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A NOTE ON THE ARENS PRODUCTS

JULIEN HENNEFELD

Arens has given a way of defining two Banach algebra products on the second dual of a Banach algebra $\mathscr{A}$. Each of these products extends the original multiplication on $\mathscr{A}$ when $\mathscr{A}$ is canonically imbedded in its second dual $\mathscr{A}^{**}$. In this paper we are concerned with characterizing those Banach algebras with the property that the two Arens products agree on the second duals.

For the case of a commutative Banach algebra $\mathscr{A}$, Gulick [2] gives a necessary and sufficient condition for the two Arens products on $\mathscr{A}^{**}$ to coincide. In § 2 we extend Gulick's result to the case of a noncommutative Banach algebra.

For an arbitrary Banach algebra Arens [1] has given a sufficient condition for the Arens products to coincide. In § 3 we prove that if the Banach algebra has a weak approximate identity, then whenever Arens' condition is satisfied the Banach algebra must be reflexive, and so, of course, the two Arens products coincide.

If $\mathscr{A}$ is a commutative Banach algebra, then $\mathscr{A}^{**}$ is commutative under either Arens product if and only if the two Arens products coincide. This result is proved in Arens [1], and is one reason for studying the question of whether the Arens products coincide. For the Banach algebra of compact operators on a Banach space, the author has shown [3, p. 45], that the question of whether the two Arens products coincide is related to an imbedding problem.

1. Preliminary definitions.

The two Arens products are defined in stages according to the following rules. Let $\mathscr{A}$ be a Banach algebra. Let $A, B \in \mathscr{A}; f \in \mathscr{A}^*; F, G \in \mathscr{A}^{**}$.

DEFINITION 1.1.

$(f *_1 A)B = f(AB)$ This defines $f *_1 A$ as an element of $\mathscr{A}^*$.
$(G *_1 f)A = G(f *_1 A)$ This defines $G *_1 f$ as an element of $\mathscr{A}^*$.
$(F *_1 G)f = F(G *_1 f)$ This defines $F *_1 G$ as an element of $\mathscr{A}^{**}$.

We will call $F *_1 G$ the first Arens product, or the m_1 product.

DEFINITION 1.2.

$(A *_2 f)B = f(BA)$ This defines $A *_2 f$ as an element of $\mathscr{A}^*$.
$(f *_2 F)A = F(A *_2 f)$ This defines $f *_2 F$ as an element of $\mathscr{A}^*$.
$(F *_2 G)f = G(f *_2 F)$ This defines $F *_2 G$ as an element of $\mathscr{A}^{**}$.

$F *_2 G$ is the second Arens product or the m_2 product.

It is proved in Arens [1] that m_1 and m_2 are both Banach algebra products on $\mathscr{A}^{**}$, which extend the original multiplication on $\mathscr{A}$ when it is imbedded in $\mathscr{A}^{**}$.

DEFINITION 1.3. A Banach algebra $\mathscr{A}$ is called Arens regular if the two Arens products coincide on $\mathscr{A}^{**}$.

LEMMA 1.4. *Each of the two Arens products is one-sided weak star continuous. More precisely,*

(a) *If* $F_\alpha \to F$ *under the weak star topology, then* $(F_\alpha *_1 G) \to F *_1 G$ *under the weak star topology.*

(b) *If* $G_\beta \to G$ *under the weak star topology, then* $(F *_2 G_\beta) \to (F *_2 G)$ *under the weak star topology.*

Proof. This is proved in Arens [1, p. 842].

LEMMA 1.5. *The two Arens products agree if one of the factors is in* $\mathscr{A}$. *That is if* $A \in \mathscr{A}$, *and* $F \in \mathscr{A}^{**}$ *then* $A *_1 F = A *_2 F$ *and* $F *_1 A = F *_2 A$.

Proof. Arens [2, p. 843] shows that $F *_1 A = F *_2 A$. The proof that $A *_1 F = A *_2 F$ is similar.

2. **A condition of Gulick's.** For a fixed $f \in \mathscr{A}^*$, let T_f be the operator from $\mathscr{A} \to \mathscr{A}^*$ given by $T_f(A) = f *_1 A$. Gulick [2, p. 123], proves the theorem that when $\mathscr{A}$ is a commutative Banach algebra, then $\mathscr{A}^{**}$ is commutative $\Leftrightarrow$ for each $f \in \mathscr{A}^*$, T_f is a weakly compact operator.

The following theorem is a generalization of the theorem of Gulick to the case of a noncommutative Banach algebra.

