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STRONG CONTINUITY OF OPERATOR FUNCTIONS

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The complex-valued functions defined on a subset S of the plane such that $(S^- - S)^- \cap S$ is empty which give strong-operator continuous mappings from the set of normal operators on a Hilbert space with spectra in S into the set of all normal operators are characterized as those which are continuous on S, bounded on bounded subsets of S and $O(z)$ (Theorem 4.2). In the process of proving this result, it is shown that the adjoint operation is strong-operator continuous on the set of normal operators (Theorem 4.1).

In proving his fundamental Density Theorem [1], Kaplansky needs and establishes the fact that continuous real-valued functions vanishing at ∞ define strong-operator continuous mappings of the set of bounded self-adjoint operators into itself. He extends this result to bounded continuous functions as well.

While the Kaplansky Density Theorem has become an indispensable tool in the study of operator algebras, the various strong-operator continuity results are themselves important and useful. The purpose of this note is to give a precise delineation of the class of functions which define strong-operator continuous mappings. The technical desirability of having these results for normal operators forces us to consider functions of n-tuples of commuting self-adjoint operators (couples would suffice, but n-tuples add no difficulties). The results for n-tuples appear in §3; their application to functions of normal operators, in §4.

The reduction from functions of normal operators to functions of pairs of commuting self-adjoint operators involves the (topological) behavior of the adjoint operation on the normal operators. Now, it is well-known that the the adjoint operation is not strong-operator continuous on the set of all bounded operators. The most familiar example illustrating this discontinuity is the "one-way shift" operator V. With $\{x_n\}_{n=1,2,\cdots}$ an orthonormal basis, V is defined by $Vx_n = x_{n+1}$, so that V maps the Hilbert space isometrically *into* itself. The same is true for V^m, for each positive integer m. Thus $\|V^m x\| = 1$ for each unit vector x and all positive m; so that (V^m) does not tend strongly to 0. However, if E_n is the orthogonal projection with range spanned by $x_{n+1}, x_{n+2}, \cdots, E_n V^n = V^n$. Thus $(V^n)^* E_n = (V^n)^*$; and $(V^n)^*$ tends to 0 strongly (since $\|(V^n)^*\| = 1$ and E_n tends strongly to 0). Despite this lack of continuity of the adjoint operation on the set of all bounded operators, it is strong-operator continuous on the normal operators. This fact (which seems to be new) is proved in Theorem 4.1.

2. Notation and preliminaries. We deal with complex Hilbert space $\mathscr{H}$. The algebra of all bounded operators on $\mathscr{H}$ is denoted by $\mathscr{B}(\mathscr{H})$. We use the notation $\boldsymbol{R}^n$ for real Euclidean n-space, and $\boldsymbol{C}$ for the set of complex numbers. The strong-operator topology on $\mathscr{B}(\mathscr{H})$ is the point-open topology on $\mathscr{B}(\mathscr{H})$ induced by the metric topology on $\mathscr{H}$ (so that (A_n) converges to A in the strong-operator topology when $(A_n x)$ converges to Ax for each x in $\mathscr{H}$). The strong-operator topology on the Cartesian product $\mathscr{B}(\mathscr{H}) \times \cdots \times \mathscr{B}(\mathscr{H})$ is the product strong-operator topology (with similar terminology for each subset of $\mathscr{B}(\mathscr{H}) \times \cdots \times \mathscr{B}(\mathscr{H})$).

DEFINITION 2.1. If $A_1, \cdots, A_n$ is a commuting set of bounded self-adjoint operators on $\mathscr{H}$, the subset $\{(\rho(A_1), \cdots, \rho(A_n))\}$ of $\boldsymbol{R}^n$, where ρ ranges through the nonzero multiplicative linear functionals on the C^*-algebra $\mathfrak{A}$ generated by $A_1 \cdots, A_n$ and I is called the spectrum of $(A_1, \cdots, A_n)$ $(= \bar{A})$ and denoted by $\sigma(\bar{A})$. If S is a subset of $\boldsymbol{R}^n$, the set of such $\bar{A}$ with $\sigma(\bar{A}) \subseteq S$ is denoted by $\mathscr{B}(\mathscr{H})_S$. Since $\mathfrak{A}$ is commutative, it is isomorphic to the algebra of continuous complex-valued functions on some compact Hausdorff space X. If $A \rightarrow \hat{A}$ is the isomorphism and f is a real-valued continuous function defined on S, we denote by $f(A_1, \cdots, A_n)$ the (self-adjoint) operator in $\mathfrak{A}$ corresponding to $x \rightarrow f(\hat{A}_1(x), \cdots, \hat{A}_n(x))$.

