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RELATIONS AMONG CONTINUOUS AND VARIOUS NON-CONTINUOUS FUNCTIONS

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In this paper a number of conditions on a function from one topological space to another are considered. Among these conditions are those of a function or its inverse preserving closedness, openness, or compactness of sets. Other conditions are having a closed graph and a concept generalizing continuity, subcontinuity, which we introduce.

Some interesting results which are uncovered are the following: (1) A function which is closed with closed point inverses and a regular space for its domain has a closed graph. (2) If a function maps into a Hausdorff space, continuity of the function is equivalent to the requirement that the function be subcontinuous and have a closed graph. (3) The usual net characterization of continuity for a function with values in a Hausdorff space is still valid if it is required only that the image of a convergent net be convergent (not necessarily to the "right" value).

Also several theorems of E. Halfar [2], [3] are extended including some sufficient conditions for continuity.

All spaces X and Y are topological spaces and no function is assumed to be continuous unless explicitly stated to be so. For any concepts which we do not define or elaborate upon, the reader is referred to Kelley's book [4]. We will denote nets by a symbol such as x_a, letting context distinguish between a net and a point of the range of the net and suppressing explicit mention of the directed set. A subnet of a net x_a will be denoted by x_{Nb} where b is a member of the domain of x_{Nb} and N is the appropriate function from the domain of x_{Nb} to the domain of x_a.

1. Functions with closed graphs.

A function $f: X \to Y$ *has a closed graph* (*relative to* $X \times Y$) if and only if $\{(x, fx) : x \in X\}$ is closed in the product topology of $X \times Y$. Using a characterization of closed sets in terms of nets (see Kelley [4, Chapter 2)] a function $f: X \to Y$ has a closed graph if and only if (x_a, fx_a) converges to (p, q) in $X \times Y$ implies that $q = fp$. The well-known example of the differentiation operator from $C^1[0, 1]$ to $C[0, 1]$ shows that a function with a closed graph need not be continuous.

Let E_a be a net of sets in a topological space, X. A point p in X belongs to lim sup E_a (lim inf E_a) if and only if E_a frequently (resp., eventually) intersects each neighborhood of p. This generalization of the lim sup and lim inf of sets from sequences to nets has been studied by Mrówka [8, p. 237].

LEMMA 1.1. *If* $f: X \to Y$ *is a function,* y_a *is a net in* Y, *and* $p \in \limsup f^{-1}[y_a]$, *then there is a net* x_{Nb} *in* X *such that* $x_{Nb} \to p$ (x_{Nb} *converges to* p) *and* fx_{Nb} *is a subnet of* y_a.

Proof. Assume y_a is a net in Y and $p \in \limsup f^{-1}[y_a]$. Then for each index a and each neighborhood of p, U, there is an index $N(a, U) \geqq a$ such that $f^{-1}[y_{N(a,U)}] \cdot U \neq \varphi$. Now direct the neighborhoods U downward by inclusion, give the pairs (a, U) the product direction, and choose $x_{N(a,U)} \in f^{-1}[y_{N(a,U)}] \cdot U$, thus obtaining a net $x_{N(a,U)}$. Finally note that $fx_{N(a,U)} = y_{N(a,U)}$ is a subnet of y_a and $x_{N(a,U)} \to p$.

The following characterization of a function with a closed graph appears in Kuratowski [5, Defn., p. 32] in a considerably restricted form.

THEOREM 1.2. *If* $f: X \to Y$ *is a function, the following conditions are equivalent*:

(a) *The function* f *has a closed graph.*

(b) *If* $y_a \to q$ *in* Y, *then* $\limsup f^{-1}[y_a] \subset f^{-1}[q]$.

(c) *If* $y_a \to q$ *in* Y, *then* $\liminf f^{-1}[y_a] \subset f^{-1}[q]$.

Proof. Assume (a) holds. Let $y_a \to q$ in Y and $q \in \limsup f^{-1}[y_a]$. By Lemma 1.1 there is a net x_{Nb} in X such that $x_{Nb} \to p$ and fx_{Nb} is a subnet of y_a. Thus we have $(x_{Nb}, fx_{Nb}) \to (p, q)$. Since f has a closed graph $q = fp$ or $p \in f^{-1}[q]$. Consequently (b) holds.

If (b) holds then, since $\liminf f^{-1}[y_a]$ is a subset of $\limsup f^{-1}[y_a]$, (c) evidently holds.

Assume then that (c) holds. Suppose $(x_a, fx_a) \to (p, q)$. Then $y_a = fx_a \to q$ and $p \in \liminf f^{-1}[y_a]$. Therefore $p \in f^{-1}[q]$ or $q = fp$ and (a) holds.