THEOREM 2.1. *Let* $\mathscr{A}$ *be a Banach algebra. Then the following conditions are equivalent.*

(1) $\mathscr{A}$ *is Arens regular.*

(2) *For each* $F, G \in \mathscr{A}^{**}$ *and* $f \in \mathscr{A}^*$, *there exists* $F_\alpha, G_\beta \in \mathscr{A}$: $F_\alpha \to F$ *weak star and* $G_\beta \to G$ *weak star and such that*

$$\lim_\alpha \lim_\beta (F_\alpha *_1 G_\beta) f = \lim_\beta \lim_\alpha (F_\alpha *_1 G_\beta) f .$$

(3) *For each* $f \in \mathscr{A}^*$, *the map* $T_f: \mathscr{A} \to \mathscr{A}^*$ *is weakly compact.*

LEMMA 2.2. *If* $F_\alpha \to F$ *and* $G_\beta \to G$ *in the weak star topology where* $F_\alpha, G_\beta \in \mathscr{A}$ *and* $F, G \in \mathscr{A}^{**}$, *then for every* $f \in \mathscr{A}^*$,

(a) $\lim_\alpha \lim_\beta [(F_\alpha *_1 G_\beta)f] = (F *_1 G)f$
(b) $\lim_\beta \lim_\alpha [F_\alpha *_1 G_\beta)f] = (F *_2 G)f$.

Proof.

(a) $\lim_\alpha \lim_\beta [(F_\alpha *_1 G_\beta)f] = \lim_\alpha \lim_\beta (F_\alpha *_2 G_\beta)f$
$= \lim_\alpha (F_\alpha *_2 G)f$ by Lemma 1.4
$= \lim_\alpha (F_\alpha *_1 G)f$ by Lemma 1.5
$= (F *_1 G)f$

(b) $\lim_\beta \lim_\alpha [(F_\alpha *_1 G_\beta)f] = \lim_\beta (F *_1 G_\beta)f$
$= \lim_\beta (F *_2 G_\beta)f = (F *_2 G)f$.

Proof of theorem. (1) $\Leftrightarrow$ (2) is evident from the lemma. Next assume (3). Then, given F, G in the unit ball of $\mathscr{A}^{**}$ and $f \in \mathscr{A}^*$, by Goldstine's theorem pick F_α, G_β in the unit ball of $\mathscr{A}$ such that $F_\alpha \to F$ and $G_\beta \to G$ in the weak star topology. Then $(f *_1 F_\alpha) = T_f(F_\alpha)$ is a net in the image of the unit ball of $\mathscr{A}$, under T_f. Since T_f is weakly compact, there exists a weakly convergent subnet, $f *_1 F_{\alpha_k}$.

In the following argument "weak limit" means in the weak topology on $\mathscr{A}^*$, whereas "lim" means as a net of complex numbers. Then,

$$\begin{aligned} &G[\text{weak} \lim_{\alpha_k} (f_1^* F_{\alpha_k})] \\ &= \lim_{\alpha_k} [G(f *_1 F_{\alpha_k})] \text{ since } \{f *_1 F_{\alpha_k}\} \text{ converges weakly,} \\ &= \lim_{\alpha_k} \lim_{\beta} [G_\beta(f *_1 F_{\alpha_k})] \text{ since } G_\beta \text{ converges weak star to } G \\ &= \lim_{\alpha_k} \lim_{\beta} [(F_{\alpha_k} *_1 G_\beta)f] \,. \end{aligned}$$

Also,

$$\begin{aligned} &G[\text{weak} \lim_{\alpha_k} (f *_1 F_{\alpha_k}] = \lim_{\beta} [G_\beta(\text{weak} \lim_{\alpha_k} (f_1^* F_{\alpha_k}))] \\ &= \lim_{\beta} [\lim_{\alpha_k} G_\beta(f *_1 F_{\alpha_k})] = \lim_{\beta} \lim_{\alpha_k} [(F_{\alpha_k} *_1 G_\beta)f] \,. \end{aligned}$$

Hence, by Lemma 2.2, (3) $\Rightarrow$ (2).

Now assume (1). Let $f \in \mathscr{A}^*$ and let F_α be an arbitrary net in the unit ball of $\mathscr{A}$. Then a subnet F_{α_k} has a weak star limit F in the unit ball of $\mathscr{A}^{**}$. Then for every $G \in \mathscr{A}^{**}$,

$$(F *_1 G)f = \lim_{\alpha_k} [(F_{\alpha_k} *_1 G)f] = \lim_{\alpha_k} G(f *_1 F_{\alpha_k}) \,.$$

However, $(F *_2 G)f = G(f *_2 F)$. Then since we are assuming (1) it follows that $\lim_{\alpha_k} G(f *_1 F_{\alpha_k}) = G(f *_2 F)$.

