COLLECTIVELY COMPACT SETS OF LINEAR OPERATORS

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A set of linear operators from one normed linear space to another is collectively compact if and only if the union of the images of the unit ball has compact closure. This paper concerns general properties of such sets. Several useful criteria for sets of linear operators to be collectively compact are given. In particular, every compact set of compact linear operators is collectively compact. As a partial converse, every collectively compact set of self adjoint or normal operators on a Hilbert space is totally bounded.

Let X and Y be real or complex normed linear spaces and $[X, Y]$ the space of bounded linear operators on X into Y. It is assumed that $[X, Y]$ has the norm topology except in Proposition 2.1(c), where a strong closure appears.

Let $\mathscr{B}$ denote the closed unit ball in X. Then $\mathscr{K} \subset [X, Y]$ is collectively compact if and only if the set $\mathscr{K}\mathscr{B} = \{Kx: K \in \mathscr{K},\ x \in \mathscr{B}\}$ has compact closure in Y. Collectively compact sets and their applications to integral equations have been treated in a number of papers [1–5, 7–9, 11–12]. Results obtained in this paper are used in a sequel [6] which relates spectral properties of operators T and T_n, $n = 1, 2, \cdots$, such that $T_n \to T$ strongly and $\{T_n - T\}$ is collectively compact.

Frequently it will be necessary to show that a set in Y or $[X, Y]$ is compact. For this purpose, recall that a subset of a metric space is compact if and only if it is closed and sequentially compact if and only if it is complete and totally bounded (for each $\varepsilon > 0$ it has a finite ε-net). An often useful fact is that a set is totally bounded whenever it has a totally bounded ε-net for each $\varepsilon > 0$. The familiar proposition that a continuous function from one topological space to another maps compact sets onto compact sets will be used several times. The following generalization of the Arzelà-Ascoli theorem will be needed.

LEMMA 1.1. *Let $\mathfrak{F}$ be an equicontinuous set of functions from a compact metric space $\mathscr{X}$ into a metric space. For each $p \in \mathscr{X}$, assume that the set $\mathfrak{F}p = \{f(p): f \in \mathscr{F}\}$ has compact closure. Then the set $\mathfrak{F}\mathscr{X} = \{f(p): f \in \mathfrak{F}, p \in \mathscr{X}\}$ has compact closure.*

2. **General properties of collectively compact sets.** Collectively compact sets of operators have a number of properties analogous to

those of sets with compact closure in arbitrary normed linear spaces. For example, any subset or scalar multiple of a collectively compact set is collectively compact. Any finite union or sum of collectively compact sets is collectively compact. A collectively compact set is necessarily bounded.

PROPOSITION 2.1. Let $\mathcal{K} \subset [X, Y]$ be collectively compact. Then the following sets are collectively compact:
  (a)   The convex hull of $\mathcal{K}$;
  (b)   The circled hull $\{\lambda K : |\lambda| \leq 1,\ K \in \mathcal{K}\}$ of $\mathcal{K}$;
  (c)   The strong closure $\overline{\mathcal{K}}^s$ and norm closure $\overline{\mathcal{K}}$ of $\mathcal{K}$;
  (d) $\{\sum_{n=1}^{N} \lambda_n K_n : K_n \in \mathcal{K},\ \sum_{n=1}^{N} |\lambda_n| \leq b\}$ for each $b > 0$, $N \leq \infty$.

Proof. Mazur's theorem [10, p. 416] yields (a). The circled hull of a compact set in Y is compact since the map f defined by $f(\lambda, y) = \lambda y$ is continuous. This yields (b). Since $\mathcal{K} \subset \overline{\mathcal{K}}^s$ and $\overline{\mathcal{K}}^s \mathcal{B} \subset \overline{\mathcal{K}\mathcal{B}}$, (c) is valid. Since $\mathcal{K}$ is bounded, the set in (d) belongs to the norm closure of the convex circled hull of $b\mathcal{K}$.