2. Subcontinuous and inversely subcontinuous functions.

The function $f: X \to Y$ is said to be *subcontinuous* if and only if $x_a \to p$ in X implies there is a subnet of fx_a, fx_{Nb}, which converges to some point q in Y. Similarly a function $f: X \to Y$ is said to be *inversely subcontinuous* if and only if $fx_a \to q$ in Y implies there is a subnet of x_a, x_{Nb} which converges to some point p in X, that is, if and only if $y_a \to q$ in Y and $x_a \in f^{-1}[y_a]$ implies that a subnet of x_a, x_{Nb}, converges to some point p in X.

Our concept of a subcontinuous (inversely subcontinuous) function is a generalization of a function whose range (resp., domain) is compact. In addition, a subcontinuous function is a generalization of a continuous function whence its name.

More generally it is clear that if $f: X \to Y$ is function and each point p in X (in Y) has a neighborhood U such that $f[U]$ (resp., $f^{-1}[U]$) is contained in a compact subset of Y(resp., of X) then f is subcontinuous (resp. inversely subcontinuous).

There are a couple more analogous pairs of facts about subcontinuous and inversely subcontinuous functions with analogous proofs of these facts. This seems to suggest that each pair of proofs could have been integrated into a single proof concerning a multiple-valued subcontinuous function. However such an approach seemed somewhat astray from the present study.

The following theorem says that a subcontinuous (inversely subcontinuous) function $f: X \to Y$ where Y(resp. X) is completely regular is very nearly such that f (resp., f^{-1}) preserves compactness. Unfortunately nearly is often not good enough. However subcontinuous (inversely subcontinuous) functions have advantages over those such that the function (resp., the inverse) preserves compactness as will be seen later.

THEOREM 2.1. *Let $f: X \to Y$ be a function. If f is subcontinuous (inversely subcontinous) and Y (resp., X) is completely regular then for each compact subset of X (resp., Y), K, $f[K]^-$ (resp. $f^{-1}[K]^-$) is compact. ($^-$denotes closure).*

Proof. We prove only the assertions for f inversely subcontinuous, the proof for f subcontinuous being entirely analogous.

Let K be a compact subset of Y. Let $\{z_a, a \in A\}$ be a net in $f^{-1}[K]^-$. Let B be a uniformity for X. Direct B downward by inclusion and let $A \times B$ have the product order. For each $(a, b) \in A \times B$ choose $x_{(a,b)}$ in $b[z_a] \cdot f^{-1}[K]$. Since K is compact, a subnet of $fx_{(a,b)}$ converges and thus a subnet $x_{(Nc,Mc)}$ of $x_{(a,b)}$ converges to some p in X. Clearly $p \in f^{-1}[K]^-$.

Now consider the net z_{Nc} which is a subnet of z_a. We proceed to show $z_{Nc} \to p$. Let $b \in B$. There is a symmetric $b_1 \in B$ such that $b_1 \circ b_1 \subset b$. By choice of the $x_{(a,b)}$, it is clear that $(z_{Nc}, x_{(Nc,Mc)})$ is eventually in b_1. Since $x_{(Nc,Mc)} \to p$, $(x_{(Nc,Mc)}, p)$ is eventually in b_1. Thus (z_{Nc}, p) is eventually in B and $z_{Nc} \to p$. Consequently $f^{-1}[K]^-$ is compact.

3. Functions and inverses which preserve closedness and compactness.

Let $f: X \to Y$ be a function. If $f(f^{-1})$ takes compact sets of X (resp., Y) onto compact sets of Y (resp., X) then f is said to be *compact-preserving* (resp., *compact*). If $f(f^{-1})$ takes closed sets of X (resp., Y) onto closed sets of Y (resp., X) then f is said to be *closed* (resp., *continuous*).

Pursuing further the similarities noted in the previous section, we characterize compact and compact preserving functions in terms of inverse subcontinuity and subcontinuity, respectively.

THEOREM 3.1. *Let* $f: X \to Y$ *be a function.*

(a) *The function* f *is compact if and only if* $f \mid f^{-1}[K]: f^{-1}[K] \to K$ *is inversely subcontinuous for each compact set* $K \subset Y$.

(b) *The function* f *is compact-preserving if and only if* $f \mid K: K \to f[K]$ *is subcontinuous for each compact set* $K \subset X$.

Proof. The proof of (b) is entirely analogous to that of (a) and so only (a) is proved.

Assume f is compact. Let K be a compact subset of Y and fx_a a net in K which converges to q in K. Then x_a, being in the compact set $f^{-1}[K]$, has a subnet x_{Nb} converging to some p in $f^{-1}[K]$. Thus $f \mid f^{-1}[K]: f^{-1}[K] \to K$ is inversely subcontinuous.

