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CHARACTERISTIC POLYNOMIALS OF SYMMETRIC MATRICES

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Let F be a field and p an F-polynomial. We say that p is F-real if and only if every real closure of F contains the splitting field of p over F. Our main purpose is to prove

THEOREM 1. Let F be an algebraic number field and p a monic F-polynomial with an odd degree factor over F. Then p is F-real if and only if it is the characteristic polynomial of a symmetric F-matrix.

That p must be F-real follows from work of Krakowski [4, Satz 3.3]. To prove the coverse we generalize results of Sapiro [6] in Lemma 1 and Theorem 3. Sapiro deals with the case in which p is a cubic. Theorem 4 considers the minimum dimension of symmetric matrices with a given root.

2. A basic lemma. In our proof we shall study congruence classes of certain symmetric matrices which are defined below. We shall denote congruence of the matrices A and B over the field F (i.e., $A = TBT'$ for some nonsingular F-matrix T) by $A \sim B(F)$.

DEFINITION. Let G be a field with subfield F. If $\lambda \in G$ is nonzero and if $\alpha_1, \cdots, \alpha_n$ form a basis for G (as a vector space) over F, define the matrices $M = \|\alpha_i^{(j)}\|$ and $D(\lambda) = \operatorname{diag}(\lambda^{(1)}, \cdots, \lambda^{(n)})$ where superscripts denote conjugacy over F. We call

$$A = A(\lambda) = MD(\lambda)M'$$

a *matrix from G to F*. Clearly

$$a_{ij} = \operatorname{tr}_{G/F}(\lambda\alpha_i\alpha_j)\ .$$

If $\mathscr{A} = \Sigma \oplus G_i$ where the G_i are extension fields of F, and if A_i is a matrix from G_i to F, then any matrix congruent to $\Sigma \oplus A_i$ over F is called a matrix from $\mathscr{A}$ to F. Note that a different choice for the basis $\alpha_1, \cdots, \alpha_n$ would lead to a matrix congruent to $A(\lambda)$ over F.

LEMMA 1. *Let F be a field and $p = q_1 \cdots q_m$ a monic F-polynomial decomposed into prime factors over F. Assume that the splitting field of p over F is a separable extension of F. If the identity is a matrix from*

$$\sum_1^m \oplus F[x]/(q_i))$$

to F, then p is the characteristic polynomial of a symmetric F-matrix.

Proof. Let $D = \Sigma \oplus D(\lambda_i)$ and $M = \Sigma \oplus M_i$ where the i^{th} component refers to $F[x]/(q_i(x))$. We have $TT' = MDM'$ for some F-matrix T. Let $E = \Sigma \oplus D(\theta_i)$ where θ_i is a zero of q_i. By separability M is nonsingular. We have $T^{-1}MD = (M^{-1}T)'$. Let

$$S = T^{-1}(MEM^{-1})T .$$

Then

$$\begin{aligned} S' &= (M^{-1}T)'E(T^{-1}M)' \\ &= (T^{-1}M)ED(T^{-1}M)' \\ &= (T^{-1}M)E(T^{-1}MD)' \\ &= (T^{-1}M)E(M^{-1}T) \\ &= S . \end{aligned}$$

Also $|S - \lambda I| = |E - \lambda I| = \pm p(\lambda)$. Finally, S is an F-matrix since $M_i^{-1} = ||\beta_i^{(j)}||$ where $\vec{\beta}$ is the complementary basis to $\vec{\alpha}$ [2, p. 437].

3. The irreducible case. In this section we shall reduce the proof of Theorem 1 to a study of the prime factors of p over F. This requires the Hasse-Minkowski Theorem. The Hilbert symbol over a local field L will be written $(a, b/L) = (a, b) = \pm 1$. If A is a symmetric L matrix and $A \sim \Sigma \oplus a_i(L)$, then

$$c(A/L) = c(A) = \prod_{i \leq j} (a_i, a_j)$$

is the Hasse invariant. If A is a nonsingular symmetric matrix over an algebraic number field F, then we have $\dim A$ and $\det A = |A|$ as global invariants, $c(A/F_{\mathfrak{p}})$ as Hasse invariants, and $\operatorname{ind}^+(A/F_{\mathfrak{p}})$ as real archimedean invariants where $\operatorname{ind}^+(A/F_{\mathfrak{p}})$ is the number of positive a_i in $A \sim \Sigma \oplus a_i(F_{\mathfrak{p}})$.

