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THE SECOND COUSIN PROBLEM WITH BOUNDED DATA

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Given a complex manifold M, an open covering $\mathscr{V} \equiv \{V_\alpha\}_{\alpha \in A}$, and, for each $\alpha \in A$ a function f_α holomorphic on V_α such that for all $\alpha, \alpha' \in A$, $f_\alpha f_{\alpha'}^{-1}$ is a zero-free holomorphic function on $V_\alpha \cap V_{\alpha'}$, the associated second Cousin problem is the problem of showing the existence of a function F holomorphic on M such that for all α, Ff_α^{-1} is a zero free function holomorphic on V_α. In the present paper we consider an analogous problem in the case that M is the open unit polycylinder $U^N = \{(z_1, \cdots, z_N) \in \mathbf{C}^N : |z_1| < 1, \cdots, |z_N| < 1\}$, that the functions f_α are required to be bounded and that the sought function F is also required to be bounded.

If Ω is an open set in $\mathbf{C}^N$, we denote by $H^\infty[\Omega]$ the algebra of functions bounded and holomorphic in Ω. In the first section of the present paper we will establish the following result.

THEOREM I.1. *Let $\mathscr{V} = \{V_\alpha\}_{\alpha \in A}$ be an open covering of $\bar{U}^N$, the closure of U^N, and for each α, let $f_\alpha \in H^\infty[V_\alpha \cap U^N]$. If for all $\alpha, \beta \in A$, $f_\alpha f_\beta^{-1}$ is an invertible element of $H^\infty[V_\alpha \cap V_\beta \cap U^N]$, then there exists $F \in H^\infty[U^N]$ such that for all α, Ff_α^{-1} is an invertible element of $H^\infty[V_\alpha \cap U^N]$.*

An analogous theorem is valid in the case that we consider functions with continuous boundary values; it is essentially contained in the thesis of Douady. In § II we consider this result briefly and give an application of it to an interpolation theorem.

I. Proof of theorem I.1.

A remark concerning the statement may be in order. Although the functions f_α are only defined on subsets of U^N and although we seek a function F defined only in U^N, it seems necessary to require that $\mathscr{V}$ be an open covering of $\bar{U}^N$ even if we require $\mathscr{V}$ to be finite. A relevant example, with $\mathscr{V}$ an open cover of $U = U^1$ which consists of two sets is as follows. Let λ be an arc with and points 1 and -1 which is contained, except for its end points, in the lower half of the open unit disc. If λ approaches the real axis with sufficient rapidity, the sequence $\{1 - 1/n\}_{n=1}^\infty$ will be the zero set of a function f_1 holomorphic and bounded on the set $V_1 = \{z \in U : z \text{ lies above } \lambda\}$. We can choose f_1 to be continuous on $V_1 \cup (\text{interior } \lambda)$ and of modulus 1 on λ. We take for V_2 the part of U lying on and below λ together with the set $\{z \in V_1 : |f_1(z)| > 1/2\}$.

Define f_2 on V_2 to be identically 1. Then $\{V_1, V_2\}$ is an open cover for U. On $V_1 \cap V_2$, $f_1 f_2^{-1}$ is bounded and bounded away from 0. But since the sequence $\{1 - 1/n\}$ does not satisfy the Blaschke condition, it is not the zero set of any $f \in H^\infty[U]$.

The proof of the theorem depends on the following lemma.

LEMMA I.2. *Let $f = \tilde{u} + i\tilde{v}$ be a function holomorphic in U^N. Let λ_1 and λ_2 be disjoint arcs in the unit circle, and define V_j, $j = 1, 2$, to be the union of U, the interior of λ_j, and the exterior, including ∞, of the unit disc. If $\tilde{u}$ is bounded we may write, for $z \in U^N$, $f(z) = f_1(z) + f_2(z)$ where f_j is holomorphic and has bounded real part in $V_j \times U^{N-1}$.*

