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D-DIMENSION, II. SEPARABLE SPACES AND COMPACTIFICATIONS

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This paper continues the discussion of a new transfinite dimension which was introduced by the author in "D-dimension, I. A new transfinite dimension," Pacific J. Math. vol. 26. In the first part of this paper we show that, for a metric space $X, D(X)$ is an ordinal if and only if each closed subset $Y \subset X$ contains a dense open (in Y) subset each of whose points has a finite-dimensional neighborhood. It follows that if X is complete and separable, then X is weakly countable-dimensional (i.e. the union of a countable number of closed finite-dimensional subsets) if and only if $D(X)$ is an ordinal. It is also shown that, if $\mathrm{Ind}(X)$ exists, then $\mathrm{Ind}(X) < D(X)$; furthermore, if X is compact and $\mathrm{Ind}(X)$ does not exists, then $D(X)$ is not an ordinal. In the second part, it is proved that each weakly infinite-dimensional separable metric space has a compactification with the same D-dimension; an example is given to show that this is not true for all separable metric spaces.

1. Separable spaces. Although the theorems in this section apply to all metric spaces, the principle results (the corollaries) apply only to certain classes of separable spaces. The notation and definitions of [1] will be used.

THEOREM 1. *Let X be any metrizable space. Then $D(X) \neq \Delta$ if and only if each closed subset $Y \subset X$ contains a dense open (in Y) subset each point of which has a finite-dimensional neighborhood.*

Proof. Let R be any space, where $D(R) \neq \Delta$. Then R has a D-representation, $R = \bigcup \{A_\alpha \mid 0 \leqq \alpha \leqq \gamma\}$. If β is the first ordinal such that $Z_\beta = A_\beta - \bigcup \{A_\beta \mid \beta + 1 \leqq \alpha \leqq \gamma\}$ is nonempty, then Z_β is a finite-dimensional open subset of R. Thus every space of D-dimension $< \Delta$ contains a finite-dimensional open subset. Define

$$S = \{x \in R \mid x \text{ has a finite-dimensional neighborhood}\} .$$

Clearly S is open. Suppose that S is not dense in R, then there is an open set $0 \subset R$ such that $0 \cap S = \varnothing$; but then, since D-dimension is monotone, 0 is a space with D-dimension $< \Delta$, and thus 0 contains a finite-dimensional open (in 0 and therefore also in R) subset, which contradicts the definition of S. Therefore, since D-dimension is monotone, we have proved the "only if" part of the Theorem.

We now prove the "if" part of the Theorem. Let

$$S_n = \{x \in X \mid x \text{ has a neighborhood of dimension } (\leqq n)\} .$$

Then each S_n, $n = 0, 1, 2, \cdots$, is open and $S = \bigcup \{S_n\}$ is dense. Define $A_0 = \varnothing$ and

$$A_n = S_n - [(1/n)\text{-neighborhood of } X - S_n] , \qquad n > 0 .$$

Then $\bigcup \{A_n \mid 0 \leqq n < \omega\} = S$. Now repeat the same with $X - S$ in place of X and call the closed (in $X - S$ and therefore in X) subsets so obtained, $A_{\omega+n}$, $n = 0, 1, 2, \cdots$. If $R_{2\omega} \equiv X - \bigcup \{A_\alpha \mid 0 \leqq \alpha \leqq 2\omega\}$ is finite-dimensional, then, setting $R_{2\omega} = A_{2\omega}$, $X = \bigcup \{A_\alpha \mid 0 \leqq \alpha \leqq 2\omega\}$ is a D-representation of X; and if $R_{2\omega}$ is not finite-dimensional, then repeat the above process with $R_{2\omega}$ in place of X and call the closed sets so obtained, $A_{2\omega+n}$, $n = 0, 1, 2, \cdots$. We can repeat this procedure for each limit ordinal, γ, if $X - \bigcup \{A_\alpha \mid 0 \leqq \alpha < \gamma\}$ is not finite-dimensional, and obtain $X = \bigcup \{A_\alpha \mid 0 \leqq \alpha < \gamma + \omega\} \cup R_{\gamma+\omega}$ which fails to be a D-representation, if at all, only because $R_{\gamma+\omega}$ is not finite-dimensional. Since at the γ-th stage, for each limit ordinal $\delta < \gamma$, the sets, $\bigcup \{A_\alpha \mid \delta \leqq \alpha < \delta + \omega\}$ are each nonempty and disjoint, it must be true that γ has cardinality at most equal to the cardinality of X. (In fact, Theorem 10 of [1], implies that the cardinality of γ is less than the weight of X.) Thus at some stage the above process terminates and we obtain a D-representation for X, and thus $D(X) < \Delta$.

