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A CHARACTERIZATION OF INTEGRAL OPERATORS ON THE SPACE OF BOREL MEASURABLE FUNCTIONS BOUNDED WITH RESPECT TO A WEIGHT FUNCTION

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Let I be a Borel set of the real line R, C the space of complex numbers, V a σ-algebra of Borel subsets of I, μ a fixed measure on V such that for any bounded set $Q \in V$, $\mu(Q) < \infty$, $g(\lambda, p)$ a nonvanishing complex valued function μ-measurable in $\lambda \in I$ such that $|g(\lambda, p)| \uparrow$ in p where p belongs to a fixed open interval (a, b), and S the set of μ-measurable functions u from I into C such that $|u(\lambda)g(\lambda, p)| \leqq m$ for some p depending on u, $p \in (a, b)$, $m \geqq 0$ and m depending on u, and for all $\lambda \in I$. The purpose of this paper is to prove the following:

THEOREM 1. **Let $c(\lambda, \delta)$ be a $\mu \times \mu$-measurable function on $I \times I$. For every function $u \in S$ the function**

$$y(\lambda) = \int_I c(\lambda, \delta)u(\delta)d\mu(\delta),\ (\lambda, \delta) \in I \times I$$

is well defined and $y \in S$ if and only if for every $p \in (a, b)$ there exists a $q \in (a, b)$ such that

$$\int_I |g(\lambda, q)c(\lambda, \delta)(g(\delta, p))^{-1}|\, d\mu(\delta) \leqq m$$

for all $(\lambda, \delta) \in I \times I$ and some $m \geqq 0$.

Two examples of the space S are:

(1) Let I be the Borel set $[0, \infty]$, C and V as before, μ the Lebesque measure, and $g(\lambda, p) = e^{\lambda p}$ where p is some real number and $\lambda \in I$. Then S is the set of all functions u which are μ-measurable from I into C and whose Laplace tranforms $\int_I e^{\lambda z}u(\lambda)d\mu(\lambda)$ exist.

(2) A complex sequence $u = \{u_n\}_{n=0}^{\infty}$ is analytic if and only if there exists some constant $M > 0$ such that $|u_n| \leqq M^{n+1}$ for $(n = 0, 1, 2, \cdots)$ if and only if the $\sup_n |p^n u_n| \leqq N$ for some $p > 0$, constant $N > 0$, and $(n = 0, 1, 2, \cdots)$. Now let I be the set of nonnegative integers, C and V as before, $\mu(Q) =$ number of elements of a set $Q \in V$ and $g(\lambda, p) = p^\lambda$ where $p \in (0, \infty)$, $\lambda \in I$. Then S is the space of all complex functions analytic at zero, or the space of analytic sequences, which will be henceforth denoted by A.

In light of Example (2), it is clear that Theorem 1 gives as a corollary a necessary and sufficient condition for infinite complex matrices to map A into itself. At the end of the paper it is shown that this corollary is equivalent to I. Heller's characterization [3, Th. 1, p. 154], namely,

PROPOSITION 1. *The transformation* $y_\lambda = \sum_{\delta=0}^{\infty} c_{\lambda\delta} u_\delta$ *maps* A *into* A *if and only if for every* $p > 0$ *there exists a* $q > 0$ *and a constant* $M > 0$ *such that* $|c_{\lambda\delta}| \leqq Mp^\delta/q^\lambda$ *for all* $(\lambda, \delta = 0, 1, 2, \cdots)$.

In [4] an alternative proof of Heller's result was given. And now the functional analysis techniques developed therein will be used to gain insight into the structure of S as a countable union of Banach spaces, and thereby to prove Theorem 1.

For every fixed $p \in (a, b)$ let $S_p = \{u \in S \mid |u(\lambda)g(\lambda, p)| \leqq m\}$ for all $\lambda \in I$ and $\|u\|_p = \sup_{\lambda \in I}\{|u(\lambda)g(\lambda, p)|\}$.

Let BM denote the set of all bounded μ-measurable functions u^* from I into C with $\|u^*\|_{BM} = \sup_{\lambda \in I}|u^*(\lambda)|$.

THEOREM 2. (1) $S = \bigcup_{n=0}^{\infty} S_{p_n}$ *where* $\{p_n\}_{n=0}^{\infty}$ *is a sequence of numbers from* (a, b) *such that* $p_n \downarrow a$, *and*

(2) *for every* $p \in (a, b)$, $(S_p, \|u\|_p)$ *is a Banach space.*

Proof. If $r < s$ where r and $s \in (a, b)$, then $S_s \subset S_r$. A set theoretic argument completes the proof.

