PACIFIC JOURNAL OF MATHEMATICS
Vol. 26, No. 1, 1968

DUAL GROUPS OF VECTOR SPACES

WILLIAM C. WATERHOUSE

Let E be a topological vector space over a field K having a nontrivial absolute value. Let E' be the dual space of continuous linear maps $E \to K$, and $\hat{E}$ the dual group of continuous characters $E \to R/Z$. $\hat{E}$ is a vector space over K by $(a\varphi)(x) = \varphi(ax)$, and composition with a nonzero character of K is a linear map of E' into $\hat{E}$. This map is always an isomorphism if K is locally compact, while if K is not locally compact it is never an isomorphism unless $\hat{E} = 0$. When K is locally compact, E' is in addition topologically isomorphic to $\hat{E}$ if each is given its topology of uniform convergence on compact sets. This leads to conditions on E which imply that E is topologically isomorphic to $(\hat{E})\hat{}$.

THEOREM 1. *Let K be a field with absolute value. Then $\hat{K}$ is one-dimensional over K if and only if K is locally compact.*

Proof. The sufficiency of local compactness is fairly well known (cf. [4, p. 92–3] for the characteristic zero case). To prove it, one takes a nonzero character π of K and considers the subspace $K\pi$ of $\hat{K}$. It is easy to check that $a \mapsto a\pi$ is a bicontinuous linear map, so $K\pi$ is complete and hence closed in $\hat{K}$. On the other hand, $K\pi$ separates the points of K, so by Pontrjagin duality it is dense in $\hat{K}$. Thus $\hat{K} = K\pi$.

Suppose conversely that $\hat{K}$ is one-dimensional, and choose a nonzero π in $\hat{K}$. The completion of K will again be a field, say L, and π extends to a character of L. Then every $a \in L$ gives a character $a\pi$ of L. If $a \neq b$, then $a - b$ is invertible, and so $\pi((a-b)c)$ cannot be zero for all c. Thus no two of the characters $a\pi$ are equal, and hence no two can agree on the dense set K. This contradicts one-dimensionality of $\hat{K}$ unless $K = L$, and we conclude that K must be complete. Hence if K is archimedean, it is locally compact.

We now assume that K is nonarchimedean. Let $A = \{x: |x| \leqq 1\}$, $M = \{x: |x| < 1\}$. Let π be a character of the discrete group A/M with $\pi(1) \neq 0$; we extend π to a character of the discrete group K/M and interpret it as an element of $\hat{K}$. Let $c > 1$ be an element of the value group, and consider the group G_c/M, where $G_c = \{x: |x| \leqq c\}$. All characters of this discrete group extend to characters of K vanishing on M, and by one-dimensionality they all come from multiples of π.

Now if $a \in A$, then $aM \subset M$, so $a\pi$ vanishes on M; conversely, if $a\pi$ vanishes on M, then $1/a \notin M$ and $a \in A$. Similarly, $a\pi$ vanishes on G_c if and only if $|a| < 1/c$. Thus the dual group of the discrete abelian group G_c/M is (algebraically) isomorphic to $A/\{a: |a| < 1/c\}$, which is isomorphic to G_c/M itself under multiplication by an element of absolute value c. A theorem of Kakutani [3, p. 396–7] shows that an infinite discrete abelian group has a dual group of strictly larger cardinality; hence G_c/M must be finite. This implies both that A/M is finite and that the value group is discrete; since K has these two properties and is complete, it is locally compact [1, p. 119].

COROLLARY. *Suppose K is not locally compact. Let E be a topological vector space over K with $\hat{E} \neq 0$. For any $\pi \in \hat{K}$, the map $E' \to \hat{E}$ given by composition with π fails to be surjective.*

Proof. If $E' = 0$ or $\pi = 0$, the statement is obvious. Suppose then there is a $0 \neq f \in E'$, and choose an $x \in E$ with $f(x) \neq 0$. The subspace Kx is topologically isomorphic to K, so its dual space is one-dimensional and is generated by the restriction of f. Hence all elements in $\hat{E}$ coming from E' restrict to multiples of $\pi \circ f$ on Kx. If τ is a character of K not a multiple of π, then $\tau \circ f \in \hat{E}$ is not in the image of E'.

REMARK. A topological field K is called *locally retrobounded* if for every pair of neighborhoods U, V of zero there is an $a \neq 0$ in K such that $a\{x^{-1}: x \notin V\} \subset U$; for example, an ordered field is locally retrobounded in its order topology. Every such field admits either an absolute value or a valuation which defines its topology [1, §5, Exerc. 2]. The proof of Theorem 1 works equally well for a valuation into any ordered abelian group, and hence Theorem 1 and its corollary hold for all locally retrobounded fields.