Now assume the subcontinuity condition. Let K be a compact subset of Y and let x_a be a net in $f^{-1}[K]$. Then fx_a has a subnet converging to a point in K and thus using the condition, x_a has a subnet converging to a point in $f^{-1}[K]$. Therefore $f^{-1}[K]$ is compact.

The following theorem giving sufficient conditions that a function be compact or compact preserving is deduced immediately from Theorem 3.1.

THEOREM 3.2. *Let* $f: X \to Y$ *be a function. If* f *is subcontinuous (inversely subcontinuous) and* $f[K]$ *(resp.,* $f^{-1}[K]$*) is closed for each compact subset,* K*, of* X *(resp.,* Y*) then* f *is compact preserving (resp., compact).*

From Theorem 3.6 it will then follow that f has a closed graph and is subcontinuous (inversely subcontinuous) implies f is compact-preserving (resp., compact).

COROLLARY 3.3. *Let* $f: X \to Y$ be a function. *If* $X(Y)$ *is Hausdorff and* f *is both closed and subcontinuous (resp., continuous and inversely subcontinuous) then* f *is compact-preserving (resp., compact).*

We turn now to gathering some more facts about functions with closed graphs and their relation to other functions. In particular, the following two theorems show that the closed graph property complements the two subcontinuities in interesting ways.

THEOREM 3.4. *Let* $f: X \to Y$ *be a function. A sufficient condition that* f *be continuous is that* f *have a closed graph and be subcontinuous. If* Y *is Hausdorff, the condition is also necessary.*

Proof. (*Sufficiency*) Let x_a be a net in X which converges to some point p. Suppose fx_a does not converge to fp. Then fx_a has a subnet, fx_{Nb}, no subnet of which converges to fp. However, by subcontinuity, some subnet of fx_{Nb}, f_{NMc}, converges to some point q. Thus we have $(x_{MNc}, fx_{NMc}) \longrightarrow (p, q)$. But by the closed graph property of f, $q = fp$ and we have a contradiction.

(*Necessity*) Assuming f is continuous then evidently f is subcontinuous. If $(x_a, fx_a) \longrightarrow (p, q)$ then $x_a \longrightarrow p$ and thus $fx_a \longrightarrow fp$. Assuming Y is Hausdorff we conclude that $q = fp$ and f accordingly has a closed graph.

THEOREM 3.5. *If the function $f: X \longrightarrow Y$ has a closed graph and is inversely subcontinuous, then f is closed.*

Proof. Let C be a closed subset of X. Suppose $f[C]$ is not closed. Then there is a q in $Y - f[C]$ and a net x_a in C such that $fx_a \longrightarrow q$. Thus by inverse subcontinuity there is a subnet of x_a, x_{Nb}, which converges to some p in X. Since C is closed $p \in C$. Consequently we have $(x_{Nb}, fx_{Nb}) \longrightarrow (p, q)$, but $q \neq fp$ since $q \notin f[C]$. This contradicts the closedness of the graph of f.

Theorem 3.4 tells us that if $f: X \longrightarrow Y$ is continuous with Y Hausdorff then f has a closed graph. The Hausdorff requirement cannot be dropped in general for if $i: X \longrightarrow X$ is the identity on X, the graph of i is closed in $X \times X$ if and only if X is Hausdorff.

We can also see from Theorem 3.4 the dependency of the closed-graphness of a function $f: X \longrightarrow Y$ upon the space Y in which $f[X]$ is imbedded.

Let $f: X \longrightarrow Y$ be a function which has a closed graph and is not continuous. Let Y^* be a compactification of Y. Then $f: X \longrightarrow Y^*$ is subcontinuous (showing also the dependency of subcontinuity on the space in which the range is embedded). Thus if $f: X \longrightarrow Y^*$ had a closed graph then it would be continuous. But continuity does not depend upon the embedding space for the range so that $f: X \longrightarrow Y$ would be continuous contradicting our assumption.

Assuming the range is embedded in a Hausdorff space, the next theorem says, roughly, that a function with a closed graph handles compact sets somewhat less successfully than a continuous function but its inverse does just as well as the inverse of a continuous functions in the treatment of compact sets. The theorem follows from an exercise in Kelley [4, Ex. A, p. 203] but we also include a proof.

THEOREM 3.6. *Let the function $f: X \longrightarrow Y$ have a closed graph. If K is a compact subset of $X(Y)$ then $f[K]$ (resp., $f^{-1}[K]$) is a closed* *subset of Y (resp., X).*

Proof. We prove only the second case, the proof of the first being entirely analogous.