The next result involves integrals of operator valued functions. Let Γ be a finite interval if X is real and a rectifiable arc if X is complex. Suppose $K_\alpha(\lambda) \subset [X, Y]$ for $\lambda \in \Gamma$ and α in an index set A. For each $\alpha \in A$ assume that $\int_\Gamma K_\alpha(\lambda) d\lambda$ is the strong or norm limit of the usual approximating sums,

$$\sum_{j=1}^{m} K_\alpha(\lambda_j')(\lambda_j - \lambda_{j-1}) .$$

PROPOSITION 2.2. With the foregoing notation, assume that $\{K_\alpha(\lambda) : \alpha \in A,\ \lambda \in \Gamma\}$ is collectively compact. Then $\left\{\int_\Gamma K_\alpha(\lambda) d\lambda : \alpha \in A\right\}$ is collectively compact.

Proof. This follows from Proposition 2.1(c), (d) and

$$\sum_{j=1}^{m} |\lambda_j - \lambda_{j-1}| \leq \text{length} (\Gamma) .$$

For the next proposition, let Z be another normed linear space.

PROPOSITION 2.3. Let $\mathcal{K} \subset [X, Y]$, $\mathcal{M} \subset [Z, X]$ and $\mathcal{N} \subset [Y, Z]$. Then
  (a) $\mathcal{K}$ collectively compact, $\mathcal{M}$ bounded $\Rightarrow$ $\mathcal{K}\mathcal{M}$ collectively compact,
  (b) $\mathcal{K}$ collectively compact, $\overline{\mathcal{N}}$ compact $\Rightarrow$ $\mathcal{N}\mathcal{K}$ collectively compact.

Proof. (a) Suppose $\| M \| < r$ for all $M \in \mathscr{M}$. Then

$$\overline{\mathscr{K}\mathscr{M}\mathscr{B}} \subset r\overline{\mathscr{K}\mathscr{B}} ,$$

so $\overline{\mathscr{K}\mathscr{M}\mathscr{B}}$ is compact and $\mathscr{K}\mathscr{M}$ is collectively compact. (b) Define a map $f \colon \mathscr{N} \times \overline{\mathscr{K}\mathscr{B}} \to Z$ by $f(N, y) = Ny$ for $N \in \mathscr{N}$, $y \in \overline{\mathscr{K}\mathscr{B}}$. Since $\mathscr{N}$ and $\overline{\mathscr{K}\mathscr{B}}$ are compact and f is continuous, its range, which contains $\mathscr{N}\mathscr{K}\mathscr{B}$, is compact. Thus $\overline{\mathscr{N}\mathscr{K}\mathscr{B}}$ is compact and $\mathscr{N}\mathscr{K}$ is collectively compact.

A collectively compact set is a bounded set of compact operators. The converse fails as can be seen by considering the set of one dimensional projections of norm one in any infinite dimensional Banach space. However we have:

THEOREM 2.4. *Every compact set* $\mathscr{K}$ *of compact operators in* $[X, Y]$ *is collectively compact.*

Proof. Define maps $f_x \colon \mathscr{K} \to Y$ by $f_x(K) = Kx$ for $K \in \mathscr{K}$, $x \in \mathscr{B}$ and let $\mathscr{F} = \{f_x \colon x \in \mathscr{B}\}$. Since $\| f_x(K_1 - K_2) \| \leq \| K_1 - K_2 \|$, $\mathscr{F}$ is equicontinuous. Since each $K \in \mathscr{K}$ is compact, the sets $\overline{\mathscr{F}K} = \overline{K\mathscr{B}}$ are compact. By hypothesis, $\mathscr{K}$ is compact. Therefore, by Lemma 1.1, $\overline{\mathscr{F}\mathscr{K}} = \overline{\mathscr{K}\mathscr{B}}$ is compact and $\mathscr{K}$ is collectively compact.

THEOREM 2.5. *If* Y *is complete, then every totally bounded set* $\mathscr{K}$ *of compact operators in* $[X, Y]$ *is collectively compact.*

Proof. In this case, $\overline{\mathscr{K}}$ is a compact set of compact operators. By Theorem 2.4, $\overline{\mathscr{K}}$ is collectively compact. Hence, $\mathscr{K}$ is collectively compact.