THEOREM 2. *Let F be an algebraic number field and q an F-real irreducible F polynomial of degree n. Let $K = F[x]/(q(x))$ and let k be a rational integer.*

(1) *If n is odd, the identity is a matrix from K to F.*

(2) *If n is even, there is a matrix A from K to F which has the same archimedean invariants as the identity and satisfies $c(A)(|A|, -1)^k = +1$ at all local completions of F.*

The next two sections develop the ideas needed in the proof of this theorem. We now prove Theorem 1 from Lemma 1 and Theorem 2.

Let $p = q_1 \cdots q_s r_1 \cdots r_t$ be the prime factorization of p over F where the degree d_i of q_i is odd and the degree e_i of r_i is even. By assumption $s \neq 0$. Let A_i be the matrix from $F[x]/(r_i(x))$ to F given by Theorem 2 (2) with

$$k = k(i) = \Big(\sum_{j=1}^{i-1} e_j + d_1 - 1\Big)\Big/2 .$$

Let B_0 be the d_1 dimensional identity matrix—a matrix from $F[x]/(q_1(x))$ to F by Theorem 2(1)—and let

$$B_i = |A_i| B_{i-1} \oplus A_i .$$

By induction, the Hasse-Minkowski Theorem gives $B_i \sim I(F)$. Thus the identity is a matrix from

$$F[x]/(q_1(x)) \oplus \sum_{i=1}^{r} \oplus F[x]/(r_i(x))$$

to F. By Theorem 2 (1), the identity is a matrix from $F[x]/(q_i(x))$, so an application of Lemma 1 proves Theorem 1.

4. The local case.

In this section we reduce the proof of theorems having the form of Theorem 2 to local considerations.

THEOREM 3. *Let F be an algebraic number field and q an F-real irreducible F-polynomial. Let $\alpha_1, \cdots, \alpha_n$ be algebraic integers in $G = F[x]/(q(x))$ which are a basis for G over F. Let $M = \| \alpha_i^{(j)} \|$ and let Ω be the set of prime spots on F which divide $2|M|^2$. Suppose that for each $\mathfrak{p} \in \Omega$ there is given a matrix $A(\lambda_\mathfrak{p})$ from $F_\mathfrak{p}[x]/(q(x))$ to $F_\mathfrak{p}$. Then there is a matrix $A = A(\lambda)$ from G to F and a local prime spot $\mathfrak{q} \notin \Omega$ on F such that*

(1) *if $\mathfrak{p} \in \Omega$, then*

$$c(A/F_\mathfrak{p}) = c(A(\lambda_\mathfrak{p})/F_\mathfrak{p}) ,$$

and

$$|A(\lambda_\mathfrak{p})|/|A| \in F_\mathfrak{p}^2 ,$$

the group of squares in $F_\mathfrak{p}$

(2) *if $\mathfrak{p} \notin \Omega$ is a local prime spot on F distinct from $\mathfrak{q}$, then $c(A/F_\mathfrak{p}) = +1$ and $|A|$ is a unit of $F_\mathfrak{p}$;*

(3) *A has the same real archimedean invariants as the identity matrix of the same dimension.*

Proof. If we change the basis used in forming $A(\lambda_\mathfrak{p})$ and change $\lambda_\mathfrak{p}$ by a square factor, then $c(A(\lambda_\mathfrak{p}))$ and $|A(\lambda_\mathfrak{p})| \cdot F_\mathfrak{p}^2$ will be unchanged.

Hence we may assume that $\alpha_1, \cdots, \alpha_n$ is the basis for all $\mathfrak{p}$ and that $\lambda_\mathfrak{p}$ is integral at $\mathfrak{p}$.