Let K be a compact subset of Y. Suppose $f^{-1}[K]$ is not closed. Then there is a p in $X - f^{-1}[K]$ and a net x_a in $f^{-1}[K]$ such that $x_a \rightarrow p$. Evidently fx_a has a subnet fx_{Nb} which converges to some q in K. Thus we have $(x_{Nb}, fx_{Nb}) \rightarrow (p, q)$ so that $p \in f^{-1}[q] \subset f^{-1}[K]$. But this contradicts the choice of p.

Having already discovered in Theorem 3.4 conditions under which a continuous function has a closed graph, we proceed to find such conditions for a closed function. The characteristic function of the interval $(0, 1]$ mapping E_1 into $\{0, 1\}$ shows us that a closed function does not always have a closed graph. One should note that this function does not have closed point inverses.

Let us call a function $f: X \rightarrow Y$ *locally closed* if for every neighborhood, U, of each point p in X, there is a neighborhood of p, V, such that $V \subset U$ and $f[V]$ is closed in Y (Whyburn [12, p. 198] has used the same term differently). It is not clear that a closed function is always locally closed but if the domain of a closed function is regular, then the function is locally closed. Also if a function $f: X \rightarrow Y$ is such that X is regular and locally compact and f takes compact sets onto closed sets, then f is locally closed.

A locally closed function need not be closed as the following example shows. Let X be the reals with the discrete topology, Y the reals with the usual topology, and $f: X \rightarrow Y$ the identity function. Then f is locally closed and, in fact, continuous but certainly not closed.

THEOREM 3.7. *If $f: X \rightarrow Y$ is a locally closed function then $y_a \rightarrow q$ in Y implies* $\limsup f^{-1}[y_a] \subset f^{-1}[q]^-$.

Proof. Let $y_a \rightarrow q$ in Y and $p \in \limsup f^{-1}[y_a]$. Suppose $p \notin f^{-1}[q]^-$. By Lemma 1.1 there is a net x_{Nb} in X such that $x_{Nb} \rightarrow p$ and fx_{Nb} is a subnet of y_a. Now $X - f^{-1}[q]^-$ is a neighborhood of p and thus there is a neighborhood of p, U, such that $U \subset X - f^{-1}[q]^-$ and $f[U]$ is closed in Y. But x_{Nb} is eventually in U so that fx_{Nb} is eventually in $f[U]$. This means fx_{Nb} is eventually in the complement of a neighborhood of q, namely $Y - f[U]$. We therefore have a contradiction to the fact that fx_{Nb} must converge to q, being a subnet of y_a.

COROLLARY 3.8. *If $f: X \rightarrow Y$ is a locally closed function and has closed point inverses then f has a closed graph.*

Proof. The statement follows immediatelly from Theorem 1.2.

COROLLARY 3.9. *If the function $f: X \to Y$ is closed with closed point inverses and X is regular then f has a closed graph.*

COROLLARY 3.10. *If the function $f: X \to Y$ is closed and subcontinuous with closed point inverses and X is regular then f is continuous.*

Proof. We have this result directly from 3.9 and Theorem 3.4.

The last corollary generalizes a theorem of Halfar's [2, Th. 3] by replacing compactness of Y with subcontinuity of f. See also in this connection Rhoda Manning [6, Th. 1.5].

THEOREM 3.11. *If $f: X \to Y$ is a function where X is regular and locally compact, the following conditions are equivalent:*

(a) *f maps compact sets onto closed sets and has closed point inverses.*

(b) *f is locally closed and has closed point inverses.*

(c) *f has a closed graph.*

Proof. We have already commented that (a) implies (b). By Corollary 3.8, (b) implies (c). Thus it remains to show (c) implies (a).

Assuming f has a closed graph, Theorem 3.6 gives us that f maps compact sets onto closed sets. Furthermore since points are compact, the same theorem yields that f has closed point inverses.

The following theorem, which should be compared with Theorem 3.2, is essentially a special case of one to be found in Bergé [1, Th. 3, p. 116]. The theorem extends one of Halfar's [2, Th. 1] by removing the requirement that f be continuous. Also of interest in this connection are Michael [7, Lemma 5.18, p. 172] and Whyburn [11, Th. 3 & Cor. 2, pp. 689-690].

THEOREM 3.12. *If $f: X \to Y$ is closed and has compact point inverses, then f is compact.*

4. Compact functions, compact-preserving functions, and k_i spaces. It is intuitively clear that a mapping $f: X \to Y$ which is compact or compact preserving will have no particular tendency to treat other topological properties nicely unless the topologies of X or Y or both are to a considerable extent dictated by the compact sets. In this section we define some topological spaces which are, in a sense, determined by their compact sets, and prove a couple of theorems concerning these spaces and compact preserving functions.