The converses of Theorems 2.4 and 2.5 are false:

EXAMPLE 2.6. Let $\mathscr{K}$ be the set of operators on $l^p (1 \leq p \leq \infty)$ defined by $K_n x = x_n \varphi_1$, $n \geq 1$. Since $\mathscr{K}\mathscr{B}$ is bounded and one-dimensional, $\mathscr{K}$ is collectively compact. But $\mathscr{K}$ is not totally bounded, for $\| K_m - K_n \| = 2^{1/p}$ if $m \neq n$.

Partial converses of Theorem 2.5 are given in the next section.

3. **Operators on a Hilbert space.** Throughout this section, let X be a Hilbert space. It will be shown that every collectively compact set of self adjoint or normal operators in $[X, X]$ is totally bounded.

We begin by considering sets of projections. Let $\mathscr{S} = \{x \colon \| x \| = 1\}$ and, for each $x \in \mathscr{S}$, let E_x be the self adjoint projection onto the subspace spanned by x.

LEMMA 3.1. *Let $\mathscr{U} \subset \mathscr{S}$ and $\mathscr{M} = \{E_x : x \in \mathscr{U}\}$. (Thus, $\mathscr{M}$ can be any set of self adjoint projections with one-dimensional ranges.) The following statements are equivalent:*
  (a) *$\mathscr{U}$ is totally bounded;*
  (b) *$\mathscr{M}$ is totally bounded;*
  (c) *$\mathscr{M}$ is collectively compact.*

Proof. Since $E_x y = (y, x)x$ for $y \in X$ and $x \in \mathscr{S}$, the map f: $\mathscr{S} \to [X, X]$ given by $f(x) = E_x$ is continuous. Since $\mathscr{M} = f(\mathscr{U})$, (a) implies (b). By Theorem 2.5, (b) implies (c). Since $\mathscr{U} \subset \mathscr{M}\mathscr{B}$, (c) implies (a).

LEMMA 3.2. *Let $\mathscr{M}$ be a collectively compact set of self adjoint projections and $\mathscr{M}'$ any subset consisting of mutually orthogonal projections. Then $\mathscr{M}'$ is finite and there is an integer n, independent of $\mathscr{M}'$, such that*

$$\sum_{E \in \mathscr{M}'} \dim EX \leqq n \, .$$

Proof. Since $\mathscr{M}\mathscr{S}$ is totally bounded, it can be covered by a finite number n of open balls of radius $1/2$. If $x, y \in \mathscr{M}\mathscr{S}$ and $x \perp y$ then $\| x - y \| = \sqrt{2}$, so that x and y lie in different balls. The lemma follows.

LEMMA 3.3. *Suppose X is a real Hilbert space and $\hat{X}$ is its complexification defined in the usual way. For $\mathscr{K} \subset [X, X]$, let $\hat{\mathscr{K}} \subset [\hat{X}, \hat{X}]$ be the set of canonical extensions of operators in $\mathscr{K}$. Then $\mathscr{K}$ is collectively compact if and only if $\hat{\mathscr{K}}$ is collectively compact.*

Since the proof is straightforward, it is omitted.
We are now ready to establish the principal results of this section.

THEOREM 3.4. *Let $\mathscr{K}$ be a set of self adjoint or normal compact operators on a Hilbert space. Then the following statements are equivalent:*
  (a) *$\mathscr{K}$ is collectively compact.*
  (b) *$\mathscr{K}^* = \{K^* : K \in \mathscr{K}\}$ is collectively compact.*
  (c) *$\mathscr{K}$ is totally bounded.*