There is a sufficiently large positive rational integer m such that

$$\lambda_0 \equiv \lambda_\mathfrak{p} \pmod{\mathfrak{p}^m} \qquad \text{for } \mathfrak{p} \in \Omega ,$$

implies

$$c(A(\lambda_0)/F_\mathfrak{p}) = c(A(\lambda_\mathfrak{p})/F_\mathfrak{p}) \qquad \text{for } \mathfrak{p} \in \Omega ,$$

and

$$| A(\lambda_\mathfrak{p}) |/| A(\lambda_0) | \in F_\mathfrak{p}^2 \qquad \text{for } \mathfrak{p} \in \Omega .$$

Choose λ_0 such that

(i) λ_0 is an integer of G

(ii) $\lambda_0 \equiv \lambda_\mathfrak{p} \pmod{\mathfrak{p}^m}$ for $\mathfrak{p} \in \Omega$

(iii) if F is formally real, λ_0 is totally positive. Let $\mathfrak{M} = \Pi_\Omega \mathfrak{p}^m$. For each local prime spot $\mathfrak{P}$ on G let $k(\mathfrak{P})$ be the largest rational integer such that $\mathfrak{P}^{k(\mathfrak{P})}$ divides λ_0. Let

$$\mathfrak{U} = \prod_{\mathfrak{P} | \mathfrak{p} \in \Omega} \mathfrak{P}^{k(\mathfrak{P})} .$$

Then $\lambda_0/\mathfrak{U}$ is prime to $\mathfrak{M}$. By the generalized arithmetic progression theorem [1, Satz 13], there is an $\alpha \in G$ and a prime spot $\mathfrak{O}$ on G such that

(i) $(\alpha\lambda_0/\mathfrak{U}) = \mathfrak{O}$,

(ii) $\alpha \equiv 1 \pmod{\mathfrak{M}}$,

(iii) if F is formally real, α is totally positive.

Let $\lambda = \alpha\lambda_0$ and let $\mathfrak{q}$ be the prime spot on F which $\mathfrak{O}$ divides. Since $\lambda \equiv \lambda_0 \equiv \lambda_\mathfrak{p}(\mathfrak{p}^m)$, part (1) holds. Since λ is totally positive if F is formally real, (3) holds. Since $A(\lambda)$ has integral entries and $| A(\lambda) | = N(\mathfrak{O}\mathfrak{U}) | M |^2$, a unit of $F_\mathfrak{p}$ for $\mathfrak{p} \notin \Omega \cup \{\mathfrak{q}\}$, part (2) holds by [5, 92: 1].

5. Local lemmas.

In this section we prove a series of lemmas. They will be used together with Theorem 3 to prove Theorem 2. Throughout this section we shall let L be a local field with prime spot $\mathfrak{p}$ and characteristic zero; further, $K = K_1, K_2, \cdots, K_m$ will be finite algebraic extensions of L.

LEMMA 2. *If $\mathfrak{p}$ is prime to 2, there is a matrix A from $\Sigma \oplus K_i$ to L with integer entries and unit determinant.*

Proof. It suffices to exhibit such a matrix from K to L. Let $\alpha_1, \cdots, \alpha_n$ be a free basis for the integers of K_i over the integers of L. Let $M = || \alpha_i^{(j)} ||$. The matrix M'^{-1} has the form $|| \beta_i^{(j)} ||$ where

$\beta_1, \cdots, \beta_n$ is the complementary basis [2, p. 437] to $\alpha_1, \cdots, \alpha_n$. Let Π be a prime of K. The ideal $(\beta_1, \cdots, \beta_n)$ equals (Π^k) for some rational integer k. Since $(\alpha_1, \cdots, \alpha_n) = (1)$, there is a matrix A, whose elements are integers of L and whose determinant is an L unit, satisfying $MD(\Pi^k) = AM'^{-1}$.

For the remainder of this section we shall assume that $\mathfrak{p}$ divides 2.

LEMMA 3. *If $[K: L]$ is odd, the identity is a matrix from K to L.*

Proof. Let T be the inertia subextension of L. Suppose that the identity is a matrix from T to L, namely $M_1D_1M_1'$, and that the identity is a matrix from K to T, namely $M_2D_2M_2'$. Then the identity is a matrix from K to L, namely

$$(M_1 \otimes M_2)(D_1 \otimes D_2)(M_1 \otimes M_2)' .$$

We first show that the identity is a matrix from T to L. Let $M_1 = \| \alpha_i^{(j)} \|$ where $\alpha_1, \cdots, \alpha_f$ is a basis for T over L. Set $A = M_1M_1'$. Since T is a cyclic extension of L, we have $A \sim I(T)$. Since $[T: L]$ is odd, it follows that $A \sim I(L)$.

