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BOUNDED SERIES AND HAUSDORFF MATRICES FOR ABSOLUTELY CONVERGENT SEQUENCES

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If f is a function from $[0, 1]$ to the complex plane and c is a complex sequence, then the Hausdorff matrix $H(c)$ for c and a sequence $L(f,c)$ are defined:

$$H(c)_{np} = \binom{n}{p}\sum_{q=0}^{n-p}(-1)^q\binom{n-p}{q}c_{p+q}$$

$$L(f, c)_n = \sum_{p=0}^{n} H(c)_{np}f(p/n)\ .$$

This paper consists of the following theorem and two converses to it.

THEOREM 1. If A is a complex sequence and $\sum_{p=0}^{\infty} A_p$ is bounded (there is a number B such that if n is a nonnegative integer then $|\sum_{p=0}^{n} A_p| < B$), f is a function from $[0, 1]$ to the complex plane such that if $0 \leqq x < 1$ then $f(x) = \sum_{p=0}^{\infty} A_p x^p$, and c is an absolutely convergent sequence ($\sum_{p=0}^{\infty}| c_{p+1} - c_p |$ converges), then $L(f,c)$ converges. Furthermore, if c has limit d, $L(f, c)$ has limit $\sum_{p=0}^{\infty} A_p(c_p - d) + f(1)\cdot d$.

Let $\mathscr{F}$ be the collection of all functions f satisfying the hypothesis of Theorem 1. $\mathscr{S}$ be the set of all absolutely convergent sequences. Theorem 1 and its converses show that $\mathscr{F}$ and $\mathscr{S}$ are related in the same way that certain sets of continuous functions are related to certain sets of sequences in [3]. There, for example, the set of functions analytic on the unit disc with power-series absolutely convergent at 1 is shown to be related to the set of bounded sequences.

In Theorem 3 we use the following result due to J. S. MacNerney [2, p. 56] and A. Jakimovski [1], which, incidentally, was used in [3] the relate the set of polynomials to the set of all sequences.

THEOREM A. *If f is a polynomial and c is a complex sequence then $L(f, c)$ converges. Furthermore, if $f(z) = \sum_{p=0}^{n} A_p z^p$ for each complex number z, then $L(f, c)$ has limit $\sum_{p=0}^{n} A_p c_p$.*

The following lemma is useful in the proofs of Theorems 1 and 2.

LEMMA 1. *If M is an infinite, complex, lower-triangular matrix, these are equivalent:*

(1) *There is a positive number B such that if each of q, n, and m is a nonnegative integer then $|\sum_{p=q}^{m} M_{np}| < B$ and there is a*

sequence A such that, for each nonnegative integer p, the sequence $M[\ ,p]$ has limit A_p.

(2) *If x is an absolutely convergent sequence with limit* 0, *then $M\cdot x$ converges* ($[M\cdot x]_n = \sum_{p=0}^{n} M_{np}x_p$).

Furthermore, if (1) *holds and x is an absolutely convergent sequence with limit* 0 *then $M\cdot x$ has limit $\sum_{p=0}^{\infty} A_p x_p$.*

Proof. First, suppose that (1) holds and that x is an absolutely convergent sequence. If each of q and m is a nonnegative integer, then $|\sum_{p=q}^{m} A_p| \leqq B$ and

$$\sum_{p=q}^{m} A_p x_p = \sum_{p=q}^{m} (x_p - x_{p+1}) \sum_{j=q}^{p} A_j + x_{m+1} \sum_{j=q}^{m} A_j \ ,$$

from which we see that $\sum_{p=0}^{\infty} A_p x_p$ converges.

If each of m and n is a positive integer, then $(M\cdot x)_n - \sum_{p=0}^{m-1} A_p x_p = \sum_{p=0}^{m-1}(M_{np} - A_p)x_p + \sum_{p=m}^{n}(x_p - x_{p+1}) \sum_{j=m}^{p} M_{nj} + x_{n+1} \sum_{j=m}^{n} M_{nj}$ and, from this, we see that $M\cdot x$ has limit $\sum_{p=0}^{\infty} A_p x_p$.