In accordance with this definition, $\mathscr{B}(\mathscr{H})_{\boldsymbol{R}}$ will denote the set of all bounded self-adjoint operators on $\mathscr{H}$. We use the notation $\mathscr{B}(\mathscr{H})_S$ to denote the set of bounded normal operators on $\mathscr{H}$ with spectra in S, when S is a subset of $\boldsymbol{C}$. Accordingly, $\mathscr{B}(\mathscr{H})_{\boldsymbol{C}}$ will denote the set of all bounded normal operators on $\mathscr{H}$. With f a continuous real-valued function defined on a subset S of $\boldsymbol{R}^n$, we use the symbol f, again, to denote the mapping of $\mathscr{B}(\mathscr{H})_S$ into $\mathscr{B}(\mathscr{H})_{\boldsymbol{R}}$ described in Definition 2.1. By means of Spectral Theory, we can ascribe a meaning to $f(A_1, \cdots, A_n)$ for certain noncontinuous functions f on S.

For a point $x = (x_1, \cdots, x_n)$ in $\boldsymbol{R}^n$, we denote by $|x|$ the sum $|x_1| + \cdots + |x_n|$ and by $\|x\|$ the number $(\sum x_j^2)^{1/2}$. We use the notation "f is $O(x)$", for a function f defined on a subset S of $\boldsymbol{R}^n$, to mean $x \rightarrow f(x)/\|x\|$ is bounded on S outside some bounded subset of S.

3. Operator functions of several variables. We determine conditions, in this section, for real-valued functions defined on certain subsets S of $\boldsymbol{R}^n$ to be strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$. Basic to this discussion is the:

REMARK 3.1. The mapping $(A_1, \cdots, A_n) \to A_1 \cdots A_n$ is strong-operator continuous on bounded subsets of $\mathscr{B}(\mathscr{H}) \times \cdots \times \mathscr{B}(\mathscr{H})$.

LEMMA 3.2. *If f is a continuous mapping of $\boldsymbol{R}^n$ into $\boldsymbol{R}$ which tends to a limit at ∞ then f is strong-operator continuous on $\mathscr{B}(\mathscr{H})_{\boldsymbol{R}}n$.*

Proof. Let X be the one-point compactification of $\boldsymbol{R}^n$; and let $\mathscr{A}$ be the algebra of finite linear combinations of products $f_1 \cdots f_n$ where f_j is a continuous real-valued function on $\boldsymbol{R}$ of bound not exceeding 1 and tending to a limit at ∞. The constant function 1 is in $\mathscr{A}$. If a and b are distinct points of X and both lie in $\boldsymbol{R}^n$, suppose a and b have distinct j th coordinates a_j, b_j. We can construct f_j on $\boldsymbol{R}$ such that $\|f_j\| = 1$, $f_j(a_j) = 1$ and f_j vanishes outside an open interval about a_j not containing b_j. Choosing f_k to be 1 for $k \neq j$, $f_1 \cdots f_n$ is in $\mathscr{A}$ has the value 1 at a and 0 at b. If a is in $\boldsymbol{R}^n$, say $a = (a_1, \cdots, a_n)$, construct f_j on $\boldsymbol{R}$ with $f_j(a_j) = 1 = \|f_j\|$ and f_j vanishing at ∞. Then $f_1 \cdots f_n$ is 1 at a and 0 at ∞. Thus $\mathscr{A}$ contains the constants and separates points of X. From the Stone-Weierstrass Theorem, $\mathscr{A}$ is uniformly dense in $C(X)$.

