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EULER CHARACTERISTICS

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Given a suitable category of R-modules, a generalized Euler characteristic is defined for each finite sequence of modules in the category, and is characterized by simple properties. For many categories, including the category of all finitely generated R-modules, this generalized characteristic has the following two properties. First, it assigns the same value to isomorphic sequences. Second, for any chain complex of R-modules in the category, the characteristic of the sequence of chain modules equals the characteristic of the sequence of homology modules. For such categories our results imply that any function having these two properties is itself a function of the characteristic so that the generalized Euler characteristic is essentially the only such function. For the special case of the category of all finitely generated modules over a principal ideal domain the generalized Euler characteristic can be identified with the integer valued function which is the classical Euler characteristic. By considering the special case of the category of all finitely generated torsion modules over the polynomial ring $F[x]$ over a field F we obtain a generalized Euler characteristic for the case of a linear endomorphism of a finite sequence of finite dimensional vector spaces over F. In this case we establish the relations between the characteristic and the sequence of Lefschez numbers of the endomorphism and its iterates.

Our results are proved in a general setting which we now discuss. For certain categories $\mathscr{M}$, a Grothendieck group G is defined, based on the following idea. We want to identify an object B of $\mathscr{M}$ with the free sum of A and C in case A is a "sub-object" of B and C is a "quotient object" of B modulo A. We also want to identify B with objects which are equivalent to B in $\mathscr{M}$. Thus G is defined as a free abelian group modulo certain relations. In all of the cases we consider, addition in the free group is identical with that induced by a sum or product in the category $\mathscr{M}$.

In particular, suppose $\mathscr{M}$ is a class of modules that is closed with respect to submodules and quotient modules. The Grothendieck group of $\mathscr{M}$ is defined as the quotient group of the free abelian group generated by the isomorphism classes $[A]$ of modules A in $\mathscr{M}$ by the subgroup generated by the elements $[B]-[A]-[C]$ for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{M}$. There is then an Euler characteristic assigning an element of G to a finite sequence of modules in $\mathscr{M}$, and this Euler characteristic has the two properties of the classical Euler characteristic. Again, any function having these two properties is itself a function of this Euler characteristic.

In addition to specializing to the case of the classical Euler characteristic this more general result also specializes to the case of a finite sequence of finite dimensional vector spaces C_i over a field F, each with a linear operator T_i. Since a finite dimensional vector space over F together with a linear operator is equivalent to a finitely generated torsion module over the polynomial ring $F[\lambda]$, the class $\mathscr{M}$ in this case is the class of finitely generated torsion modules over $F[\lambda]$. The corresponding Euler characteristic can be identified with the rational form

$$\prod_i P(T_{2i})/\prod_i P(T_{2i+1})$$

where $P(T_j)$ is the characteristic polynomial of T_j (see Corollary 5.3). For every k the Lefschetz number $\Lambda_k = \sum(-1)^j \operatorname{Tr}(T_j)^k$ has the two properties of the Euler characteristic. It follows that Λ_k is a function of the above rational form. We determine the precise form of this relationship and also show that in case F has characteristic zero the numbers Λ_k determine the rational form.

We actually define the Euler characteristic of a finite sequence of modules of a class $\mathscr{M}$ which need not be closed with respect to submodules or quotient modules. This Euler characteristic is characterized by two other properties which are equivalent to the previously mentioned properties in case $\mathscr{M}$ is closed with respect to submodules and quotient modules. This more general Euler characteristic is related to the obstruction to finiteness of a CW-complex which has been considered by Wall [3]. We show in Theorem 5.6 that a finitely generated projective chain complex over an associative ring R with a unit is chain equivalent to a finitely generated free chain complex over R if and only if the image of its Euler characteristic in the projective class group of R is zero.

The first two sections are devoted to general considerations of Grothendieck groups. Various special cases are considered, and we compute the Grothendieck groups of all finitely generated modules (in Example 2.5) and of all finitely generated torsion modules (in Example 2.6) over a principal ideal domain. The third section is devoted to special results concerning the Grothendieck group of linear operators on finite dimensional vector spaces over a field and to connections with the sequence of traces of powers of the operator. The fourth section introduces Euler equivalence of chain complexes, the corresponding equivalence classes forming an abelian group called the Euler group. In the fifth section it is shown (Theorem 5.1) that the Euler characteristic is an isomorphism of the Euler group of chain complexes with the Grothendieck group of corresponding modules. The above mentioned results are then derived from this isomorphism.