Second, suppose that (2) holds. Sequences having the value 1 at one nonnegative integer and 0 at the others show us that there is a sequence A such that, for each nonnegative integer p, the sequence $M[\ ,p]$ has limit A_p.

Let S be the set of all absolutely convergent sequences with limit 0 and let N be a function from S to the numbers such that if x is in S then $N(x) = \sum_{p=0}^{\infty} |x_p - x_{p+1}|$. $\{S, N\}$ is a complete, normed, linear space.

For each nonnegative integer n, let T_n be a function from S to the complex numbers such that if x is in S then $T_n(x) = (M\cdot x)_n$, and note that T_n is a continuous linear transformation.

For each x in S the sequence $T(x)$ converges, so that by the "principle of uniform boundedness" there is a number B such that if n is a nonnegative integer and x is in S and $N(x) \leqq 1$ then $|T_n(x)| \leqq B$.

If each of q and m is a nonnegative integer, let $z(q, m)$ be the sequence such that if p is a nonnegative integer, then $z(q, m) = 1/2$ if $q \leqq p \leqq m$ and $z(q, m)_p = 0$ otherwise, and notice that $z(q, m)$ is in S and $N(z(q, m)) \leqq 1$.

If each of m and q is a nonnegative integer,

$$\left|\frac{1}{2} M_{mq}\right| = |T_m(z(q, q))| \leqq B \ ,$$

and if n is a nonnegative integer,

$$\left|\frac{1}{2} \sum_{j=q}^{m} M_{nj}\right| = \left| T_n(z(q, m+1)) - M_{n,m+1}\cdot\frac{1}{2}\right| \leqq 2B \ ,$$

and Lemma 1 is proved.

LEMMA 2. *Suppose that $B > 0$ and v is a nondecreasing nonnegative-number-sequence and b is a complex sequence such that if each of n and q is a nonnegative integer then $|\sum_{p=q}^{n} b_p| \leqq B$. Then, if m is a nonnegative integer, $|\sum_{p=0}^{m} b_p v_p| \leqq v_m B$.*

Proof. The lemma is true if m is 0. Suppose that m is a positive integer such that, for each sequence b as described above, $|\sum_{p=0}^{m-1} b_p v_p| \leqq v_{m-1} B$.

Let a be a complex sequence such that if each of n and q is a nonnegative integer, then $|\sum_{p=q}^{n} a_p| \leqq B$. Let b be the sequence such that if p is a nonnegative integer, then $b_p = a_p$ if $p < m - 1$, $b_{m-1} = a_{m-1} + a_m$, and $b_p = 0$ if $p \geqq m$. Then

$$\begin{aligned}\left|\sum_{p=0}^{m} a_p v_p\right| &= \left|\sum_{p=0}^{m-1} a_p v_p + a_m v_{m-1} + a_m(v_m - v_{m-1})\right| \\ &\leqq \left|\sum_{p=0}^{m-1} b_p v_p\right| + |a_n|(v_m - v_{m-1}) \\ &\leqq B v_{m-1} + B(v_m - v_{m-1}) = B v_m ,\end{aligned}$$

and Lemma 2 is proved.

Let us define a matrix Y such that if each of p and k is a nonnegative integer, then

$$Y_{pk} = \sum_{q=0}^{p} (-1)^{p+q}\binom{p}{q} q^k ,$$

where we interpret 0^0 as 1. Without proof we state

LEMMA 3. *If each of p and k is a nonnegative integer, then $Y_{p+1,k+1} = (p+1)(Y_{pk} + Y_{p+1,k})$; $Y_{pp} = p!$; $Y_{pk} \geqq 0$ for $p > k$, and, if n is a positive integer $Y_{n,k+1} n^{-k-1} \geqq Y_{nk} n^{-k}$; $\lim_{k\to\infty} Y_{nk} n^{-k} = 1$; and, therefore, $Y_{nk} n^{-k} \leqq 1$.*