(2) It suffices to observe that $(S_p, \|u\|_p)$ is isometrically isomorphic with the Banach space $(BM, \|u^*\|_{BM})$. The operator E_p from S_p into BM establishing this maps u into u^* where $u^*(\lambda) = u(\lambda)g(\lambda, p)$ for all $\lambda \in I$.

THEOREM 3. *Let* $c^*(\lambda, \delta)$ *be defined on* $I \times I$ *such that*

$$(C^*(u^*))(\lambda) = y^*(\lambda) = \int_I c^*(\lambda, \delta)u^*(\delta)d\mu(\delta)$$

is well defined for all $u^* \in BM$, $(\lambda, \delta) \in I \times I$ *and the obtained function* $y^* = C^*(u^*) \in BM$. *Then* (1) C^* *is a linear continuous operator from BM into BM, and*

(2) $\|C^*\| = \sup_{\lambda \in I} \int_I |c^*(\lambda, \delta)| d\mu(\delta) < \infty$.

Proof. (1) For each $\lambda \in I$, let $h_\lambda(u^*) = \int_I c^*(\lambda, \delta)u^*(\delta)d\mu(\delta)$. It is now shown that for each $\lambda \in I$, h_λ is a continuous linear functional from BM into C with $\|h_\lambda\| = \int_I |c^*(\lambda, \delta)| d\mu(\delta)$. For each $\lambda \in I$, and nonnegative integer n define $c_n^*(\lambda, \delta) = c_n(\lambda, \delta)\chi_{Q_n}(\delta)$ where

$$c_n(\lambda, \delta) = \begin{cases} c^*(\lambda, \delta) & \text{if } |c^*(\lambda, \delta)| \leqq n \\ 0 & \text{if } |c^*(\lambda, \delta)| > n\,, \end{cases}$$

where $Q_n = I \cap [-n, n]$, and χ_{Q_n} is the characteristic function of Q_n. Define for each $\lambda \in I$, $(h_\lambda(u^*))_n = \int_I c_n^*(\lambda, \delta)u^*(\delta)d\mu(\delta)$. Clearly for each $\lambda \in I$, $(h_\lambda(u^*))_n$ is a continuous linear functional on BM. And now as the hypotheses of the Dominated Convergence Theorem are satisfied, $(h_\lambda(u^*))_n \xrightarrow{n \to \infty} h_\lambda(u^*)$. That $h_\lambda(u^*)$ is linear and continuous follows from [2, Th. 17, p. 54].

From this last property, it follows that $|h_\lambda(u^*)| \leqq \|h_\lambda\| \cdot \|u^*\|_{BM}$ for all $u^* \in BM$. In particular, for each $\lambda \in I$, let

$$u_{0\lambda}^*(\delta) = \frac{c^*(\lambda, \delta)!}{|c^*(\lambda, \delta)|}\chi_B$$

where $B = I - \{\delta \in I \mid c^*(\lambda, \delta) = 0\}$ and ! denotes complex conjugation. So $u_{0\lambda}^*$ is a bounded μ-measurable function such that $\|u_{0\lambda}^*\| \leqq 1$. Substituting $u_{0\lambda}^*$ for u^* yeilds $|h_\lambda(u_{0\lambda}^*)| = \int_I |c^*(\lambda, \delta)| d\mu(\delta) \leqq \|h_\lambda\|$.

Conversely, for any $u^* \in BM$,

$$|h_\lambda(u^*)| \leqq \|u^*\|_{BM} \int_I |c^*(\lambda, \delta)| d\mu(\delta)$$

or $\|h_\lambda\| \leqq \int_I |c^*(\lambda, \delta)| d\mu(\delta)$. And so $\|h_\lambda\| = \int_I |c^*(\lambda, \delta)| d\mu(\delta)$.