If we have established the strong-operator continuity of each function in $\mathscr{A}$ on $\mathscr{B}(\mathscr{H})\boldsymbol{R}^n$, then that of f will follow. In fact, given $A_1, \cdots, A_n$ commuting self-adjoint operators and x in $\mathscr{H}$, select vectors $y_k^{(j)}$, $k = 1, \cdots, m$; $j = 1, \cdots, n$ in $\mathscr{H}$ such that if $\|[A_j - B_j]y_k^{(j)}\| < 1$, $k = 1, \cdots, m$; $j = 1, \cdots, n$, with $B_1, \cdots, B_n$ commuting self-adjoint operators, then

$$\|[g(A_1, \cdots, A_n) - g(B_1, \cdots, B_n)]x\| < 1/3 ,$$

where g is a function in $\mathscr{A}$ such that $\|f - g\| < 1/3\|x\|$. For this $(B_1, \cdots, B_n)$,

$$\begin{aligned} &\|[f(A_1, \cdots, A_n) - f(B_1, \cdots, B_n)]x\| \\ &\quad \leqq \|[f(A_1, \cdots, A_n) - g(A_1, \cdots, A_n)]x\| \\ &\qquad + \|[g(A_1, \cdots, A_n) - g(B_1, \cdots, B_n)]x\| \\ &\qquad + \|[g(B_1, \cdots, B_n) - f(B_1, \cdots, B_n)]x\| < 1 . \end{aligned}$$

The continuity of g in $\mathscr{A}$ will follow from that of the products $f_1 \cdots f_n$ used in the definition of $\mathscr{A}$. Since each f_j is strong-operator continuous on $\mathscr{B}(\mathscr{H})_{\boldsymbol{R}}$ [1; Lemma 5] and $(A_1, \cdots, A_n) \to A_1 \cdots A_n$ is strong-operator continuous on $\mathscr{B}(\mathscr{H})_1 \times \cdots \times \mathscr{B}(\mathscr{H})_1$, where $\mathscr{B}(\mathscr{H})_1$ is the unit ball in $\mathscr{B}(\mathscr{H})$, the composite mapping

$$\begin{aligned} &(B_1, \cdots, B_n) \to (f_1(B_1), \cdots, f_n(B_n)) \to f_1(B_1) \cdots f_n(B_n) \\ &\quad = (f_1 \cdots f_n)(B_1, \cdots, B_n) \end{aligned}$$

(recall that $\|f_j\| \leqq 1$ so that $\|f_j(B_j)\| \leqq 1$, and compare Remark 3.1) is continuous.

With f a real-valued function defined on a subset S of $\boldsymbol{R}^n$, a *jump point* for f is a point in S^- for which $(\overline{\lim} f)(p) - (\underline{\lim} f)(p) > 0$. If f is continuous, the jump points for f lie in $S^- - S$. We shall need the following lemma whose proof is a slight variation of the proof of [1; Th. 2] to suit the present circumstances.

LEMMA 3.3. *If h is a bounded, real-valued function on the subset S of $\boldsymbol{R}^n$ and the set J of jump points for h is such that $J^- \cap S$ is null, then h is strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$.*

Proof. Suppose $(A_1, \cdots, A_m)(= \bar{A})$ is in $\mathscr{B}(\mathscr{H})_S$. Then $\sigma(\bar{A})$ is a compact subset of $\boldsymbol{R}^n$ disjoint from J^- (by assumption). Let O be a bounded open set containing $\sigma(\bar{A})$ with closure O^- disjoint from J^-. Since no jump point for h lies in $O^- \cap S^-$, assigning $(\overline{\lim}\, h)(p)$ to each p of this set defines a continuous extension of h to it. Finally, let h_0 be the function on $\boldsymbol{R}^n$ which is some continuous extension h_1 to O^- of this function (Tietze Extension Theorem), h on S and 0 elsewhere. We note that, with k continuous on $\boldsymbol{R}^n$, 1 on $\sigma(\bar{A})$ and 0 outside O, $h_0k(= p)$ and $1 - k + h_0k(= q)$ are continuous on $\boldsymbol{R}^n$. On the complement of O^-, k and hence p are 0; so that p is continuous at points of this complement (an open set). On $O^- - O$, k is 0; so that p is 0 and continuous at points of $O^- - O$, since $p = h_1k$ on O^- with h_1 continuous, hence bounded, on O^-. On O, an open set, p is the product of the two continuous functions h_1 and k. Since p and $q - 1$ vanish outside O and are continuous on $\boldsymbol{R}^n$, they are strong-operator continuous on $\mathscr{B}(\mathscr{H})\boldsymbol{R}^n$ (from Lemma 3.2).

