3.4 and 3.5, we have $\phi - \varepsilon < \phi_r \leqq \phi \leqq \phi^r < \phi + \varepsilon$ on Γ for large r. Also $\phi_r \leqq g = h \leqq \phi^r$ on Γ, by Propositions 4.1 and 4.5. Since $h \mid \Gamma \in B(\Gamma)$, the proof is complete.

5. **Proof of** (I): $\bar{B}(\Gamma) = \Lambda(\Gamma)$. In order to see that $\bar{B}(\Gamma) \subset \Lambda(\Gamma)$, let Λ be a maximal flat piece of Γ. Then Λ is a convex set, say, of dimension $k, 0 < k < n$. Let x_0 be an interior point of Λ and let $\phi \in B(\Gamma)$. Let $\pi^{\pm}(x)$ be linear functions of x satisfying (1.0). Suppose that ϕ is not a linear function, then $(a^+ - a^-)\cdot(x - x_0) = 0$ defines a hyperplane in R^n supporting Ω at $x = x_0$. This hyperplane contains Λ and, therefore, $\phi(x) = a^{\pm}\cdot(x - x_0) + \phi(x_0)$ for $x \in \Lambda$. Thus $\phi \in B(\Gamma) \Rightarrow \phi \in \Lambda(\Gamma)$. Hence $B(\Gamma) \subset \Lambda(\Gamma)$ and, therefore, $\bar{B}(\Gamma) \subset \Lambda(\Gamma)$.

In order to prove the opposite inclusion, let $\phi(x) \in \Lambda(\Gamma)$. In the R^{n+1} space with coordinates $(x, x^{n+1}) = (x^1, \cdots, x^n, x^{n+1})$, let Ω_0 be an open, bounded convex set such that its boundary $\Gamma_0 = \partial\Omega_0$ has the properties that $\Gamma_0 \cap \{x^{n+1} = 0\} = \Gamma$ and every point $(x, x^{n+1}) \in \Gamma_0$ with $x^{n+1} \neq 0$ is an extreme point of Γ_0. Extend $\phi(x) = \phi(x, 0)$ to a continuous function $\phi_0(x, x^{n+1})$ on Γ_0. Then $\phi_0 \in \Lambda(\Gamma_0)$ since the only flat pieces of Γ_0 are contained in Γ.

Since the set of extreme points of Γ_0 contains $\Gamma_0 - \Gamma$, they are dense on Γ_0. Hence, by Proposition 4.6, there exists, for every $\varepsilon > 0$, a function $h_0 \in B(\Gamma_0)$ satisfying $|\phi_0 - h_0| < \varepsilon$ on Γ_0. Thus $h = h_0 \mid \Gamma$ satisfies $h \in B(\Gamma)$ and $|\phi - h| < \varepsilon$ on Γ. This completes the proof.

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