We shall defer the proof of the lemma for the moment and proceed to show how the lemma implies the theorem. It is more convenient to work in the polycube than in the polycylinder. We set

$$\Delta^+ = \{z \in \mathbf{C}^N : z_j = x_j + iy_j,\ -\frac{1}{2} < x_1 < 1,\ |x_2|, \cdots, |x_N|,\ |y_1|, \cdots, |y_N| < 1\},$$

and

$$\Delta^- = \{z \in \mathbf{C}^N : -1 < x_1 < \frac{1}{2},\ |x_2|, \cdots, |x_N|, |y_1|, \cdots, |y_N| < 1\}.$$

Assume given F an invertible element of $H^\infty[\Delta^+ \cap \Delta^-]$. We assert that on $\Delta^+ \cap \Delta^-$,

$$F = F^+ F^- \tag{I.1.1}$$

where F^+ and F^- are invertible elements of $H^\infty[\Delta^+]$ and $H^\infty[\Delta^-]$ respectively.

We may write $F = e^G$ where G is holomorphic in $\Delta^+ \cap \Delta^-$ and where $Re\ G$ is bounded; in general Im G will not be bounded. The lemma implies that $G = G_+ + G_-$ where G_+ and G_- are holomorphic in Δ^+ and Δ^- respectively and have bounded real parts: This can be proved in the following way.

Let α be a conformal map of U onto $\{z = x + iy : |x|, |y| < 1\}$ and β one from U onto $\{z = x + iy : |x| < 1/2, |y| < 1\}$. Then the map $\Phi : U^N \to \Delta^+ \cap \Delta^-$ given by $\Phi(z) = (\beta(z_1), \alpha(z_2), \cdots, \alpha(z_N))$ is an analytic homeomorphism. Let

$$A_1 = \left\{\frac{1}{2} + iy : |y| \leqq 1\right\},$$

$$A_2 = \left\{-\frac{1}{2} + iy : |y| \leqq 1\right\},$$

and let λ_1, λ_2 be respectively the preimages of Λ_1 and Λ_2 under β. Let V_1 and V_2 be the domains in the Riemann sphere constructed from λ_1 and λ_2 respectively as in the statement of the lemma. Given a point $w, |w| > 1$, we define $\beta_1(w)$ to be $1 - [\beta(1/\bar{w})]^-$. (Here $[\cdot]^-$ is used to indicate complex conjugation.) This definition effects a continuation of β to a conformal one-to-one mapping of V_1 into C in such a way that the range of the continued map, β_1, contains

$$\left\{z = x + iy : |y| < 1, -\frac{1}{2} < x < 1\right\}.$$

In the same way β continues to a conformal map β_2 of V_2 into C such that $\beta_2[V_2]$ contains

$$\left\{z = x + iy : |y| < 1, -1 < x < \frac{1}{2}\right\}.$$

Define $\Phi_j : V_j \times U^{N-1} \to \mathbf{C}^N$ to be the natural continuation of Φ from U^N into the set $V_j \times U^{N-1}$. The range of Φ_1 contains Δ^+, that of Φ_2, Δ^-.

Apply the lemma to the function $G \circ \Phi$ to get $G \circ \Phi = G_1 + G_2$ where G_j is holomorphic and has bounded real part in $V_j \times U^{N-1}$. But then $G_j \circ \Phi_j^{-1}$ is holomorphic and has bounded real part in Δ^+ if $j = 1$, in Δ^- if $j = 2$, and we have, since $\Phi_1^{-1} = \Phi_2^{-1} = \Phi^{-1}$ on $\Delta^+ \cap \Delta^-$, that in $\Delta^+ \cap \Delta^-$, $G = G_1 \circ \Phi_1^{-1} + G_2 \circ \Phi_2^{-1}$. This is a decomposition of G of the desired kind.

For the function F_+, we take $\exp(G_1 \circ \Phi_1^{-1})$, and for F_- we take $\exp(G_2 \circ \Phi_2^{-1})$. This choice of F_+ and F_- gives the decomposition of F which we seek.