We now show that the identity is a matrix from K to T. Let Π be a prime of K such that $\Pi^e = \pi$, a prime of T, where $e = [K: T]$ is odd. Let $\alpha_i = \Pi^{i-1}$ and $M_2 = \| \alpha_i^{(j)} \|$ and $a = (e^2 - 1)/8$. There are two cases.

(i) If $(-1, -1/T)^a = +1$, let $\lambda = 1/e$,

(ii) If $(-1, -1/T)^a = -1$, let $\lambda = (1 + \Pi^{-1} + 4\Pi^{-2})/e$.

Set $A = |B| \cdot B$ where $B = M_2D(\lambda)M_2'$. In case (i) it is easily verified that $c(A) = +1$.

We consider case (ii). Since $(-1, -1)^a = -1$, it follows that $e \equiv \pm 3 \pmod 8$. Also, as

$$-\left(\frac{1 - \sqrt{-3}}{2}\right)^2 - \left(\frac{1 + \sqrt{-3}}{2}\right)^2 = 1 ,$$

we have $f(T(\sqrt{5})/T) = 2$ (see [5, 63: 3]). Thus $(\pi, 5) = -1$ and $(\varepsilon, 5) = +1$ for any unit ε of T. When $e = 3$ it is easily shown that $c(A) = +1$. Assume $e > 3$. The matrix B has the form shown in Figure I. We shall use the formula [3, p. 31]:

$$c(C_m) = (-1, | C_m |) \prod_{i=1}^{m-1} (| C_i |, - | C_{i+1} |) ,$$

if $\Pi_{i=1}^m | C_i | \neq 0$, where $C_i = \| c_{st} \|$ $(1 \leqq s, t \leqq i)$.

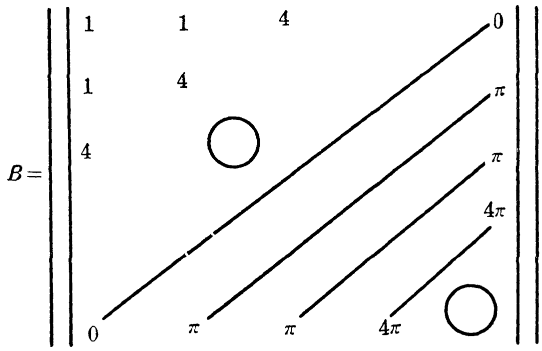

FIGURE I.

We must transform B. Let X be the $e \times e$ matrix such that premultiplication by X adds π^{-1} times the $(e-k+2)^{\text{nd}}$ row to the k^{th} row for $k = 4, 6, 8, \cdots, 4[e/8]+2$ and leaves the remaining rows unchanged. Let $C = XBX'$. By studying $(X_i)^{-1}C_i(X_i)'^{-1}$, we find that

(i) $|C_{2i+1}| \in (-1)^i T^2$ for $2i+1 < e$,
(ii) $|C_{e-1}| \in (-1)^{(e-1)/2} 5T^2$,
(iii) $|C_e| = \pi^{e-2}\varepsilon$ for some unit ε of T,
(iv) $\Pi_1^e |C_i| \neq 0$.

It therefore follows that

(i) $(|C_{2i-1}|, -|C_{2i}|)(|C_{2i}|, -|C_{2i+1}|) = (-1)^{i-1}$ for $2i+1 < e$,
(ii) $(|C_{e-2}|, -|C_{e-1}|) = (-1)^{(e-3)/2}$,
(iii) $(|C_{e-1}|, -|C_e|) = (-1)^{(e+1)/2}(-1, |C_e|)^{(e-1)/2}$.