Moreover, for all $\lambda \in I$ and all $u^* \in BM$, $|h_\lambda(u^*)| = |y^*(\lambda)| \leqq \|y^*\|_{BM}$. By the Unifrom Boundness Theorem, $\|h_\lambda\| \leqq P$ for all $\lambda \in I$ and so $a = \sup_{\lambda \in I}\left\{\int_I |c^*(\lambda, \delta)| d\mu(\delta)\right\} \leqq P$. But $|h_\lambda(u^*)| = |y^*(\lambda)| \leqq a\|u^*\|_{BM}$. And thus for all $u^* \in BM$, $\|y^*\|_{BM} = \|C^*(u^*)\|_{BM} \leqq a\|u^*\|_{BM}$. This implies that C^* is continuous from BM into itself and that $\|C^*\| \leqq a$.

(2) As C^* is a linear continuous operator from BM into BM $\left|\int_I c^*(\lambda, \delta)u^*(\delta)d\mu(\delta)\right| \leqq \|C^*\| \cdot \|u^*\|_{BM}$ for all $u^* \in BM$. Substituting $u_{0\lambda}^*$ (defined above in (1)) for u^* yeilds $\int_I |c^*(\lambda, \delta)| d\mu(\delta) \leqq \|C^*\|$ for each $\lambda \in I$. Thus $a \leqq \|C^*\|$. And so $\|C^*\| = a$.

THEOREM 4. *Let $c(\lambda, \delta)$ be a function defined on $I \times I$ such that for all $u \in S$, $y(\lambda) = \int_I c(\lambda, \delta)u(\delta)d\mu(\delta)$ is well defined and $y \in S$. Put $y = C(u)$. For each p and q fixed and belonging to (a, b), let $S_{pq} = \{u \in S_p \mid C(u) \in S_q\}$. Then*

(1) $S_p = \bigcup_{n=0}^{\infty} S_{pq_n}$ *where $q_n \downarrow a$ for any $p \in (a, b)$, and*

(2) $(S_{pq}, \|u\|_{pq} = \|u\|_p + \|C(u)\|_q)$ *is a Banach space.*

Proof. (2) If the graph of C is closed in $S_p \times S_q$, then

$$(Z, \|u\|_p + \|C(u)\|_q)$$

where $Z = \{(u, C(u)) \mid u \in S_p\}$ is a Banach space. And as the mapping from S_{pq} into Z defined by $u \to (u, C(u))$ establishes an isometric isomorphism between S_{pq} and Z, it suffices to prove that the graph of C is closed in $S_p \times S_q$.

For each $\lambda \in I$, let

$$k_\lambda(u) = \int_I c(\lambda, \delta)u(\delta)d\mu(\delta) = (C(u))(\lambda) = k_\lambda(E_p^{-1}(u^*)) .$$

Here $E_p: u \to u^*$ is the isometric isomorphism from S_p into BM, and $u^*(\lambda) = u(\lambda)g(\lambda, p)$ for all $\lambda \in I$. As $k_\lambda E_p^{-1}$ is a linear continuous functional on BM, it follows that k_λ is a linear continuous functional on S_p for each $\lambda \in I$. This with the uniqueness of limits in S_q and the Closed Graph Theorem, prove that C is closed in $S_p \times S_q$.

THEOREM 5. *Let $c(\lambda, \delta)$ be a function defined on $I \times I$ such that for all $u \in S$, $y(\lambda) = \int_I c(\lambda, \delta)u(\delta)d\mu(\delta)$ is well defined and $y \in S$. Put $y = C(u)$. Then*

(1) *for every $p \in (a, b)$ there exists a $q \in (a, b)$ such that $u \in S_p$ implies $C(u) \in S_q$.*

The operator C from S_p into S_q generated by $c(\lambda, \delta)$ is

(2) *linear and continuous, and*

(3) *its norm, $\| C \| = \sup_{\lambda \in I} \int_I | g(\lambda, q)c(\lambda, \delta)(g(\delta, p))^{-1} | d\mu(\delta) < \infty$.*

Proof. (1) From Theorem 2. (1), $S = \bigcup_{n=0}^\infty S_{q_n}$ where $\{q_n\}_{n=0}^\infty \in (a, b)$ and $q_n \downarrow a$. As C maps S into itself, for any $p \in (a, b)$, C maps S_p into $\bigcup_{n=0}^\infty S_{q_n}$. But $S_p = \bigcup_{n=0}^\infty S_{pq_n}$ by Theorem 4. (1). Now as the injective maps from S_{pq_n} into S_p are continuous for all p and q_n, by [5, Corollary 6, p. 205] or [6, Satz 4.6, p. 472] there exists an index $q_k \in (a, b)$ such that $S_p = S_{pq_k}$. So q_k is the desired number.