As soon as we have the decomposition (I.1.1.) for invertible elements of $H^\infty[\Delta^+ \cap \Delta^-]$, the theorem can be established by a patching argument familiar in this context. (See, e.g. [1].) For the rest of the argument, let us set $\Delta_1^1 = \Delta^+, \Delta_1^2 = \Delta^-$. Suppose given $\mathscr{V} = \{V_\alpha\}$, an open covering of the closure of the polycube $\Delta = \Delta_1^1 \cup \Delta_1^2$, and for each $\alpha, f_\alpha \in H^\infty[V_\alpha \cap \Delta]$ such that $f_\alpha f_\beta^{-1}$ is invertible in $H^\infty[\Delta \cap V_\alpha \cap V_\beta]$. Suppose the theorem is false so that for no $F \in H^\infty[\Delta]$ is it the case that for all α, Ff_α^{-1} is invertible in $H^\infty[V_\alpha \cap \Delta]$. Then it cannot be that the induced problems on Δ_1^1 and Δ_1^2 are both solvable. That is to say, it cannot be the case that there exist F_1 and F_2 in $H^\infty[\Delta_1^1]$ and $H^\infty[\Delta_1^2]$ respectively such that $F_j f_\alpha^{-1}$ is, for all α, an invertible element of $H^\infty[V_\alpha \cap \Delta_1^j]$.

Suppose that such F_1 and F_2 exist. The function $F_1F_2^{-1}$ is then an invertible element of $H^\infty[\Delta_1^1 \cap \Delta_1^2]$. Since $\mathscr{V}$ is an open cover for the closed polycube $\bar{\Delta}$, finitely many of the elements of $\mathscr{V}$, say $V_1, \cdots, V_p$ cover $\Delta_1^1 \cap \Delta_1^2$. Thus, there are ε and $M > 0$ such that for

$j = 1, 2, \cdots, p$, $|F_1 f_j^{-1}|$ and $|F_2 f_j^{-1}|$ are at least ε but no more than M on $\Delta_1^1 \cap V_j$ and $\Delta_1^2 \cap V_j$ respectively. Thus, if $z \in \Delta_1^1 \cap \Delta_1^2$ so that $z \in V_j$, say, then

$$|F_1(z)F_2(z)^{-1}| = |(F_1(z)f_j(z))(F_2(z)f_j(z))^{-1}|$$

and this is no more than M/ε and no less than ε/M. Consequently, the decomposition (I.1.1.) applies to $F_1F_2^{-1}$ and provides G_1 and G_2, invertible elements of $H^\infty[\Delta_1^1]$ and $H^\infty[\Delta_1^2]$ respectively, such that on $\Delta_1^1 \cap \Delta_1^2$ we have

$$F_1F_2^{-1} = G_1G_2$$

whence

$$F_1G_1^{-1} = F_2G_2 .$$

Thus, we can define $F \in H^\infty[\Delta]$ by setting

$$F(z) = \begin{cases} F_1(z)G_1^{-1}(z) & z \in \Delta_1^1 \\ F_2(z)G_2(z) & z \in \Delta_1^2 . \end{cases}$$

Consider now Ff_α^{-1} on V_α. On $V_\alpha \cap \Delta_1^1$, this is $(FF_1^{-1})(F_1f_\alpha^{-1})$ and so is an invertible element of $H^\infty[V_\alpha \cap \Delta_1^1]$ and on $V_\alpha \cap \Delta_1^2$ it is $(FF_2^{-1})(F_2f_\alpha^{-1})$ and so is invertible in $H^\infty[V_\alpha \cap \Delta_1^2]$. Thus, for all α, Ff_α^{-1} is invertible in $H^\infty[V_\alpha \cap \Delta]$. By hypothesis no such F exists, so we must conclude that either F_1 or F_2 does not exist.