THEOREM 2. *Suppose K is locally compact, $0 \neq \pi \in \hat{K}$. Let E be a topological vector space over K. Then the map $E' \to \hat{E}$ given by $f \mapsto \pi \circ f$ is a vector space isomorphism. It is a homeomorphism if E' and $\hat{E}$ have their topologies of uniform convergence on compact sets.*

Proof. If $0 \neq f$, then $f(E) = K$, so $\pi \circ f \neq 0$; thus the map is injective. Now let $\varphi \in \hat{E}$. For each $x \in E$ there is a unique linear functional on Kx inducing $\varphi | Kx$, since $Kx \cong K$ and $\hat{K}$ is one-dimensional. We define $f(x)$ to be this functional evaluated at x; this gives us a homogeneous function $f: E \to K$. For any $x, y \in E$ and $a \in K$, we have

$$\begin{aligned} 0 &= \varphi(ax) + \varphi(ay) - \varphi(ax + ay) = \pi f(ax) + \pi f(ay) - \pi f(ax + ay) \\ &= \pi[f(ax) + f(ay) - f(ax + ay)] = \pi(a[f(x) + f(y) - f(x + y)]); \end{aligned}$$

hence $f(x) + f(y) - f(x + y) = 0$, and f is linear. If finally f were not continuous, then $f^{-1}(a)$ would be dense in E for every $a \in K$. Hence $f(U) = K$ for any neighborhood U of zero, so $\varphi(U) = \pi \circ f(U) = \pi(K)$ for all such U and φ would not be continuous.

Now the map $E' \to \hat{E}$ is an isomorphism, and it is obviously continuous; we need only prove it is open. The map $K' \to \hat{K}$ is a homeomorphism, since (as we noted in the proof of Theorem 1) $\hat{K} \cong K$. Hence, given any neighborhood U of zero in K, we can find an open V and a compact set B such that, for g in K', $\pi \circ g(B) \subset V$ implies $g(a) \in U$ for $|a| \leqq 1$. But if C is any compact set in E, BC will again be compact. It is easy to see then that if $f \in E'$ and $\pi \circ f(BC) \subset V$, then $f(C) \subset U$; this means that $E' \to \hat{E}$ is open.

Let K again be locally compact, and let E be a locally convex topological vector space over K. (In the archimedean case, the requisite theory is standard, cf. [2]; van Tiel has shown that exactly the same theory holds in the nonarchimedean case [6].) In view of Theorem 2, we identify E' and $\hat{E}$ furnished with the topology of uniform convergence on compact sets.

THEOREM 3. *If E is quasi-complete and barrelled, then E is topologically isomorphic to $(\hat{E})\hat{}$.*

Proof. Since E is locally convex, the map $E \to (\hat{E})\hat{}$ is injective. Since E is quasi-complete, the closed convex hull of a compact set is compact; thus the topology on $\hat{E}$ is that of uniform convergence on convex compact sets. This is weaker than the Mackey topology, and hence the map $E \to (\hat{E})\hat{}$ is bijective.

If S is a compact balanced set in $\hat{E}$, then its polar S° is a barrel in E, and hence is a neighborhood of 0 in E. These polars are a neighborhood basis at 0 in $(\hat{E})\hat{}$, so the map $E \to (\hat{E})\hat{}$ is continuous.

Finally, if U is a neighborhood of 0 in E, U° is equicontinuous and therefore compact in $\hat{E}$; hence $U^{\circ\circ}$ is a neighborhood of 0 in $(\hat{E})\hat{}$. But E has a neighborhood basis at 0 consisting of closed absolutely convex sets U, and for them $U = U^{\circ\circ}$. Thus the map is open.

As particular cases of Theorem 3, we get

COROLLARY. *If E is either complete and metrizable or reflexive, E is topologically isomorphic to $(\hat{E})\hat{}$.*

For the real and complex fields, Theorem 2 and these two cases of Theorem 3 were proved by M. F. Smith [5].

REFERENCES

1. N. Bourbaki, *Algèbre Commutative*, Hermann, Paris, 1964.

2. N. Bourbaki, *Espaces Vectoriels Topologiques*, Hermann, Paris, 1964.

3. E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Vol. 1, Springer-Verlag, Berlin, 1963

4. S. Lang, *Algebraic Numbers*, Addison-Wesley, Reading, Mass., 1964.

5. M. F. Smith, *The Pontrjagin duality theorem in linear spaces*, Ann. of Math. (2) **56** (1952), 248-253.

6. J. van Tiel, *Espaces localement K-convexes I, II, III, IV*, Nederl. Akad. Wetensch. Proc. Ser. A **68** (1965), 249-289.

Received December 20, 1966. During work on this paper, the author held a National Science Foundation Graduate Fellowship.

HARVARD UNIVERSITY