As $p = q = h$ on $\sigma(\bar{A})$, $p(\bar{A}) = q(\bar{A}) = h(\bar{A})$. Combining this with the identity $h_0 = (1 - h_0)p + h_0q$ which becomes $h = (1 - h)p + hq$ on S; we have, for each $\bar{B}$ in $\mathscr{B}(\mathscr{H})_S$,

$$h(\bar{B}) - h(\bar{A}) = [1 - h(\bar{B})][p(\bar{B}) - p(\bar{A})] + h(\bar{B})[q(\bar{B}) - q(\bar{A})] .$$

The strong-operator continuity of h on $\mathscr{B}(\mathscr{H})_S$ follows from that of p and q, this last identity and the fact that h is bounded on S.

THEOREM 3.4. *If f is a real-valued function defined and $O(x)$ on a subset S of $\boldsymbol{R}^n$, bounded on bounded subsets of S and such that $J^- \cap S$ is null, where J is the set of jump points of f, then f is strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$.*

Proof. We note, first, that if g is bounded, with jump points in J, and real-valued on S, and h is strong-operator continuous on

$\mathscr{B}(\mathscr{H})_S$, then gh is strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$. This follows from the strong-operator continuity of h, of g (from Lemma 3.3), and the inequality:

$$\begin{aligned}&\|[g(\bar{A})h(\bar{A}) - g(\bar{B})h(\bar{B})]x\| \\ &\qquad \leqq \|g(\bar{A})\| \cdot \|[h(\bar{A}) - h(\bar{B})]x\| + \|[g(\bar{A}) - g(\bar{B})]h(\bar{B})x\| ,\end{aligned}$$

where $\bar{A} = (A_1, \cdots, A_n)$ and $\bar{B} = (B_1, \cdots, B_n)$ are in $\mathscr{B}(\mathscr{H})_S$.

Let $g(x)$ be $f(x)/(1 + |x|)$ for x in S,

$$\begin{aligned}x = (x_1, \cdots, x_n),\ |x| = |x_1| + \cdots + |x_n| (&\geqq \|x\| \\ &= (\textstyle\sum |x_j|^2)^{1/2}) .\end{aligned}$$

From the hypothesis, g is bounded on S; and its set of jump points is contained in J. Once we note that $x \to |x|$ is strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$, the strong-operator continuity of g on $\mathscr{B}(\mathscr{H})_S$ (Lemma 3.3) and the argument of the first paragraph gives the strong-operator continuity on $\mathscr{B}(\mathscr{H})_S$ of h defined by $h(x) = (1 + |x|)g(x)$, for x in S. Since $|(A_1, \cdots, A_n)| = |A_1| + \cdots + |A_n|$, the strong-operator continuity of $x \to |x|$ on $\mathscr{B}(\mathscr{H})\boldsymbol{R}^n$ will follow from that of $A \to |A|$ on $\mathscr{B}(\mathscr{H})_{\boldsymbol{R}}$. Let $r(x)$ be x for $|x| \leqq 1$ and $|x|/x$ for $1 \leqq |x|$; $s(x)$ be $xr(x)$; and $t(x)$ be $|x| - s(x)$. Since r is bounded, t vanishes outside $[-1, 1]$ and both are continuous on $\boldsymbol{R}$, [1; Th. 2, Lemma 5] shows that both are strong-operator continuous on $\mathscr{B}(\mathscr{H})_{\boldsymbol{R}}$. So is s, from the argument of the first paragraph. Thus, $x \to |x| = s(x) + t(x)$ is strong-operator continuous on $\mathscr{B}(\mathscr{H})_{\boldsymbol{R}}$.