Suppose F_1 does not exist. Set

$$\Delta_2^1 = \left\{z \in \Delta_1^1: \quad -\frac{1}{2} < y_1 < 1\right\}$$

$$\Delta_2^2 = \left\{z \in \Delta_1^1: \quad -1 < y_1 < \frac{1}{2}\right\} .$$

Arguing as in the last paragraph, there do not exist functions $F_2^{(1)}$ and $F_2^{(2)}$ in $H^\infty[\Delta_2^1]$ and $H^\infty[\Delta_2^2]$ respectively such that for all α, $F_2^{(j)}f_\alpha^{-1}$ is invertible in $H^\infty[\Delta_2^j]$, $j = 1, 2$. Thus, the induced problem is not solvable on both Δ_2^1 and Δ_2^2.

Iterating this procedure, proceeding cyclicly through the real coordinates of $\mathbf{C}^N$ we obtain a nested sequence $\Delta_1^1 \supset \Delta_2^{k_2} \supset \Delta_3^{k_3} \supset \cdots$ of cubes with diameter decreasing to zero on none of which we are able to solve the induced problem. This leads to a contradiction, though, for $\mathscr{V}$ is an open cover for the closure of Δ so for some N and some α, $\Delta_n^{k_n} \subset V_\alpha$ if $n \geqq N$, and the function f_α is then a solution to the induced problem on $\Delta_n^{k_n}$ for all $n \geqq N$.

Thus, it remains only to prove the lemma.

Proof of Lemma 1.2. It is convenient to establish at the outset certain notations which will be useful throughout the proof. We will denote by Z^N the set of N-tuples of integers. Let $E = \{n = (n_1, \cdots, n_N) \in Z^N: n_1 \geqq 0, \cdots, n_N \geqq 0 \text{ or else } n_1 \leqq 0, \cdots, n_N \leqq 0\}$. Given $n \in Z^N$, let $E^+(n) = \{m \in Z^N: n_1 \leqq m_1, \cdots, n_N \leqq m_N\}$, $E^-(n) = \{m \in Z^N: n_1 \geqq m_1, \cdots, n_N \geqq m_N\}$. If $z = (r_1 e^{i\theta_1}, \cdots, r_N e^{i\theta_N})$ is in U^N and $\varphi = (\varphi_1, \cdots, \varphi_N) \in R^N$, put

$$P(z, \varphi) = \sum_{n \in Z^N} r_1^{|n_1|} \cdots r_N^{|n_N|} e^{in\cdot(\theta - \varphi)}$$

and

$$K(z, \varphi) = \sum_{n \in E} r_1^{|n_1|} \cdots r_N^{|n_N|} e^{in\cdot(\theta-\varphi)} .$$

Here we use $n\cdot(\theta - \varphi)$ as abbreviation for $n_1(\theta_1 - \varphi) + \cdots + n_N(\theta_N - \varphi_N)$. The kernel P is the N dimensional Poisson kernel. A short calculation shows that

$$\text{(I.2.1)} \qquad K(z, \varphi) = Re\left\{\frac{2}{(1 - z_1 e^{-i\varphi_1}) \cdots (1 - z_N e^{-i\varphi_N})}\right\} - 1 .$$

A preliminary reduction of the problem seems desirable. The function f of the lemma may be written as the sum $f = f_0 + f_1$ where $f_0(z_1, \cdots, z_N) = f(0, z_2, \cdots, z_N)$ and $f_1 = f - f_0$. The function f_0 is holomorphic in the product of the Riemann sphere and U^{N-1}, and it has bounded real part. Thus, it is enough to prove the lemma with f replaced by f_1. Let $f_1 = u + iv$; the function u is bounded. For $0 \leqq \rho_1, \cdots, \rho_N < 1$, we may write

$$\text{(I.2.2)} \qquad u(\rho_1 e^{i\theta_1}, \cdots, \rho_N e^{i\theta_N}) = \sum_{k \in E} \hat{u}(k) \rho_1^{|k_1|} \cdots \rho_N^{|k_N|} e^{ik\cdot\theta}$$

where $\hat{u}(k)$ denotes the k^{th} Fourier coefficient of the function u:

$$\hat{u}(k) = \left(\frac{1}{2\pi}\right)^N \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} u(e^{i\varphi_1}, \cdots, e^{i\varphi_N}) e^{-ik\cdot\varphi} d\varphi_1 \cdots d\varphi_N .$$