Proof. Without loss of generality, X is complex. Assume

$$\mathscr{K} = \{K_\alpha : \alpha \in A\}$$

collectively compact. Then every K_α is compact. For each $\alpha \in A$,

the spectral theorem yields a decomposition

$$K_\alpha = \sum_n \lambda_{\alpha n} E_{\alpha n}$$

with the self adjoint projections $E_{\alpha n}$ multually orthogonal and with $\dim E_{\alpha n} X = 1$ (thus, the $\lambda_{\alpha n}$ are not necessarily distinct). Since $\mathscr{K}$ is bounded, there exists $b < \infty$ such that

$$| \lambda_{\alpha n} | \leqq \| K_\alpha \| \leqq b , \qquad \alpha \in A, \, n \geqq 1 .$$

For $\alpha \in A$ and $\varepsilon > 0$, let

$$
\begin{aligned}
N_{\alpha\varepsilon} &= \{n : | \lambda_{\alpha n} | \geqq \varepsilon \} , \\
\mathscr{M}_\varepsilon &= \{E_{\alpha n} : \alpha \in A, \, n \in N_{\alpha\varepsilon} \} , \\
K_{\alpha\varepsilon} &= \sum_{n \in N_{\alpha\varepsilon}} \lambda_{\alpha n} E_{\alpha n} , \\
\mathscr{K}_\varepsilon &= \{K_{\alpha\varepsilon} : \alpha \in A\} .
\end{aligned}
$$

Then $K_\alpha E_{\alpha n} = \lambda_{\alpha n} E_{\alpha n}$ and, for $n \in N_{\alpha\varepsilon}$, $E_{\alpha n} \mathscr{B} = \lambda_{\alpha n}^{-1} K_\alpha E_{\alpha n} \mathscr{B} \subset \varepsilon^{-1} K_\alpha \mathscr{B}$. Therefore, $\mathscr{M}_\varepsilon \mathscr{B} \subset \varepsilon^{-1} \mathscr{K} \mathscr{B}$ and $\mathscr{M}_\varepsilon$ is collectively compact. By Lemma 3.1, $\mathscr{M}_\varepsilon$ is totally bounded. By Lemma 3.2, there exists n such that, for each $\alpha \in A$, $N_{\alpha\varepsilon}$ contains no more than n_ε elements. Then $\mathscr{K}_\varepsilon$ is in the convex circled hull of $b \, n_\varepsilon \mathscr{M}_\varepsilon$, so $\mathscr{K}_\varepsilon$ is totally bounded. Since $\| K_\alpha - K_{\alpha\varepsilon} \| < \varepsilon$ for all $\alpha \in A$, $\mathscr{K}_\varepsilon$ is an ε-net for $\mathscr{K}$. Therefore, $\mathscr{K}$ is totally bounded.

This result and Theorem 2.5 give:

$\mathscr{K}$ collectively compact if and only if $\mathscr{K}$ totally bounded .

Since $\| T^* \| = \| T \|$ for all $T \in [X, X]$,

$\mathscr{K}$ totally bounded if and only if $\mathscr{K}^*$ totally bounded .

The theorem follows.

THEOREM 3.5. *Let $\mathscr{K}$ be a set of compact operators on a Hilber space. Then $\mathscr{K}$ is totally bounded if and only if both $\mathscr{K}$ and $\mathscr{K}^*$ are collectively compact.*

Proof. As above, $\mathscr{K}$ totally bounded implies $\mathscr{K}$ and $\mathscr{K}^*$ collectively compact. Now assume $\mathscr{K}$ and $\mathscr{K}^*$ collectively compact. Then the sets

$$\mathscr{R} = \{K + K^* : K \in \mathscr{K}\} , \qquad \mathscr{I} = \{K - K^* : K \in \mathscr{K}\}$$

are collectively compact. By Theorem 3.4, $\mathscr{R}$ and $\mathscr{I}$ are totally bounded. Since $\mathscr{K} \cup \mathscr{K}^* \subset (\mathscr{R} + \mathscr{I}) \cup (\mathscr{R} - \mathscr{I})$, both $\mathscr{K}$ and $\mathscr{K}^*$ are totally bounded.

For the collectively compact set $\mathscr{K}$ in Example 2.6 with $p = 2$, it is easily verified that $\mathscr{K}^*$ is not collectively compact. This also follows from Theorem 3.5, since $\mathscr{K}$ is not totally bounded.

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