Our thanks are due to R. J. Blattner for suggesting '$1 + |x|$' in place of '$|x|$' to define g thereby correcting and simplifying the argument.

LEMMA 3.5. *With S a subset of $\boldsymbol{R}^n$, if the real-valued function f is strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$ it is continuous on S, bounded on bounded subsets of S, and $O(x)$.*

Proof. Assuming f is defined on $\mathscr{B}(\mathscr{H})_S$ (by Spectral Theory) and restricting f to $\{(a_1I, \cdots, a_nI) : (a_1, \cdots, a_n) \text{ in } S\}$, we see that f must be continuous on S if it is to be strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$. With x_0 in S, the translated set, $S - x_0$, contains 0; and g defined on $S - x_0$ by $g(x) = f(x + x_0) - f(x_0)$ is bounded on bounded subsets of $S - x_0$ and $O(x)$ if and only if f is bounded on bounded subsets of S and $O(x)$. We may assume that 0 lies in S and $f(0)$ is 0.

Suppose that f is not $O(x)$. Then there is a sequence (x_m) in S with $\|x_m\| \to \infty$ such that $m\|x_m\| \leqq |f(x_m)|$. Taking $\mathscr{L}_2(0, 1)$ for $\mathscr{H}$ (relative to Lebesgue measure), we show that f is not strong-operator continuous at $(0, \cdots, 0)$ on n-tuples of multiples (by coordinates of the x_m's) of a projection in the multiplication algebra of $\mathscr{L}_2(0, 1)$. More specifically, given $\psi_1^{(j)}, \cdots, \psi_m^{(j)}$, $j = 1, \cdots, n$, in $\mathscr{L}_2(0, 1)$, we find a subset X of $(0, 1)$ having positive measure and r such that, with $g_j = a_j$ on X and 0 on the complement of X, where $x_r = (a_1, \cdots, a_n)$, $\int | g_j \psi_p^{(j)} |^2 \leqq 1$ for $j = 1, \cdots, n$; $p = 1, \cdots, m$; while $\int | f \circ g |^2 \geqq 1$, where $g(s) = (g_1(s), \cdots, g_n(s))$ for s in $(0, 1)$. With M_{g_j} the multiplication operator (on $\mathscr{L}_2(0, 1)$) corresponding to g_j, $(M_{g_1}, \cdots, M_{g_n}) \in \mathscr{B}(\mathscr{H})_S$ and $f(M_{g_1}, \cdots, M_{g_n}) = M_{f \circ g}$. Thus

$$\| f(M_{g_1}, \cdots, M_{g_n})(1) \| \geqq 1$$

despite the fact that $\| M_{g_j} \psi_p^{(j)} \| \leqq 1$ for $p = 1, \cdots, m$; $j = 1, \cdots, n$. Hence f is not strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$.

It remains to locate X and r as described. With $\psi = \sum_{j,p} | \psi_p^{(j)} |$, let X_k be the subset of $(0, 1)$ at which $|\psi|$ does not exceed k, for $k = 1, 2, \cdots$. Since ψ is in $\mathscr{L}_2(0, 1)$, X_k has positive measure a for some k. Choose r larger than k so that $\| x_r \|^2 ak^2 \geqq 1$; and let b be $(\| x_r \|^2 ak^2)^{-1}$. Then $0 < b \leqq 1$, and there is a subset X of X_k with measure ab. Defining g_j to be a_j at points of X and 0 at points of the complement of X, where $x_r = (a_1, \cdots, a_n)$, we have

$$\int | g_j \psi |^2 = \int_X | g_j \psi |^2 \leqq k^2 \int | g_j |^2 = k^2 \, | a_j |^2 ab \leqq k^2 \, \| x_r \|^2 ab = 1 \, ,$$

while

$$\int | f \circ g |^2 \geqq \int_X | f \circ g |^2 = f(x_r)^2 ab \geqq r^2 \, \| x_r \|^2 ab = r^2 k^{-2} \geqq 1 \, .$$

Since

$$\sum_{p=1}^{m} \int | g_j \psi_p^{(j)} |^2 \leqq \int | g_j \psi |^2 \leqq 1 \, ,$$

we have $\int | g_j \psi_p^{(j)} |^2 \leqq 1$, for $p = 1, \cdots, m$ and $j = 1, \cdots, n$.