Since $u = Re\, f_1$ and since $f_1(0, z_2, \cdots, z_N) = 0$, it follows that $\hat{u}(0, k_2, \cdots, k_N) = 0$ for all choices of $k_2, \cdots, k_N$. Thus the summation in (I.2.2) extends over the set $E^+(1, 0, \cdots, 0) \cup E^-(-1, 0, \cdots, 0)$.

Let h be a real function of class $\mathscr{C}^\infty$ on the circle which vanishes identically on a neighborhood of λ_1 and is identically one on a neighborhood of λ_2. Let h have Fourier expansion $h(e^{i\varphi_1}) = \sum_{n=-\infty}^{\infty} c_m e^{im\varphi_1}$. Note that since h is real, $c_{-m} = \bar{c}_m$. Define g_1 and g_2 by means of

$$g_1(z) = \left(\frac{1}{2\pi}\right)^N \int u(\varphi) h(e^{i\varphi_1}) \left\{\frac{2}{(1 - z_1 e^{-i\varphi_1}) \cdots (1 - z_N e^{-i\varphi_N})} - 1\right\} d\varphi$$

and

$$g_2(z) = \left(\frac{1}{2\pi}\right)^N \int u(\varphi)\{1 - h(e^{i\varphi_1})\} \left\{\frac{2}{(1 - z_1 e^{-i\varphi_1}) \cdots (1 - z_N e^{-\varphi_N})} - 1\right\} d\varphi .$$

Here we have written $u(\varphi)$ instead of $u(e^{i\varphi_1}, \cdots, e^{i\varphi_N})$ and $d\varphi$ instead of $d\varphi_1 \cdots d\varphi_N$. The integrals extend over $-\pi \leqq \varphi_1, \cdots, \varphi_N \leqq \pi$. Since h is identically zero on λ_1, it follows that the function g_1 is holomorphic in $V_1 \times U^{N-1}$. Similarly, g_2 is holomorphic in $V_2 \times U^{N-1}$. We shall show that g_1 and g_2 have bounded real parts. We have $\operatorname{Re} g_1 + \operatorname{Re} g_2 = u$ whence for some purely imaginary constant γ, $f_1 = g_1 + (g_2 + \gamma)$ is a decomposition of f_1 of the kind we seek. It is enough to establish the boundedness of $\operatorname{Re} g_1$.

Since u and h are real, we have

$$\operatorname{Re} g_1(z) = \left(\frac{1}{2\pi}\right)^N \int u(\varphi) h(e^{i\varphi_1}) K(z, \varphi) d\varphi$$

$$(\text{I.2.3}) \qquad = c_0 \left(\frac{1}{2\pi}\right)^N \int u(\varphi) K(z, \varphi) d\varphi + \sum_{m=1}^{\infty} \left(\frac{1}{2\pi}\right)^N \int u(\varphi)[c_m e^{im\varphi_1} + \bar{c}_m e^{-im\varphi_1}] K(z, \varphi) d\varphi \,.$$

Let μ_m be the measure on T^N whose Fourier transform $\hat{\mu}_m$ is the characteristic function of $\{n \in \boldsymbol{Z}^N\colon -m + 1 \leqq n_1 \leqq -1\}$ and let ν_m be such that $\hat{\nu}_m$ is the characteristic function of $\{n \in \boldsymbol{Z}^N\colon 1 \leqq n_1 \leqq m - 1\}$; the measures μ_m and ν_m are of norm no more than $C \log(1 + m)$ for some absolute constant C. Define A_m on T^N by

$$A_m(\varphi) = c_m e^{im\varphi_1}(u - u * \mu_m)(\varphi) + \bar{c}_m e^{-im\varphi_1}(u - u * \nu_m)(\varphi)$$