Let X be a topological space and $p \in X$. X is said to have *property* k_1 *at* p and only if for each infinite subset A having p as an accumulation point, there is a compact subset of $A + p, B$, such that $p \in B$ and p is an accumulation point of B. X is a k_1 *space* if it has property k_1 at each of its points.

X has *property* k_2 *at* p if and only if for each set A having p as an accumulation point, there is a subset of A, B, and a compact set $K \supset B + p$ such that p is an accumulation point of B. We call X a k_2 *space* if it has property k_2 at each of its points.

X is a k_3 *space* if and only if U is an open set in X precisely whenever $U \cdot K$ is open in K for each compact set K in X.

Halfar defines property k_1 at a point in one of his papers [3, Defn. 2]. Property k_2 at a point is a slight variation of a definition I believe is due to S. B. Myers. A definition differing slightly from that of a k_3 space is discussed by Kelley in his book [4, p. 230]. If X is Hausdorff, the k_2 and k_3 definitions agree with those of Myers and Kelley respectively.

It is immediate that X is k_1 at p implies X is k_2 at p. Also it is not difficult to show that a k_2 space is always a k_3 space, but I do not know whether k_2 space and k_3 space are equivalent concepts. It is easy to see that a locally compact or first countable space is a k_2 space. Finally the following example shows that X being k_2 at a point p does not necessitate X being k_1 at p.

EXAMPLE 4.1. This example provides a k_2 space which does not have property k_1 at any point.

Let F be the space of all functions mapping $[0, 1]$ into $[0, 1]$ with the topology of point-wise convergence. Let $\mathscr{A}$ be the collection of all finite subsets of $[0, 1]$ and ω the set of positive integers. For each A in $\mathscr{A}$ and n in ω let f_{nA} be the function in F defined by $f_{nA}x = 1/n$ for x in A and $f_{nA}x = 1$ otherwise. Letting $\mathscr{A}$ be directed upward by inclusion, ω have the usual order, and $\omega \times \mathscr{A}$ have the product order, then $\{f_{nA}, (n, A) \in \omega \times \mathscr{A}\}$ is a net in F.

It is easy to see that f_{nA} converges to the zero function, 0^*, and thus that 0^* is an accumulation point of $\{f_{nA} : (n, A) \in \omega \times \mathscr{A}\}$. We proceed to show F lacks property k_1 at 0^* and the proof that F has property k_1 at no point will then be clear.

Let $p = \{f_{nA} : (n, A) \in \omega \times \mathscr{A}\}$ and let B be any subset of P which has 0^* as an accumulation point. Then $\{n, A) : f_{nA} \in B\}$ must be cofinal in $\omega \times \mathscr{A}$ for if for any (n_0, A_0) there is no $(n_1, A_1) \geqq (n_0, A_0)$ such that $f_{n_1 A_1} \in B$ then there is no member of B in the neighborhood of 0^* given by $\{f \in F : |fx| < 1/n_0 \text{ for } x \in A_0\}$.

Now let $f_{n_0 A_0} \in B$. For every k in ω choose $f_{n_k A_k} \in B$ such that $n_k > n_{k-1}$ and $A_k \supset A_{k-1}$. Consider $Q = \Sigma\{A_k : k \in \omega\}$. Let $f \in F$ be defined by $fx = 0$ for x in Q, $fx = 1$ otherwise. The sequence $f_{n_k A_k}$ converges to f and $f \neq 0^*, 0, f \notin P$ since Q is countable. Thus $B + 0^*$ is not closed and since F is Hausdorff it is consequently not compact.

We conclude F does not have property k_1 at any point, but on the other hand since by the Tychonoff theorem F is compact, it is clearly a k_2 space.

The next theorem extends one of Halfar [2, Th. 2] by requiring that f have only a closed graph instead of being continuous and by requiring that Y only be a k_3 space instead of locally compact Hausdorff. The theorem should be compared with a similar one in Whyburn [11, p. 690].

THEOREM 4.2. *Let $f: X \to Y$ be a compact function and Y a k_3 space. A sufficient condition that f be closed is that f have a closed graph. If X is regular Hausdorff, the condition is also necessary.*

Proof. Assume f has a closed graph. Let C be a closed subset of X. Since Y is a k_3 space, to show that $f[C]$ is closed we have only to show that the intersection of $f[C]$ with each compact set Y, K, is closed in K.

Let K be a compact subset of Y. Then $f^{-1}[K]$ is compact, and it follows that $C \cdot f^{-1}[K]$ is compact. By Theorem 3.6 we have that $f[C \cdot f^{-1}[K]]$ is closed. Finally since

$$f[C \cdot f^{-1}[K]] = f[C] \cdot K$$

$f[C] \cdot K$ is closed in Y and thus in K.