1. Grothendieck equivalence. In a category $\mathscr{C}$ with sums (or products) the Grothendieck equivalence relation is generated by requiring that for certain sequences $A \to B \to C$ in $\mathscr{C}$ the object B is equivalent to the sum (or product) of A and C. The corresponding equivalence classes usually form an abelian semi-groups with respect to sum (or product). In this section we study this semigroup and compute it for the case of finitely generated modules over a principal ideal domain.

Given a set S and a subset $\mathscr{R}$ of the cartesian product $S \times S$ the *equivalence relation on S generated by* $\mathscr{R}$ is the smallest equivalence relation $\sim$ such that if $(a, b) \in \mathscr{R}$ then $a \sim b$. It is characterized by the following property. Given $a, b \in S$ then $a \sim b$ if and only if $a = b$ or there is a finite sequence $a_0, a_1, \cdots, a_n \in S$ with $n \geqq 1$ such that $a = a_0$, $a_n = b$ and for $1 \leqq j \leqq n$ either $(a_j, a_{j-1}) \in \mathscr{R}$ or $(a_{j-1}, a_j) \in \mathscr{R}$. We shall have occasion to consider equivalence relations generated in this way in the sequel.

Let $\mathscr{M}$ be a category with initial objects, one being denoted by 0, and having finite sums, the sum of A and B being denoted by $A \vee B$. Assume there is given a collection $\mathscr{S}$ of sequences of morphisms $A \to B \to C$ in $\mathscr{M}$ such that the following hold:

(1) If $B \xrightarrow{\beta} C$ is an equivalence in $\mathscr{M}$, then $0 \to B \xrightarrow{\beta} C$ is in $\mathscr{S}$.

(2) If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is in $\mathscr{S}$ and X is in $\mathscr{M}$, then

$$A \vee X \xrightarrow{\alpha \vee 1} B \vee X \xrightarrow{\beta \vee 0} C$$

is also in $\mathscr{S}$.

The *Grothendieck equivalence relation* on $\mathscr{M}$ corresponding to $\mathscr{S}$ is the equivalence relation generated by the collection $R = \{(B, A \vee C) \mid A \to B \to C \text{ is in } \mathscr{S}\}$. It follows from (1) that if A, B are equivalent in $\mathscr{M}$ then $A \sim B$. It follows from (2) and (1) that there is a well-defined operation $+$ on the collection $[\mathscr{M}, \mathscr{S}]$ of equivalence classes such that

$$[A] + [B] = [A \vee B]$$

making $[\mathscr{M}, \mathscr{S}]$ into an abelian semigroup with [0] as unit.

Similarly let $\mathscr{M}$ be a category with terminal objects, one being denoted by P, and having finite products, the product of A and B being denoted by $A \wedge B$. Assume there is given a collection $\mathscr{S}'$ of sequences of morphisms $A \to B \to C$ in $\mathscr{M}$ such that:

(1)′ If $A \xrightarrow{\alpha} B$ is an equivalence in $\mathscr{M}$, then $A \xrightarrow{\alpha} B \to P$ is in $\mathscr{S}'$.

(2)′ If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is in $\mathscr{S}$ and X is in $\mathscr{M}$, then

$$A \xrightarrow{\alpha \wedge 0} B \wedge X \xrightarrow{\beta \wedge 1} C \wedge X$$

is in $\mathscr{S}'$.

The *Grothendieck equivalence relation* on $\mathscr{M}$ corresponding to $\mathscr{S}'$ is the equivalence relation generated by the collection $R = \{(B, A \wedge C) \mid A \rightarrow B \rightarrow C \text{ is in } \mathscr{S}'\}$. The collection of equivalence classes is denoted by $[\mathscr{M}, \mathscr{S}']$ and for A in $\mathscr{M}$ its equivalence class is denoted by $[A]$. There is a well-defined operation $+$ on $[\mathscr{M}, \mathscr{S}']$ such that

$$[A] + [B] = [A \wedge B]$$

and making $[\mathscr{M}, \mathscr{S}']$ into an abelian semigroup with $[P]$ as unit element.