where $u * \mu$ denotes the convolution of the function u and the measure μ. The Fourier series of the function u is

$$\sum\{\hat{u}(k) e^{ik\cdot\varphi}\colon k \in E^+(1, 0, \cdots, 0) \cup E^-(-1, 0, \cdots, 0)\}$$

so that of A_m is

$$c_m \sum\{\hat{u}(k) e^{i(k\cdot\varphi + m\varphi_1)}\colon k \in E^+(1, 0, \cdots, 0) \cup E^-(-m, 0, \cdots, 0)\}$$
$$+ \bar{c}_m \sum\{\hat{u}(k) e^{i(k\cdot\varphi - m\varphi_1)}\colon k \in E^+(m, 0, \cdots, 0) \cup E^-(-1, 0, \cdots, 0)\} \,.$$

Since $\hat{u} = 0$ off $E^+(1, 0, \cdots, 0) \cup E^-(-1, 0, \cdots, 0)$, it follows from the definition of $K(z, \varphi)$ and $P(z, \varphi)$, that

$$\left(\frac{1}{2\pi}\right)^N \int u(\varphi)[c_m e^{im\varphi_1} + \bar{c}_m e^{-im\varphi_1}] K(z, \varphi) d\varphi$$
$$= \left(\frac{1}{2\pi}\right)^N \int A_m(\varphi) P(z, \varphi) d\varphi \,.$$

Also,
$$\left(\frac{1}{2\pi}\right)^N \int u(\varphi) K(z, \varphi) = \left(\frac{1}{2\pi}\right)^N \int u(\varphi) P(z, \varphi) d\varphi \,.$$

Thus the series (I.2.3) is bounded term by term by the series $2C \| u \|_\infty \sum_{m=0}^{\infty} | c_m | (1 + \log(1 + m))$ provided $z \in U^N$. Since $c_m = O(m^{-2})$, it follows that $\operatorname{Re} g_1$ is bounded in U^N.

We must show that $\operatorname{Re} g_1(z_1, z_2, \cdots, z_N)$ is bounded for $|z_1| > 1$, $|z_2|, \cdots, |z_N| < 1$.[1] If $|z_1| < 1$, set $z_1^* = 1/\bar{z}_1$ so $|z_1^*| > 1$. From the definition of g_1, we find, after a short calculation, that

$$g_1(z_1, z_2, \cdots, z_N) - g_1(z_1^*, z_2, \cdots, z_N)$$

$$= \left(\frac{1}{2\pi}\right)^N \int_{-\pi}^{\pi} h(\varphi_1)P(z_1, \varphi_1)d\varphi_1 \int \frac{2u(e^{i\varphi_1}, e^{i\varphi_2}, \cdots, e^{i\varphi_N})}{(1 - z_2e^{-i\varphi_2}) \cdots (1 - z_Ne^{-i\varphi_N})} d\varphi_2 \cdots d\varphi_N .$$

Here $P(z_1, \varphi_1)$ is the one dimensional Poisson kernel. The real part of the inner integral is bounded in φ_1, as may be seen from the fact that if we replace in the first part of our argument the function h by the function identically 1 and then restrict the resulting function of $z_1, \cdots, z_N$ to the hyperplane $z_1 = 0$, our argument shows that the resulting function of $z_2, \cdots, z_N$ is uniformly bounded in U^{N-1}. Thus,

$$\operatorname{Re}\{g_1(z_1, z_2, \cdots, z_N) - g_1(z_1^*, z_2, \cdots, z_N)\}$$

is bounded, and the lemma is established.