Thus

$$\begin{aligned} c(A) &= C(B)(-1, |B_e|)^{(e+1)/2} \\ &= c(C_e)(-1, |C_e|)^{(e+1)/2} \\ &= +1 \quad \text{since} \quad e \equiv \pm 3 \pmod 8) . \end{aligned}$$

LEMMA 4. *If $L^2 \supseteqq N(K/L)$, the norm group of K over L, then the identity is a matrix from K to L.*

Proof. We make some preliminary observations. Let T_i be a subfield of K (to be specified later) such that $N(K/T_i) \subseteqq T_i^2$. Let T_i^* be the multiplicative group of T_i. Let H be the maximum abelian subextension of T_i in K of type $(2, 2, \cdots, 2)$. By the reciprocity and limitation theorems of class field theory [7, pp. 177, 180], the Galois group of H over T_i is isomorphic to

$$(T_i^*/N(K/T_i))/(T_i^*/N(K/T_i))^2 .$$

Since $N(K/T_i) \subseteqq T_i^{*2}$, this is isomorphic to T_i^*/T_i^{*2}. Hence $[T_i^*: T_i^{*2}] =$

$[H: T_i]$ which divides $[K: T_i]$. By [5, 63: 9], 8 divides $[T_i^*: T_i^{*2}]$. Thus

(i) $[K: T_i] \equiv 0 \pmod 8$.

Since $N(H/T_i) \subseteqq T_i^2$, we have that $f(H/T_i) > 1$. Since $[H: T_i]$ is a power of 2 and $K \supseteqq H$, we have

(ii) $f(K/T_i) \equiv 0 \pmod 2$.

Suppose $K = T_i(\theta)$. Let $\alpha_i = \theta^{i-1}$ and $M = \| \alpha_i^{(j)} \|$. If $\lambda \in K$ we have

$$| MD(\lambda)M' | = N_{K/T_i}\Big(\lambda \prod_{i \neq 1} (\theta^{(1)} - \theta^{(i)})\Big) \in T_i^2 ,$$

by the formula for a van der Monde determinant and $N(K/T_i) \subseteqq T_i^2$. Hence

(iii) if C is a matrix from K to T_i, then $|C| \in T_i^2$.

We now apply the above observations. Let T be the inertia subextension of L. Construct the tower

$$L = T_0 \subset T_1 \subset \cdots \subset T_k \subseteqq T ,$$

where $[T_j: T_{j-1}] = 2$ for $1 \leqq j \leqq k$ and $[T: T_k]$ is odd. Since $f(K/T_k)$ is odd, we have $N(K/T_k) \not\subseteqq T_k^2$ by (ii). Hence we may choose i such that $N(K/T_i) \subseteqq T_i^2$ and $N(K/T_{i+1}) \not\subseteqq T_{i+1}^2$. (Actually $i = k - 1$, but this is irrelevant.) Suppose the identity is a matrix from K to T_i. Let B be a matrix from T_i to L. Then $A = I \otimes B$ is a matrix from K to L. By (i) we have $\dim I \equiv 0 \pmod 4$. Hence $|A| \in L^2$ and $c(A/L) = +1$ by the formula.

$$(*) \quad c(X \otimes Y) = c(X)^y c(Y)^x (-1, |X|)^{y(y-1)/2} (-1, |Y|)^{x(x-1)/2} (|X|, |Y|)^{xy+1}$$

where X, Y are symmetric matrices, $x = \dim X$ and $y = \dim Y$. It suffices to show that the identity is a matrix from K to T_i.

Let C be a matrix from K to T_{i+1} with $|C| \notin T_{i+1}^2$. (This can be done since $N(K/T_{i+1}) \not\subseteqq T_{i+1}^2$.) We have $C \sim I \oplus -1 \oplus s \oplus t$ where $s, t \in T_{i+1}$ by [5, 63: 17]. Let $e \in T_i$ be such that $T_{i+1} = T_i(\sqrt{e})$. Let $M = \left\| \begin{matrix} 1 & 1 \\ \sqrt{e} & -\sqrt{e} \end{matrix} \right\|$ and $E(q) = MD(q)M'$ for $q \in T_{i+1}$. We have that

$$S(r) = (I \oplus -1) \otimes E(r) \oplus E(rs) \oplus E(rt)$$

is a matrix from K to T_i for nonzero $r \in T_{i+1}$. By (iii) we have $|S(r)| \in T_i^2$. Since

$$\dim (I \oplus -1) = \dim S(r)/2 - 2 \equiv 2 \pmod 4 \text{ by (i)} ,$$

we have

$$|(I \oplus -1) \otimes E(r)| \in T_i^2 .$$

Hence $|E(rs)| \in |E(rt)| \cdot T_i^2$. Thus

$$\begin{aligned} c(S(r)) &= c((I \oplus -1) \otimes E(r))c(E(rs))c(E(rt))(|E(rs)|, -1) \\ &= (-1, -1)c(E(rs))c(E(rt))(|E(rs)|, -1) \text{ by } (^*) . \end{aligned}$$