Of particular interest is the case where $\mathscr{M}$ is a category of modules containing zero and closed with respect to finite sum, over an associative ring R with a unit and $\mathscr{S}$ is the class of all sequences $A \rightarrow B \rightarrow C$ in $\mathscr{M}$ such that $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact. In this case, because finite sums and products coincide, the two types of Grothendieck equivalence agree and the corresponding semigroup is denoted by $[\mathscr{M}]$. Let $\mathscr{M}'$ be a full subcategory of $\mathscr{M}$ such that, given any short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $\mathscr{M}$, if two of the three modules A, B, C are in $\mathscr{M}'$, then the third is also. In this case there is an imbedding of the semigroup $[\mathscr{M}']$ into the semigroup $[\mathscr{M}]$ sending $[A]_{\mathscr{M}'}$ to $[A]_{\mathscr{M}}$ for all A in $\mathscr{M}$.

We calculate an example. Let R be a principal ideal domain and let $\mathscr{M}$ be the category of all finitely generated modules over R. If $\mathscr{M}'$ is the full subcategory of finitely generated torsion modules, it has the property stated above so that $[\mathscr{M}']$ is a subsemigroup of $[\mathscr{M}]$.

There is a well-defined epimorphism,

$$\text{rank}: [\mathscr{M}] \longrightarrow \boldsymbol{Z}^+ ,$$

where $\boldsymbol{Z}^+$ is the semigroup of nonnegative integers, such that $\text{rank}\,[A] = \text{rank}\, A$. (It is standard that if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact then $\text{rank}\, B = \text{rank}\, A + \text{rank}\, C$.) Clearly $[\mathscr{M}'] =$ kernel of rank.

LEMMA 1.1. *If* $\text{rank}\, A = \text{rank}\, B > 0$, *then* $A \sim B$.

Proof. It suffices to prove that if $\text{rank}\, A = m > 0$ then A is Grothendieck equivalent to a free module of rank m. By the structure theorem, if A has rank m, then A is isomorphic to

$$R^m \oplus R/s_1R \oplus \cdots \oplus R/s_qR$$

where $s_1, \cdots, s_q$ are nonzero elements of R. Thus, it suffices to show that $R \oplus R/sR \sim R$, but this follows from the exactness of the sequence

$$0 \longrightarrow R \xrightarrow{\alpha} R \xrightarrow{\beta} R/sR \longrightarrow 0$$

where α is multiplication by s and β is the canonical quotient map.

It follows from the lemma that rank is an isomorphism of the semigroup $[\mathscr{M}] - [\mathscr{M}']$ onto the semigroup of positive integers. To complete the description of $[\mathscr{M}]$ we need to compute $[\mathscr{M}']$.

Let R' be the multiplicative semigroup of R modulo its subsemigroup U of units of R. If A is a finitely generated torsion module, then A is isomorphic to $R/s_1R \oplus \cdots \oplus R/s_qR$ where $s_1, \cdots, s_q$ are nonzero elements of R. We define the *characteristic* of A, $\operatorname{char} A \in R'$, by

$$\operatorname{char} A = (s_1 \cdots s_q)U .$$

Then $\operatorname{char} 0 = U$. We present another description of the characteristic to show that it is well-defined and is invariant under Grothendieck equivalence.

It φ is an endomorphism of a finitely generated free R module F, then the determinant of a matrix representing φ in some basis is independent of the choice of basis. We let $|\varphi|$ denote the corresponding element of R'. The map sending φ to $|\varphi|$ is a homomorphism,

$$\operatorname{End}(F) \longrightarrow R' ,$$

and if φ is an automorphism, $|\varphi| = U$. Note that $\operatorname{rank}(\operatorname{im}\varphi) = \operatorname{rank} F$ if and only if $|\varphi| \neq 0$.