II. The case of continuous boundary values. If K is a set in $\mathbf{C}^N$, let $\mathscr{A}[K]$ denote the algebra of functions continuous on $\bar{K}$ and holomorphic in its interior. Similarly, if E is a commutative Banach algebra with identity, let $\mathscr{A}[K, E]$ be the algebra of all continuous E-valued functions on $\bar{K}$ which are analytic in its interior. There are several formally different definitions of analytic E-valued functions which are in fact equivalent. For a discussion of these see [4]. For the sake of definiteness, let us say that $F: \Omega \to E$, Ω an open set in $\mathbf{C}^N$, is analytic if given $z^0 = (z_1^0, \cdots, z_N^0) \varepsilon \Omega$, there is an expansion

$$F(z) = \sum e_{j_1 \cdots j_N}(z_1 - z_1^0)^{j_1} \cdots (z_N - z_N^0)^{j_N}$$

with coefficient $e_{j_1 \cdots j_N}$ elements of E, where the summation extends over all N tuples of nonnegative integers, and where $\sum \| e_{j_1 \cdots j_N} \| \rho^{j_1 + \cdots + j_N}$ is convergent for some $\rho > 0$. The following theorem obtains.

THEOREM II.1. *Let $\mathscr{V} = \{V_\alpha\}_{\alpha \in A}$ be an open covering of the closed unit polycylinder $\bar{U}^N$ in $\mathbf{C}^N$, and for each α let $f_\alpha \in \mathscr{A}[V_\alpha, E]$. If for all $\alpha, \beta \in A$ there exists $h_{\alpha\beta}$, an invertible element of $\mathscr{A}[V_\alpha \cap V_\beta]$, such that on $\bar{V}_\alpha \cap \bar{V}_\beta$ $f_\alpha = f_\beta h_{\alpha\beta}$, then there exists $F \in \mathscr{A}[\bar{U}^N, E]$ such that for all $\alpha \in A$, $F = f_\alpha h_\alpha$, h_α an invertible element of $\mathscr{A}[V_\alpha, E]$.*

[1] This argument was suggested by W. Rudin as an alternative to a more complicated argument of the author.

The proof of this result depends on the fact proved by Douady[2] [2, p. 48] that if Δ, Δ^+ and Δ^- are as in the proof of Theorem I.1, and if F is an invertible element of $\mathscr{A}[(\Delta^+ \cap \Delta^-), E]$ then there exist invertible $F_+ \in \mathscr{A}[\Delta^+, E]$ and $F_- \in \mathscr{A}[\Delta^-, E]$ such that on $\Delta^+ \cap \Delta^-$, $F = F_+F_-$.

Once one has this fact, it is possible to argue by contradiction and establish Theorem II.1 just as Theorem I.1 was established.

As an application of Theorem II.1, we will show that certain sets in $T^N = \{z \in \boldsymbol{C}^N: |z_1| = 1, \cdots, |z_N| = 1\}$ are zero sets for elements of $\mathscr{A}[U^N]$.

THEOREM II.2. *Let $\beta_1, \cdots, \beta_N$ be strictly positive real numbers. Let Λ be a compact set of Lebesgue measure zero in the real line, and let $G_\Lambda = \{t \in \boldsymbol{R}^N: t \cdot \beta = t_1\beta_1 + \cdots + t_N\beta_N \in \Lambda\}$. If E is a compact subset of $\{(e^{it_1}, \cdots, e^{it_N}): t \in G_\Lambda\}$, then E is the zero set of an element of $\mathscr{A}[U^N]$.*

Proof. Define a map $\Upsilon: \boldsymbol{C}^N \to \boldsymbol{C}^N$ by means of $\Upsilon(z_1, \cdots, z_N) = (e^{iz_1}, \cdots, e^{iz_N})$. Regarding $\boldsymbol{R}^N$ as the set of points in $\boldsymbol{C}^N$ with real coordinates, we have that Υ carries $\boldsymbol{R}^N$ onto T^N. If Q_+^N denotes the set $\{z \in \boldsymbol{C}^N: \operatorname{Im} z_j \geqq 0 \text{ for } j = 1, \cdots, N\}$, then Υ carries Q_+^N onto a dense subset of $\bar{U}^N$. The Jacobian of the map Υ is $e^{i(z_1+\cdots+z_N)}$ which never vanishes, so Υ is a local homeomorphism at every point of $\boldsymbol{C}^N$.