Now assume that X is regular Hausdorff and f is closed. Then f being compact, point inverses are closed. Thus by Corollary 3.9, f has a closed graph.

Comparing Theorem 3.5 and the sufficiency part of the one immediately preceding, we see that in the former theorem "f is inversely subcontinuous" replaces "f is compact" and no requirements are put on the space Y. The example which follows shows that the requirement that Y be a k_3 space cannot be dropped in Theorem 4.3. These observations illustrate that in some instances f being inversely subcontinuous is effectively a stronger requirement than f being compact.

EXAMPLE 4.3. We will display a function which is compact and has a closed graph but which is not closed.

Let X be an uncountable set. Let $\mathscr{U}_1$ be the topology for X consisting of all compliments of countable sets (plus the empty set). Let $\mathscr{U}_2$ be the discrete topology on X. It is known that the only compact subsets of either $(X, \mathscr{U}_1)$ or $(X, \mathscr{U}_2)$ are the finite sets.

Consider the identity map $i : (X, \mathscr{U}_2) \to (X, \mathscr{U}_1)$. The function i has a closed graph as we now show. Let $(x_a, ix_a) \to (p, q)$. Then x_a must eventually be the constant p in $(X, \mathscr{U}_1)$ and so also must $ix_a = x_a$. If ix_a converged to $q \neq q$ then we would have a contradiction to the fact the points are closed in $(X, \mathscr{U}_1)$.

The inverse of i carries compact (finite) subsets of $(X, \mathscr{U}_1)$ onto compact (finite) subsets of $(X, \mathscr{U}_2)$ and i is consequently compact. Finally since $\mathscr{U}_2$ is a strictly large topology than $\mathscr{U}_1$ i is not closed.

THEOREM 4.4. *Let $f : X \to Y$ be a function where X and Y are Hausdorff and X has property k_2 at a point p. If f is compact preserving and has closed point inverses, then f is continuous at p.*

Proof. Suppose f is not continuous at p. Then there is a neighborhood V^* of fp such that for each neighborhood of p, U, there exists a point x_U in U with the property that $fx_U \notin V^*$. The collection $A = \{x_U : U \text{ is a neighborhood of } p\}$ has p as our accumulation point. Thus A has a subset B and a compact set $K \supset B + p$ such that p is an accumulation point of B.

Consider the function $f \mid K : K \to Y$. The function $f \mid K$ is (strongly) closed since Y is Hausdorff. As

$$(f \mid K)^{-1}[p] = f^{-1}[p] \cdot K$$

$f \mid K$ has closed point inverses. The range of $f \mid K$ is compact and thus $f \mid K$ is subcontinuous. Finally, with the observation that K being compact Hausdorff is regular, we may conclude from Corollary 3.10 that $f \mid K$ is continuous.

However if we choose a net x_a in $B \subset K$ such that $x_a \to p$ then fx_a is never in the neighborhood V^* of fp. This contradiction proves the theorem.

The last theorem, 4.4, is a generalization of two theorems of Halfar [3, Ths. 2 and 5]. Halfar's Theorem 5 is the same as our Theorem 4.4 except that Halfar requires X to have property k_1 at p instead of k_2 at p. Our Example 4.1 shows then that Halfar's Theorem 5 is not as strong as our 4.4 and in particular does not apply to all locally compact spaces.

5. Another characterization of continuity. In this section we give a second characterization of continuity (The first occurred in Theorem 3.4). This characterization was discovered while pursuing the question "How much must the usual net characterization of continuity (see Kelley [4, Th. 1, *p.* 86]) be relaxed in order that something less than continuity be achieved?" One reasonable answer to this question is the subcontinuity concept defined in § 2. The principal theorem of this section seems to suggest this answer.

THEORM 5.1. *Let $f: X \to Y$ be a function where Y is Hausdorff. The following conditions are equivalent:*

(a) *The function f is continuous.*

(b) *If $x_a \to p$ in X, then there is a q in Y such that $fx_a \to q$.*

(c) *If $x_a \to p$ in X, then there is a subnet of x_a, x_{Nb}, such that $fx_{Nb} \to fp$.*

(d) *For each p in X there is a q in Y such that $x_a \to p$ implies there is a subnet of x_a, x_{Nb}, such that $fx_{Nb} \to q$.*

Note that (b), probably the most interesting equivalence, says that the usual net characterization of continuity is still valid even if it is not required that the image of a convergent net converge to the "right" value. In order to prove that (b) implies (a) we will use the following lemmas which we state separately since they may be of interest in other applications of nets.