If n is a positive integer, f is a function from $[0, 1]$ to the complex plane and c is a complex sequence then

$$L(f, c)_n = \sum_{p=0}^{n} c_p \binom{n}{p} \sum_{q=0}^{p} (-1)^{p+q} \binom{p}{q} f(q/n) ,$$

and we let M^f be a matrix such that if p is a nonnegative integer, then

$$M^f_{np} = \binom{n}{p} \sum_{q=0}^{p} (-1)^{p+q} \binom{p}{q} f(q/n) .$$

Proof of Theorem 1. Suppose that A, f, B and c are as in the theorem.

Let n be a positive integer.

$$\begin{aligned} M^f_{nn} &= f(1) + \sum_{q=0}^{n-1} (-1)^{n+q}\binom{n}{q} f(q/n) \\ &= f(1) + \sum_{q=0}^{n-1} (-1)^{n+q}\binom{n}{q} \sum_{k=0}^{\infty} A_k q^k n^{-k} \\ &= f(1) - \sum_{k=0}^{\infty} A_k(1 - n^{-k} Y_{nk}) \,. \end{aligned}$$

If each of m and q is a nonnegative integer $|\sum_{p=q}^{m} A_p| \leqq 2B$, so that by Lemma 2 and Lemma 3,

$$\left|\sum_{k=0}^{m} A_k n^{-k} Y_{nk}\right| \leqq n^{-m} Y_{nm}(2B) \leqq 2B \,,$$

and

$$| M^f_{nn} | \leqq | f(1) | + B + 2B = |f(1)| + 3B \,.$$

Suppose, now, that m is a nonnegative integer less than n.

$$\begin{aligned} \sum_{p=0}^{m} M^f_{np} &= \sum_{p=0}^{m} \binom{n}{p} \sum_{k=p}^{\infty} A_k n^{-k} Y_{pk} \\ &= \sum_{k=0}^{\infty} A_k \sum_{p=0}^{m} \binom{n}{p} n^{-k} Y_{pk} \,. \end{aligned}$$

For each nonnegative integer k let G_k be $\sum_{p=0}^{m} \binom{n}{p} n^{-k} Y_{pk}$ and note that

$$\begin{aligned} n^{k+1}[G_k - G_{k+1}] &= \sum_{p=0}^{m} n\binom{n}{p} Y_{pk} - \sum_{p=0}^{m} \binom{n}{p} Y_{p,k+1} \\ &= \sum_{p=0}^{m} n\binom{n}{p} Y_{pk} - \sum_{p=1}^{m} \binom{n}{p} p Y_{pk} - \sum_{p=1}^{m} \binom{n}{p} p Y_{p-1,k} \\ &= \sum_{p=0}^{m} \left[(n-p)\binom{n}{p} - \binom{n}{p+1}(p+1) \right] Y_{pk} \\ &\qquad + (n-m)\binom{n}{m} Y_{mk} \\ &= (n-m)\binom{n}{m} Y_{mk} \geqq 0 \,, \end{aligned}$$

so that G is a nonincreasing sequence. $G_0 = 1$. The sequence $1 - G$ is nondecreasing and nonnegative valued, so that, for each nonnegative integer r,

$$\left|\sum_{k=0}^{r} A_k(1 - G_k)\right| \leqq 2B \,,$$

$$\left|\sum_{k=0}^{r} A_k G_k \right| \leqq 4B \,,$$

and

$$\left|\sum_{p=0}^{m} M_{np}^{f}\right| \leqq 4B ,$$

and M^f satisfies condition (1) of Lemma 1.

Let c have limit d. $M\cdot(c-d)$ converges, $L(f, c) = M\cdot(c-d) + L(f, d)$, so that $L(f, c)$ converges with limit $\sum_{p=0}^{\infty} A_p(c_p - d) + d\cdot f(1)$.