Assume initially that there exists a compact set $K \subset G_\Lambda$ such that $\Upsilon[K] = E$. In general no such set K will exist, but we shall excise this difficulty later. If $t \in \boldsymbol{R}^N$, let

$$\Delta(t, \varepsilon) = \{z \in Q_+^N: |t_1 - z_1| < \varepsilon, \cdots, |t_N - z_N| < \varepsilon\} ;$$

this is a certain product of half discs. For each $t^0 \in K$, let $\varepsilon(t^0)$ be such that Υ carries $\Delta(t^0, \varepsilon(t^0))^-$ homeomorphically onto a closed (relative) neighborhood $V(t^0)$ of the point $s^0 = \Upsilon(t^0)$. There is $\delta > 0$ such that $\Delta'(s^0, \delta) = \{z \in \bar{U}^N: |s_1^0 - z_1| < \delta, \cdots, |s_N^0 - z_N| < \delta\}$ is contained in $V(t^0)$.

Let $\Upsilon_{t_0}^{-1}: V(t^0) \to \Delta(t^0, \varepsilon(t^0))^-$ be inverse to Υ. By compactness, finitely many of the sets $\Upsilon_t^{-1}[\Delta'(s^0, \delta/2)]$ will cover K. Let the $\Delta'(s^0, \delta/2)$ corresponding to such a cover be $\Delta'(s^j, \delta/2)$, $j = 1, \cdots, q$, and let the corresponding $\Upsilon_{t_0}^{-1}$ be Υ_j^{-1}, $j = 1, \cdots, q$.

The set Λ is a compact Lebesgue null set, so the Rudin-Carleson theorem [6, p. 81] applied to Q_+, the upper half plane, yields a function F continuous on $\bar{Q}_+$, holomorphic on Q_+, which vanishes exactly on the set Λ. Define H on Q_+^N by $H(z) = F(\beta_1 z_1 + \cdots + \beta_N z_N)$. The zero set of H is the set G_Λ.

Define F_j on $\Delta'(s^j, \delta)$ by $f_j = H \circ \Upsilon_j^{-1}$. The function f_j is in

[2] I am indebted to the referee for this reference.

$\mathscr{A}[\Delta'(s^j, \delta)]$ and vanishes on $\Upsilon[K \cap \Delta(t^j, \varepsilon(t^j))^-]$. Consequently [8], there is $g_j \in \mathscr{A}[\Delta'(s^j, \delta)]$ which vanishes exactly on the part of $\Upsilon[K \cap \Delta(t^j, \varepsilon(t^j))^-]$ which lies in $\Delta'(s^j, \delta/2)$.

Let $W_j = \{z \in \bar{U}^N\colon \text{for some } k = 1, \cdots, N, |z_k - s_k^j| > 3\delta/4\}$. Let $\tilde{g}_j$ be the function identically one on $\bar{W}_j$. Since on $W_j \cap \Delta'(s^j, \delta)$, g_j is zero free, the functions g_j and $\tilde{g}_j$ constitute a set of Cousin II data to which Theorem II.1 can be applied. We conclude that there exists $F_j \in \mathscr{A}[U^N]$ which vanishes exactly on the part of $\Upsilon[K \cap \Delta(t^j, \in(t^j))^-]$ which is contained in $\Delta'(s^j, \delta/2)$. Define F to be the product of the finitely many functions F_j so constructed. The function F has $\gamma[K] = E$ as its zero set.

In the case that the set D does not exist, we may write $E = \cup E_k$ where $E_k = \gamma[K_k]$, K_k a compact set in G_Λ. The remark on page 435 of [5] or the simpler Corollary 1.2 of [8] now implies that E is the zero set of some element of $\mathscr{A}[U^N]$.

Let us note that in the light of Theorem 1.1 of [8], this theorem generalizes Theorem 4.6 of [7], the case that $\Lambda = \{0\}$.

It should be noted that Theorem II.2 is contained as a special case of more general results of Forelli [3] which are obtained by studying measures orthogonal to $\mathscr{A}[U^N]$.

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