Since G was arbitrary in $\mathscr{A}^{**}$ and $f *_2 F$ is in $\mathscr{A}^*$, it follows that $f *_2 F$ is the weak limit of $f *_1 F_{\alpha_k}$.

Hence, T_f is weakly compact and (1) $\Rightarrow$ (3).

3. A condition of Arens'. Arens gives a condition [1] involving weakly compact sets of the base algebra which guarantees that a Banach algebra be Arens regular. The condition is: for every $\varepsilon > 0$ there exists a weakly compact set Γ of the unit ball S such that for each $x \in S$, there is an $x' \in \Gamma$ so that if y is any element of the Banach algebra, then $\|(x - x')y\| \leqq \varepsilon \|y\|$. Gulick [2] gives an example of a Banach algebra which is Arens regular but fails to satisfy Arens' sufficient condition. His example is $C(X)$ for X an infinite, compact Hausdorff space. Our next theorem shows that Arens' condition is indeed too restrictive. We show that for a large class of Banach algebra, whenever Arens' condition is satisfied the Banach algebra must be reflexive.

DEFINITION 3.1. Let $\mathscr{A}$ be a Banach algebra. A net y_α of elements of $\mathscr{A}$ of unit norm is called a weak approximate identity if for each $x \in \mathscr{A}$, both $y_\alpha x$ and $x y_\alpha$ approach x in the weak topology.

THEOREM 3.2. *Let $\mathscr{A}$ be a Banach algebra with a weak approximate identity and which satisfies Arens' condition. Then $\mathscr{A}$ is reflexive.*

LEMMA 3.3. *Let X be a Banach space and f_α a net in X^*. If f_α approaches f in X^* in either the weak star topology or the weak topology, then for every $\varepsilon > 0$, eventually $\|f_\alpha\| \geqq \|f\| - \varepsilon$.*

Proof. First, let $f_\alpha \to f$ in the weak topology and suppose there exists an $\varepsilon > 0$ and a subnet f_{α_k} such that $\|f_{\alpha_k}\| < \|f\| - \varepsilon$. By the Hahn Banach theorem there exists an $F \varepsilon X^{**}$ of norm one: $F(f) = \|f\|$. The $F(f_{\alpha_k}) < \|f\| - \varepsilon$ for each α_k and hence $F(f_{\alpha_k})$ cannot approach $F(f)$. This is contradiction. The case of the weak star topology is similar.

Proof of theorem. Suppose $\mathscr{A}$ is not reflexive. Choose x^{**} in $\mathscr{A}^{**}$ of norm one, but not in $\mathscr{A}$, where $\mathscr{A}$ is imbedded in $\mathscr{A}^{**}$ in the canonical way. Then since $\mathscr{A}$ is a norm closed subspace of $\mathscr{A}^{**}$, the distance from x^{**} to $\mathscr{A}$ is >0. That is,

$$d = \operatorname*{glb}_{x \in \mathscr{A}} \|x^{**} - x\| > 0 \, .$$

Since we are assuming that Arens' condition is satisfied, there exists a weakly compact set Γ in $\mathscr{A}$ corresponding to $(1/2)d$. Hence, for each x in the unit ball of $\mathscr{A}$ there exists an x' in Γ: for all y, $\|(x - x')y\| \leqq (1/2)d \|y\|$. Let y_β range over the weak approximate identity. By Lemma 3.3 it follows that for each $\varepsilon > 0$, eventually

$\| x - x' \| - \varepsilon \leqq \| (x - x')y_\beta \|$. Hence $\| x - x' \| \leqq (1/2)d$.

By Goldstine's theorem there exists x_α in the unit ball of $\mathscr{A}$: $x_\alpha \to x^{**}$ in the weak star topology on $\mathscr{A}^{**}$. Let x'_α be an element of Γ such that $\| x_\alpha - x'_\alpha \| \leqq (1/2)d$. By the weak compactness of Γ, a subnet x'_{α_k} converges weakly to $x' \in \Gamma \subset \mathscr{A}$. Hence, $(x_{\alpha_k} - x'_{\alpha_k})$ converges to $x^{**} - x'$ in the weak star topology on $\mathscr{A}^{**}$. We have elements in $\mathscr{A}^{**}$ of norm $\leqq (1/2)d$ converging in the weak star topology to an element of norm $\geqq d$. This is a contradiction.

Correction. Arens [2, p. 846], claims that (c_0) with pointwise multiplication satisfies his sufficient condition and therefore is Arens regular. It is true that (c_0) is Arens regular but it cannot follow from Arens' sufficient condition since c_0 has an approximate identity and is not reflexive.

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COLUMBIA UNIVERSITY