THEOREM 2. *Suppose that f is a function from $[0, 1]$ to the complex plane and f is continuous on $[0, 1)$. Suppose that, for each absolutely convergent sequence c, $L(f, c)$ converges. Then there is a complex sequence A such that $\sum_{p=0}^{\infty} A_p$ is bounded and, if x is in $[0, 1)$, $f(x) = \sum_{p=0}^{\infty} A_p x^p$.*

Proof. Since each sequence dominated by a geometric sequence with ratio less than 1 is absolutely convergent, we know from [3, Th. 3] that there is a complex sequence A such that if x is in $[0, 1)$ then $f(x) = \sum_{p=0}^{\infty} A_p x^p$, and A_p is the limit of the sequence $M^f[\ \ , p]$.

By Lemma 1 there is a positive number B such that if each of n and m is a positive integer then $|\sum_{p=0}^{m} M_{np}^{f}| \leqq B$, and, consequently, $|\sum_{p=0}^{m} A_p| \leqq B$.

THEOREM 3. *Suppose that c is an infinite complex sequence such that, for each function f, analytic on the unit disc and defined at 1, such that $\sum_{p=0}^{\infty} f^{(p)}(0)/p!$ is bounded, $L(f, c)$ converges. Then c is absolutely convergent.*

Proof. Suppose that $\sum_{p=0}^{\infty} |c_{2p+1} - c_{2p}|$ is not bounded.

Let $\mathscr{F}_0$ be the set of all functions f as described in the theorem such that $f(1) = 0$. For each member f of $\mathscr{F}_0$ let $N(f)$ be the least number L such that if n is a nonnegative integer then

$$\left|\sum_{p=0}^{n} f^{(p)}(0)/p!\right| \leqq L .$$

$\{\mathscr{F}_0, N\}$ is a complete, normed linear space.

For each positive integer n let T_n be the continuous linear transformation from $\mathscr{F}_0$ to the plane such that if f is in $\mathscr{F}_0$ then $T_n(f) = L(f, c)_n$. By the "principle of uniform boundedness" there is a number B such that if f is in $\mathscr{F}_0$ and $N(f) \leqq 1$ then $|T_n(f)| \leqq B$ for each positive integer n.

Let m be a positive integer such that $\sum_{p=1}^{m} |c_{2p+1} - c_{2p}| > 2B$. Let A be a sequence such that $A_0 = A_1 = 0$ and if p is a positive integer then $A_{2p+1} = -A_{2p} = 0$ if $c_{2p+1} = c_{2p}$ or $p > m$ and

$$A_{2p+1} = -A_{2p} = |c_{2p+1} - c_{2p}|/(c_{2p+1} - c_{2p})$$

otherwise.

Let f be the polynomial such that if z is a complex number then

$$f(z) = \sum_{p=0}^{m} \{A_{2p+1}z^{2p+1} + A_{2p}z^{2p}\} .$$

f is in $\mathscr{F}_0$ and $N(f) \leqq 1$. By Theorem A there is a positive integer n such that

$$\left| L(f, c)_n - \sum_{p=0}^{2m+1} A_p c_p \right| < \sum_{p=1}^{m} | c_{p2+1} - c_{2p} | - 2B$$

so that

$$| L(f, c)_n | > \left| \sum_{p=0}^{2m+1} A_p c_p \right| - \sum_{p=1}^{m} \left| c_{2p+1} - c_{2p} \right| + 2B = 2B > B ,$$

which is a contradiction. So $\sum_{p=0}^{\infty} | c_{2p+1} - c_{2p} |$ is bounded.

Similarly $\sum_{p=1}^{\infty} | c_{2p} - c_{2p-1} |$ is bounded. Hence $\sum_{p=0}^{\infty} | c_p - c_{p+1} |$ converges and c is absolutely convergent.

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