LEMMA 5.2. *Let $(A, >_a)$ and $(B, >_b)$ be disjoint directed sets which are isomorphic (i.e., there is a one-to-one function h from A onto B such that $\alpha_1 >_a \alpha_2$ if and only if $h\alpha_1 >_b h\alpha_2$). Then there is a directed set $(C, >_c)$ such that A and B are cofinal subsets of C and $C = A + B$.*

Proof. Let $C = A + B$ and define $>_c$ as follows: If $\gamma_1, \gamma_2 \in A$ then $\gamma_1 >_c \gamma_2$ if and only if $\gamma_1 >_a \gamma_2$ and similarly if $\gamma_1, \gamma_2 \in B$. If $\gamma_1 \in A$ and $\gamma_2 \in B$ then $\gamma_1 >_c \gamma_2$ if and only if $h\gamma_1 >_b \gamma_2$ and $\gamma_2 >_c \gamma_1$ if and only if $\gamma_2 >_b h\gamma_1$. In brief, we order C by leaving the order of A and B the same and identifying points in A with their images in B.

It is not difficult to show that $(C, >_c)$ is directed set, and it is clear that A and B are cofinal in C.

LEMMA 5.3. *Let X be a topological space and $\{x_a, a \in A\}$ and $\{z_b, b \in B\}$ nets in X with disjoint directed sets which are isomorphic. Then there is a net $\{w_c, c \in C\}$ in X such that x_a and z_b are subnets and $w_c \to p$ if and only if $x_a \to p$ and $z_b \to p$.*

Proof. Let C be the directed set constructed as in the previous lemma and define $w_c = x_c$ if $c \in A$, $w_c = z_c$ if $c \in B$. The assertions of the lemma are then clear.

Proof of Theorem 5.1. Since each of conditions (b), (c), and (d) are clearly implied by continuity, we have only to show that each of these conditions implies continuity.

Assume condition (b) holds. Let $x_a \to p$. Then $fx_a \to q$ for some q in Y. Suppose $q \neq fp$. Let z_b be a net which is constantly p and whose directed set is disjoint and isomorphic to that of x_a. Then by Lemma 5.3 there is a net w_c such that x_a and z_b are subnets and $w_c \to p$. However $fx_a \to q$ and $fz_b \to fp$. Since Y is Hausdorff, fw_c cannot converge and we have a contradiction.

Assume condition (d) holds. We will show condition (c) holds. Let $p \in X$ and q be the corresponding point assured by the condition. We wish to show that $q = fp$. But this is evident for if x_a is a net constantly p then $x_a \to p$ and thus $fx_a \to fp$. Since Y is Hausdorff $q = fp$.

Finally we show condition (c) implies continuity. Suppose condition (c) holds and f is not continuous. Then there is a p in X and an open neighborhood of fp, V^*, such that for each neighborhood of p, U, there exists a point x_U in U such that $fx_U \notin V^*$. Now the function x_U defined on the collection of neigborhoods of p directed downward by inclusion is a net converging to p. On the other hand it is clear that for no subnet of x_U, x_{Nb}, is it true that $fx_{Nb} \to fp$. This contradiction completes our proof.

The following corollary has as a corollary to it a theorem announced by Yu-Lee Lee to the effect that if f is a function from uniform space to another which preserves Cauchy nets, then f is continuous.

COROLLARY TO 5.1 (b) 5.4. *Let $f: X \to Y$ be a function where $(Y, \mathscr{V})$ is a Hausdorff uniform space. If f maps convergent nets onto Cauchy nets then f is continuous.*

Proof. Let $(h, Y^*, \mathscr{V}^*)$ be the Hausdorff completion of $(Y, \mathscr{V})$ where h is a uniform isomorphism of Y into Y^*. Note $h \circ f$ maps convergent nets onto Cauchy nets. Since Y^* is complete, it follows from 5.1(b) that $h \circ f$ is continuous. But then so is f since h is a homeomorphism.

6. Open functions and continuity of their inverses.

One feels intuitively that openness of a function should be related to continuity (in some sense) of the inverse function. In this section we find a sense and a setting in which this is indeed the case for the set-valued inverse function.

By an *open function*, we will mean a function $f: X \to Y$ such that U is open in X implies $f[U]$ is open in Y.