Any $q \in T_{i+1}$ has the form $a + b\sqrt{e}$ with $a, b \in T_i$. Write $q_1 = a$. If $q_1 \neq 0$, then

$$c(E(q)) = (2q_1, -|E(q)|)(-1, |E(q)|) .$$

If $(rs)_1(rt)_1 \neq 0$, we have

$$c(S(r)) = (-(rs)_1(rt)_1, -|E(rs)|) .$$

We may choose $r = s^{-1}(l + \sqrt{e})^2\sqrt{e}$ with $l = 0, 1$, or 4 such that $(rs)_1(rt)_1 \neq 0$. Since $-|E(rs)| \in T_i^2$, we have $c(S(r)) = +1$.

LEMMA 5. *If $\sum_1^m [K_i: L]$ is odd, the identity is a matrix from $\sum_1^m \oplus K_i$ to L.*

Proof. By Lemmas 3 and 4 we are done unless $[K_i: L] = d$ is even and $N(K_i/L) \nsubseteq L^2$ for some i. Suppose that this is the case. Since $N(K_i/L) \nsubseteq L^2$, there is a matrix B from K_i to L such that $(-1)^{d/2}|B| \notin L^2$. Let C be a matrix from $\Sigma_{j\neq i} \oplus K_j$ to L. Let

$$A = |B|\cdot|C|\cdot C \oplus aB$$

where $a \in L$ is chosen so that

$$c(A) = c(|B|\cdot|C|\cdot C)(|B|, -1)c(B)(a, (-1)^{d/2}|B|) = +1 .$$

LEMMA 6. *If $\Sigma_1^m[K_i: L]$ is even, $N(K_1/L) \nsubseteq L^2$, and k is a rational integer, then there is a matrix A from $\Sigma_1^m \oplus K_i$ to L such that $c(A)(|A|, -1)^k = +1$.*

Proof. Let B be a matrix from $\Sigma_1^m \oplus K_i$ to L such that $(-1)^n|B| \notin L^2$ where $n = \Sigma_1^m[K_i: L]/2$. Let $A = aB$ where $a \in L$ is chosen so that $c(A)(|A|, -1)^k = c(B)(|B|, -1)^k(a, (-1)^n|B|) = +1$.

6. Proof of Theorem 2. If n if odd, apply Lemmas 2 and 5. Let B be the matrix given by Theorem 3. Define $A = |B|\cdot B$. If n is even, apply Lemmas 2, 3, 4 and 6. Let A be the matrix given by Theorem 3. In both cases, behavior at the exceptional spot is handled by the Hilbert reciprocity formula [5, p. 190].

7. Matrices with given roots. We prove

THEOREM 4. *Let F be an algebraic number field. Let θ be the root of an irreducible F-polynomial q of degree n. Then θ is the* *characteristic root of some symmetric F-matrix if and only if q is F-real. When such a matrix exists, it may be chosen to have dimension n or n + 1, whichever is odd. This dimension is the least possible*

(1) *if n is odd or*

(2) *if* $n \equiv 2 \pmod 4$ *and* $(-1) \notin N(F(\theta)/F) \cdot F^2$.

Proof. Use Theorem 1 with $p(x) = q(x)$ or $xq(x)$. The result is clearly best possible when n is odd. Suppose $n \equiv 2(4)$ and n is least possible. Let $\alpha_i = \theta^{i-1}$ and $M = \|\alpha_i^{(j)}\|$. By the converse of Lemma 1 when p does not have repeated roots (see [6, Lemma 1.1] for a proof), there is an F-matrix T and a $\lambda \in F(\theta)$ such that $MD(\lambda)M' = TT'$. Noting that $|MM'| = -N(p'(\theta))$, we get

$$-1 \in N(F(\theta)/F) \cdot F^2 .$$

By class field theory, for all $n \equiv 2(4)$ there exist F and θ such that $n + 1$ is the least possible dimension.

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