DEFINITION. A space X is *weakly countable-dimensional* if it is the union of a countable number of closed finite-dimensional subsets. Some authors call this strongly countable-dimensional, but I am sticking to what seems to me to be the most widely used term.

COROLLARY. *Let X be a complete separable metric space. $D(X) \neq \Delta$ if and only if X is weakly countable-dimensional.*

Proof. If $D(X) \neq \Delta$, then $D(X)$ is countable ([1], Th. 10); and thus its $D(X)$-D-representation gives X as the countable union of closed finite-dimensional subsets. If X is weakly countable-dimensional then the Baire Category Theorem easily leads to the conclusion that has a dense open subset, each point of which has a finite-dimensional neighborhood. Since each closed subset of X is also complete and weakly countable-dimensional, the corollary follows.

REMARK. The space, $\mathscr{E}$, in § 2 of [4] is weakly countable-dimensional, but $D(\mathscr{E}) = \Delta$.

THEOREM 2. *Let X be any metric space. If* Ind(X) *exists, then* Ind$(X) \leqq D(X)$.

Proof. The proof will be by transfinite induction on Ind(X). By Theorem 1 of [1], Ind$(X) = D(X)$, if Ind(X) is finite. Assume that the theorem is true for all $\beta < \alpha$, and let X be a space such that Ind$(X) = \alpha$. We shall show that $D(X) \geqq \alpha$ in one of two cases depending on whether α is or is not a limit ordinal.

If α is a limit ordinal, then for each $\beta < \alpha$ there is a closed subset $X_\beta \subset X$ such that Ind$(X_\beta) = \beta$. By induction, $D(X_\beta) \geqq \beta$, for each $\beta < \alpha$. It follows that $D(X) \geqq \alpha$.

If α is not a limit ordinal then $\alpha = \gamma + n$, where γ is a limit ordinal and n is a positive integer; and there are in X two disjoint closed sets, E and F, such that every set which separates E from F has dimension at least $\gamma + (n-1)$. Let γ be the collection of all closed subsets of X which have D-dimension $\gamma + (n-1)$. For each $V \in \mathscr{V}$, pick a reduced $D(V)$-D-representation (see Theorem 6 of [1]) and let $A(V)$ be the last (or γ-th) set in this representation. Thus the closure of each open (in V) set which intersects $A(V)$ has D-dimension at least γ. Let closure $(\cup\{A(V) \mid V \in \mathscr{V}\}) = A$. Then Theorem 7 of [1] will give us that $D(X) \geqq \alpha = \gamma + n$, if we show that A contains a closed n-dimensional subset. Since Ind(X) exists, so does Ind(A), and thus it suffices to show that Ind$(A) \geqq n$. The latter will follow if we show that every set S that separates $E \cap A$ from $F \cap A$ in A has dimension at least $n-1$. Let S be such a separating set, then there is a set S' which separates E from F in X and such that $S = S' \cap A$. (The existence of such an S' follows easily from the fact that X is completely normal.) Thus Ind$(S') \geqq \gamma + (n-1)$ and, by the induction hypothesis, $D(S') \geqq \gamma + (n-1)$. If $D(S') \geqq \gamma + n = \alpha$, then the theorem follows because D-dimension is monotone. If $D(S') = \gamma + (n-1)$, then $S' \in \mathscr{V}$ and $A(S') \subset S' \cap A = S$; therefore, Ind$(S) \geqq$ Ind$(A(S')) = n - 1$. This finishes the proof.

REMARK. Let X be the disjoint union, $\bigcup\{I^n \mid n = 1, 2, 3, \cdots\}$. Then by Axiom II of [1] $D(X) = \omega$; but, Ind(X) does not exist (see [3], p. 177).

COROLLARY. *If X is compact, then* Ind$(X) \leqq D(X)$, *where we let* Ind$(X) = \Delta$ *if* Ind(X) *is not an ordinal.*

Proof. The inequality is clearly true if $D(X) = \Delta$. If $D(X) \neq \Delta$, then, by the corollary to Theorem 1, X is weakly countable dimensional, but then it is well known (see [4], Th. 4) that Ind(X) exists $(\neq \Delta)$ and thus the corollary follows from Theorem 2.

REMARK. Smirnov ([4], Th. 3) describes a compact metric space, X, which is not weakly countable-dimensional such that Ind(X) exists.

Therefore, $D(X) = \Delta$, but $\mathrm{Ind}(X) < \Delta$.

2. Compactifications.

DEFINITION. A space X is *weakly infinite-dimentsonal* if, for every countable number of pairs $\{F_i, G_i\}$, $i = 1, 2, \cdots$ of disjoint closed subsets of X, there exist open sets $U_i, i = 1, 2, \cdots$ such that $F_i \subset U_i \subset X - G_i$, and

$$\bigcap \{\text{boundary}\,(U_i) \mid i = 1, 2, \cdots, k\} = \varnothing\ ,$$

for some k. (See [3], p. 161.)