Suppose, next, that f is not bounded on some bounded subset of S. Then there is a sequence (x_m), with x_m in S, tending to some point x_0 in $\boldsymbol{R}^n$ such that $m \leqq | f(x_m) |$. As before, translating by $-x_0$, we may assume that $x_0 = 0$. Select $(b_1, \cdots, b_n)$ in S with $| b_j | \leqq 1$, $j = 1, \cdots, n$.

We shall show that f is not strong-operator continuous at $(b_1 I, \cdots, b_n I)$ on $\mathscr{B}(\mathscr{H})_S$. Given $\psi_k^{(j)}$, $j = 1, \cdots, n$; $k = 1, \cdots, m$ in $\mathscr{L}_2(0, 1)$; let $\psi = \sum_{p,k} | \psi_k^{(p)} |$. There is a subset X of $(0, 1)$ with positive Lebesgue measure a such that $\int_X | \psi |^2 \leqq 1/4$. Choose r so that $| a_j | \leqq 1$, $j = 1, \cdots, n$, where $x_r = (a_1, \cdots, a_n)$; and so that $a \, | f(x_r) - f(b_1, \cdots, b_n) |^2 \geqq 1$. Let

g_j be a_j on X and b_j on the complement of X in $(0, 1)$.

As before, $f(M_{g_1}, \cdots, M_{g_n}) = M_{f \circ g}$, where $g(s) = (g_1(s), \cdots, g_n(s))$ for s in $(0, 1)$. Thus $\| [f(M_{g_1}, \cdots, M_{g_n}) - f(b_1 I, \cdots, b_n I]1 \| \geqq 1$, since

$$\int | f \circ g - f(b_1, \cdots, b_n) |^2$$
$$= \int_X | f(x_r) - f(b_1, \cdots, b_n) |^2 = a \, | f(x_r) - f(b_1, \cdots, b_n) |^2 \geqq 1 \, .$$

But

$$\int | (b_j - g_j)\psi_k^{(j)} |^2 \leqq \int | b_j - g_j |^2 (\textstyle\sum_{p, k} | \psi_k^{(p)} |)^2$$
$$= \int | (b_j - g_j)\psi |^2 = \int_X | (b_j - a_j)\psi |^2 \leqq 4 \int_X | \psi |^2 \leqq 1 \, ,$$

so that $\| (b_j I - M_{g_j})\psi_k^{(j)} \| \leqq 1$, for $j = 1, \cdots, n$ and $k = 1, \cdots, m$. As $(M_{g_1}, \cdots, M_{g_n}) \in \mathscr{B}(\mathscr{H})_S$, f is not strong-operator continuous at $(b_1 I, \cdots, b_n I)$ on $\mathscr{B}(\mathscr{H})_S$, completing the proof of this lemma.

Combining Theorem 3.4 with the foregoing lemma, we have:

THEOREM 3.6. *If S is a subset of $\boldsymbol{R}^n$ such that $(S^- - S)^- \cap S$ is empty then a real-valued function f defined on S is strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$ if and only if it is continuous on S, bounded on bounded subsets of S, and $O(x)$.*

Proof. In view of Theorem 3.4 and Lemma 3.5, we need note only that the set of jump points of a function continuous on S is a subset of $S^- - S$.