(2) The linearity of C is clear. And by definition of the Banach norm $\| u \|_{pq}$ on S_{pq}, C is continuous from S_p into S_q.

(3) Map S_p into BM by the operator $E_p: u \to u^*$ where $u^*(\lambda) = u(\lambda)g(\lambda, p)$ for all $\lambda \in I$. Define the operator C^* to be $E_q C E_p^{-1}$ where $p, q \in (a, b)$. C^* is a linear and continuous operator from BM into itself whose norm is given by Theorem 3 (2). But $\| C^* \| = \| C \|$.

Proof of Theorem 1. Necessity follows immediately from (1) and (3) of Theorem 5.

Conversely, let $u \in S$ and $y(\lambda) = \int_I c(\lambda, \delta)u(\delta)d\mu(\delta)$. Now for any $p, q \in (a, b)$

$$\begin{aligned} | y(\lambda)g(\lambda, q) | &\leqq \int_I | g(\lambda, q)c(\lambda, \delta)(g(\delta, p))^{-1} | \cdot | g(\delta, p)u(\delta) | \, d\mu(\delta) \\ &\leqq \| C \| \cdot \| u(\delta)g(\delta, p) \|_{BM} \leqq M . \end{aligned}$$

Moreover as (I, V, μ) is a totally σ-finite measure space, the μ-measurable function $u(\delta)$ defined to be $u''(\lambda, \delta)$ is $\mu \times \mu$-measurable. An application of Tonelli's Theorem completes the proof that $y(\lambda)$ is μ-measurable. And so $y(\lambda) \in S$.

If N is the set of nonnegative integers, $c(\lambda, \delta)$, where $(\lambda, \delta) \in N \times N$, can be identified with an infinite complex matrix $(c_{\lambda\delta})$. Clearly $(c_{\lambda\delta})$ is $\mu \times \mu$-measurable on $N \times N$.

COROLLARY TO THEOREM 1. *The transformation C generated by an infinite complex matrix $(c_{\lambda\delta})$, $(\lambda, \delta) \in N \times N$ defined by $y_\lambda = \sum_{\delta=0}^{\infty} c_{\lambda\delta} u_\delta$ maps the space A of analytic sequences into itself if and only if for every $p > 0$ there exists a $q > 0$ such that*

$$\sup_{\lambda \in I} \sum_{\delta=0}^{\infty} q^\lambda \mid c_{\lambda\delta} \mid p^{-\delta} \leqq k \qquad \textit{for } (\lambda, \delta) \in N \times N\text{ , constant } k > 0\text{ .}$$

The next proposition shows that this corollary is equivalent to Heller's characterization, Proposition 1.

PROPOSITION 2. For each $p > 0$ there exists a $q > 0$ and a constant $k > 0$ such that $\sup_{\lambda \in I} \sum_{\delta=0}^{\infty} q^\lambda \mid c_{\lambda\delta} \mid p^{-\delta} \leqq k$ if and only if for each $p > 0$ there exists a $r > 0$ and a constant $m > 0$ such that $\mid c_{\lambda\delta} \mid \leqq mp^\delta / r^\lambda$.

Proof. Sufficiency. Let $p > 0$ and let p' be such that $0 < p' < 1$. Then $pp' > 0$. Given there exists a $r > 0$ such that $\mid c_{\lambda\delta} \mid \leqq m(pp')^\delta / r^\lambda$ for all $(\lambda, \delta) \in N \times N$, and so $\sum_{\delta=0}^{\infty} r^\lambda \mid c_{\lambda\delta} \mid p^{-\delta} \leqq m(1 - p')^{-1}$, for each $\lambda \in N$.

In conclusion, it is natural to ask: (1) which analytic functions f in the half plane $\mathrm{Re}\,(z) \leqq r$ can be represented by the integral $f(z) = \int_I u(\lambda) e^{\lambda z} d\mu(\lambda)$ where the determining function $u \in S$, I is a Borel set of the real line and μ is the Lebesque measure; and (2) to which classes of measurable functions can Theorem 1 be generalized? It is thought that Theorem 1 can be generalized to (a) Bochner measurable functions bounded with respect to a weight function simply by using a Fubini theorem in place of a Tonelli theorem in the sufficiency proof of Theorem 1, and (b) Borel measurable functions essentially bounded with respect to a weight function, where two functions are equal if and only if they coincide everywhere, by using the lifting property of A. and C. Ionescu-Tulcea.

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