THEOREM 3. *Every weakly-infinite-dimensional separable metric space is contained as a dense subset of a compact metric space of the same D-dimension.*

Proof. Let X be a weakly-infinite-dimensional separable metric space. We now make use of a theorem of Sklyarenko, [3], Th. VI. 6, which asserts that X is equal to the union $Y \cup \{A_n \mid 0 \leqq n < \omega\}$, where Y is compact (and weakly-infinite-dimensional), each A_n is open and finite-dimensional, and each sequence of points in $X - Y$ without a cluster point is eventually contained in one of the A_n. Define F_i, $0 \leqq i < \omega$, to be the set of all points which are at least a distance of $(1/2^i)$ from Y. Clearly each F_i is closed in X. Suppose that F_i is not finite-dimensional, then for each positive integer j there is a point $p \in F_i - \bigcup \{A_n \mid 0 \leqq n \leqq j\}$ because $\bigcup \{A_n \mid 0 \leqq n \leqq j\}$ is finite-dimensional. The sequence $\{p_j\}$ must have a cluster point, p, because it is not eventually in any of the A_n. But, since E_i is closed, $p \in F_i$ and thus, for some k, $p \in A_k$; but this contradicts the fact that A_k is open. Thus each F_i must be closed and finite dimensional. Let $X = \bigcup \{C_\alpha \mid 0 \leqq \alpha \leqq \gamma\}$ be a $D(X)$-D-representation of X if $D(X) \neq \Delta$, and define $G_i = F_i \cup C_i$; if $D(X) = \Delta$, let $G_i = E_i$. According to [2], (§ 40, VII, 5), X can be considered as a dense subset of a compact metric space X^* such that, if G_i^* denotes the closure of G_i in X^*, then G_i^* has the same dimension of G_i. Every sequence of points in X either has a cluster point in Y or is eventually contained in G_j, for some j. Therefore, $X^* = \bigcup \{G^*{}_i \mid 0 \leqq i < \omega\} \cup Y$. For essentially the same reason $\bigcup \{G^*{}_i \mid n \leqq i < \omega\} \cup Y$ is compact, for each n. Thus $X^* = \bigcup \{G_i^* \mid 0 \leqq i < \omega\} \cup \bigcup \{C_\alpha \cup Y \mid \omega \leqq \alpha \leqq \gamma\}$ is a $D(X)$-D-representation of X^* and therefore $D(X^*) = D(X)$.

REMARK. Let $Q^\omega = \{p\} \cup \bigcup \{I^n \mid 0 < n < \omega\}$ be the one-point-compactification of the disjoint union of n-cells, one for each n. Let X be the subset of $Q^{\omega+1} = [0, 1] \times Q^\omega$, defined as

$$(\{1\} \times \{p\}) \cup (\{0\} \times \{p\}) \cup ([0, 1] \times \bigcup \{I^n \mid 0 < n < \omega\}) .$$

It is easy to show that $D(X) = \omega$. Let X^* be any compact metric space that contains X as a dense subset. Then each of the closed sets $X^* - ([0, 1] \times \bigcup \{I^n \mid 0 < n < k\})$ connects $\{0\} \times Q^\omega$ to $\{1\} \times Q^\omega$ in X^*. Therefore their intersection $X^* - ([0, 1] \times \bigcup \{I^n \mid 0 < n < \omega\})$ also connects and is therefore at least 1-dimensional. It can now be checked that X^* satisfies the hypothesis of Theorem 7 of [1] with the conclusion that $D(X^*) > \omega + 1$. Also note that $Q^{\omega+1}$ is a compactification of X and that $D(Q^{\omega+1}) = \omega + 1$ ([1], Th. 12). This leads to the following.

CONJECTURE. *Every separable metric space X is contained as a dense subset of a compact metric space X^*, such that $D(X^*) = D(X)$ or $D(X) + 1$.*

A proof of the conjecture might be constructed as follows, for spaces with have a $D(X)$-D-representation $X = \bigcup \{A_\alpha \mid 0 \leqq \alpha \leqq \omega\}$ (i.e. $\omega \leqq D(X) < 2\omega$). Embed A_ω in a contractable compact space B with $\mathrm{Ind}\,(B) \leqq \mathrm{Ind}\,(A_\omega) + 1$ (e.g. B might be the cone over a compactification of A_ω of dimension $\mathrm{Ind}\,(A_\omega)$) in such a way that $Y = \bigcup \{A_\alpha \mid 0 \leqq \alpha < \omega\} \cup B$ is weakly-infinite-dimensional.

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