For a closed set S, $S^- - S$ is empty; and, for an open set S, $S^- - S$ is closed. In both cases $(S^- - S)^- \cap S$ is empty; from which we have:

COROLLARY 3.7. *If S is a closed or open subset of $\boldsymbol{R}^n$, a real-valued function defined on S is strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$ if and only if it is continuous on S, bounded on bounded subsets of S, and $O(x)$.*

Of course, the continuity assumption makes the hypothesis of boundedness on bounded subsets superfluous when S is a closed set.

4. Functions of normal operators.

The key to applying the results of §3 to the normal operators $\mathscr{B}(\mathscr{H})_C$ is:

THEOREM 4.1. *The adjoint operation is strong-operator continuous on $\mathscr{B}(\mathscr{H})_C$.*

Proof. The assertion follows from:

$$\begin{aligned} \| (B^* - A^*)x \|^2 &= \| Bx \|^2 - \| Ax \|^2 + (x, (A - B)A^*x) \\ &+ ((A - B)A^*x, x) \leqq \| (A - B)x \| (\| Ax \| + \| Bx \|) \\ &+ 2 \| A - B)A^*x \| \| x \| . \end{aligned}$$

(Our original proof of Theorem 4.1 was somewhat longer. A. Hoppenwasser found a simpler proof which led us to the argument above.)

THEOREM 4.2. *With f a complex-valued function defined on a subset S of $\boldsymbol{C}$ for which $(S^- - S)^- \cap S$ is empty (in particular, for S open or closed), f is strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$ if and only if f is continuous, bounded on bounded subsets of S, and $O(z)$.*

Proof. Adopting the usual identification of $\boldsymbol{C}$ with $\boldsymbol{R}^2$, we may view S as a subset of $\boldsymbol{R}^2$. With $z = a + ib$, a and b real, let $f(z) = g(a, b) + ih(a, b)$, $g(a, b)$ and $h(a, b)$ real. Then g and h are defined on S. Moreover, g and h are continuous on S, bounded on bounded subsets of S, and $O(z)$, if and only if the same are true for f. This is the case if and only if g and h are strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$, from Theorem 3.6.

We conclude the proof by showing that

$$A_1 + iA_2 \rightarrow g(A_1, A_2) + ih(A_1, A_2) = f(A_1 + iA_2)$$

is strong-operator continuous if and only if g and h are. Since

$$A_1 + iA_2 \rightarrow \Big(\frac{1}{2}[A_1 + iA_2 + (A_1 + iA_2)^*] , \\ \frac{1}{2i}[A_1 + iA_2 - (A_1 + iA_2)^*]\Big) = (A_1, A_2)$$

is a strong-operator homeomorphism of $\mathscr{B}(\mathscr{H})_C$ with $\mathscr{B}(\mathscr{H})\boldsymbol{R}^2$, from Theorem 4.1, it will suffice to show that $(A_1, A_2) \rightarrow g(A_1, A_2) + ih(A_1, A_2)$ is strong-operator continuous if and only if g and h are. All that requires proof is the strong-operator continuity of g and h on $\mathscr{B}(\mathscr{H})_S$ from that of $(A_1, A_2) \rightarrow g(A_1, A_2) + ih(A_1, A_2)$ on $\mathscr{B}(\mathscr{H})_S$. From Theorem 4.1, $(A_1, A_2) \rightarrow [g(A_1, A_2) + ih(A_1, A_2) + (g(A_1, A_2) + ih(A_1, A_2))^*]/2 = g(A_1, A_2)$ is strong-operator continuous on $\mathscr{B}(\mathscr{H})_S$, and similarly for $(A_1, A_2) \rightarrow h(A_1, A_2)$.

We have made no distinction between $\mathscr{B}(\mathscr{H})_S$ with S a subset of $\boldsymbol{C}$, referring to the normal operators on $\mathscr{H}$ with spectra in S, and $\mathscr{B}(\mathscr{H})_S$ with S a subset of $\boldsymbol{R}^2$, referring to pairs of commuting self-adjoint operator with joint spectrum in S. The context makes clear the sense in which this notation applies; and the argument indicates that there is no essential distinction between the sets designated. Of course, a theorem analogous to Theorem 3.4 holds for functions of normal operators.

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