THEOREM 6.2. *If $f: X \to Y$ is a function, the following conditions are equivalent*:

(a) *f is open.*
(b) *$y_a \to q$ implies $f^{-1}[q] \subset \liminf f^{-1}[y_a]$.*
(c) *$y_a \to q$ implies $f^{-1}[q] \subset \limsup f^{-1}[y_a]$.*
(See § 1, second paragraph for definitions)

Proof. Assume (a) holds. Let $y_a \to q$ and $p \in f^{-1}[q]$. Suppose $p \notin \liminf f^{-1}[y_a]$. Then there is an open set, U, containing p such that frequently $f^{-1}[y_a] \cdot U = \varphi$. Thus y_a is frequently outside $f[U]$. But this is absurd since $f[U]$ is a neighborhood of q. Thus (b) holds.

Clearly if (b) holds then (c) holds. Hence assume (c) holds. Suppose f is not open. Then there is an open set, U, in X and a net y_a in $Y - f[U]$ such that $y_a \to q$ for some q in $f[U]$. Thus $f^{-1}[q] \subset \lim \sup f^{-1}[y_a]$. Let $p \in f^{-1}[q] \cdot U$. Then U is a neighborhood of p and thus $f^{-1}[y_a] \cdot U$ is frequently nonempty. But this means y_a is frequently in $f[U]$ in contradiction to our choice. Therefore (a) holds.

If E_a is a net of sets in a topological space X such that lim inf $E_a = \limsup E_a = E$, then we say that the *limit of E_a exists* and we write $\lim E_a = E$. With this definition the following theorem follows from the previous theorem and Theorem 1.2.

THEOREM 6.3. *The function $f : X \to Y$ is open and has a closed graph if and only if $y_a \to q$ in Y implies* $\lim f^{-1}[y_a] = f^{-1}[q]$.

Proof. By the theorems cited above, f is open and has a closed graph if and only if $y_a \to q$ in Y implies

$$f^{-1}[q] \subset \liminf f^{-1}[y_a] \subset \limsup f^{-1}[y_a] \subset f^{-1}[q] .$$

Thus the theorem holds.

A theorem similar to the one immediately preceding is given in Whyburn [10, Th. 4.32, p. 130] for metric spaces.

Now let X be a locally bicompact space (i. e. one which has a basis of open sets with compact closures), and 2^X be the collection of all nonempty closed subsets of X. Consider all sets of the form

$$\{A \in 2^X : A \cdot U_i = \varphi \text{ and } A \cdot \mathrm{Cl}\,(V_j) = \varphi \text{ for } i = 1, \cdots, n \text{ and } j = 1, \cdots, m\}$$

where the U_i and V_j are open and have a compact closure in X.

These sets from a basis for a topology for 2^X which Mrówka calls the $l\,bc$ topology. Mrówka proves in his paper [9, Th. 4] that for a locally bicompact space X, convergence in 2^X relative to the $l\,bc$ topology is the same as the convergence of sets previously described (i. e. when lim inf = lim sup). Since by Theorem 3.6 if $f : X \to Y$ has a closed graph then $f^{-1}[y]$ is closed for each y, the following theorem follows from the preceding comments and Theorem 6.3.

6.4 THEOREM. *Let $f: X \to Y$ be a function where X is locally bicompact. Let F be the function on Y defined by $Fy \equiv f^{-1}[y]$. Let 2^X have the 1bc topology. If f is open and has a closed graph then $f: Y \to 2^X$ is continuous. Conversely, if f has closed point inverses and $F: Y \to 2^X$ is continuous then f is open and has a closed graph.*

THEOREM 6.5. *Let f be an open, continuous function of X onto Y where X is locally bicompact and Y is Hausdorff. Let*

$$\mathscr{D} = \{f^{-1}[y]: y \in Y\}$$

and let $\mathscr{D}$ have the relativized 1bc topology. Then Y and $\mathscr{D}$ are topologically equivalent.

Proof. By Theorem 3.4, since f is continuous and Y is Hausdorff, f has a closed graph. Thus by the previous theorem $F: Y \to D$ defined by $Fy = f^{-1}[y]$ is continuous. Thus it remains only to show that F^{-1} is continuous.

Let $f^{-1}[y_a]$ converge to $f^{-1}[q]$ in $\mathscr{D}$. Then

$$\limsup f^{-1}[y_a] = f^{-1}[q] .$$

By Lemma 1.1, if $p \in \limsup f^{-1}[y_a]$ there is a net x_{Nb} in X such that $x_{Nb} \to p$ in X and fx_{Nb} is a subnet of y_a. Now f is continuous and thus fx_{Nb} converges to $fp = q$. But

$$F^{-1}(f^{-1}[fx_{Nb}]) = fx_{Nb} .$$

Since $f^{-1}[fx_{Nb}]$ is a subnet of $f^{-1}[y_a]$, it follows by Theorem 5.1(c) that F^{-1} is continuous.

Theorems somewhat similar to the previous two theorems may be found in a paper of E. Michael [8] (see in particular Theorem 5.10.2).

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