Represent a torsion module A as the quotient F/G where F is a free finitely generated module and G is a submodule (necessarily free and of the same rank as F). Let $\varphi\colon F \to G$ be any epimorphism. Regarding φ as an element of $\operatorname{End}(F)$ we obtain the element $|\varphi| \in R'$. If $\psi\colon F \to G$ is another epimorphism, it is necessarily an isomorphism and so $\psi^{-1} \circ \varphi$ is an automorphism of F. Therefore,

$$|\psi| = |\psi|\,|\psi^{-1} \circ \varphi| = |\psi \circ \psi^{-1} \circ \varphi| = |\varphi| .$$

Therefore $|\varphi|$ is independent of the choice of the epimorphism φ. Finally, if $e_1, \cdots, e_n$ is a basis for F such that $a_1e_1, \cdots, a_ne_n$ is a basis for G, we can choose $\varphi(e_i) = a_ie_i$, in which case $|\varphi| = a_1 \cdots a_nU$. Since $A \approx R/a_1R \oplus \cdots \oplus R/a_nR$, we see that $|\varphi| = \operatorname{char} A$. We therefore have the following description of the characteristic:

LEMMA 1.2. *If A is a finitely generated torsion module isomorphic to F/G, where F is a free finitely generated module, then* $\operatorname{char} A = |\varphi|$ *where $\varphi\colon F \to G$ is any epimorphism.*

COROLLARY 1.3. *If A is a finitely generated torsion module and $A' \subset A$, then*

$$\text{char } A = (\text{char } A')(\text{char } A/A') .$$

Proof. Let $A \approx F/G$ where F is a free finitely generated module and choose F' with $G \subset F' \subset F$ such that $A' \approx F'/G$. Then $F/F' \approx A/A'$. Let $\varphi: F \to F'$ and $\psi: F' \to G$ be epimorphisms. Then $\psi \circ \varphi: F \to G$ is an epimorphism and since φ and $\varphi^{-1} \circ \psi \circ \varphi$ are endomorphisms of F we have

$$\text{char } A = |\psi \circ \varphi| = |\varphi \circ (\varphi^{-1} \circ \psi \circ \varphi)| = |\varphi| \, |\varphi^{-1} \circ \psi \circ \varphi| .$$

Since φ is an isomorphism of F with F', we see that for the endomorphism ψ of F' and the endomorphism $\varphi^{-1} \circ \psi \circ \varphi$ of F we have $|\psi| = |\varphi^{-1} \circ \psi \circ \varphi|$. Therefore,

$$\text{char } A = |\varphi| \, |\psi| = \text{char } (A/A') \text{ char } A' .$$

It follows from the corollary that there is a homomorphism

$$\text{char: } [\mathscr{M}'] \longrightarrow R'$$

such that $\text{char } [A] = \text{char } A$ (it is a homomorphism because $\text{char } (A \oplus B) = \text{char } A \text{ char } B$, trivially). It is easy to see that char is an epimorphism of $[\mathscr{M}']$ onto R'; indeed, for each nonzero member r of R, $\text{char } R/rR = rU$. The following proposition shows that char is an isomorphism of $[\mathscr{M}']$.

PROPOSITION 1.4. *If A is a torsion module and $r \in \text{char } A$, then A is Grothendieck equivalent to R/rR.*

Proof. Notice that if s and t are nonzero members of R then the Noether isomorphism theorem yields an exact sequence

$$0 \longrightarrow sR/stR \longrightarrow R/stR \longrightarrow R/sR \longrightarrow 0 .$$

Consequently $R/stR \sim sR/stR \oplus R/sR$, and since $sR/stR \approx R/tR$, we see that $R/stR \sim R/sR \oplus R/tR$. It follows by induction that if $s_1, \cdots, s_q$ are nonzero members of R, then

$$R/s_1R \oplus \cdots \oplus R/s_qR \sim R/(s_1 \cdots s_q)R .$$

But every finitely generated torsion module A is isomorphic to such a direct sum of cyclic modules, and the proposition follows.

Summing up our results:

THEOREM 1.5. *Let $\mathscr{M}$ be the category of finitely generated* *modules over a principal ideal domain R, and let $\mathscr{M}'$ be the subcategory of torsion modules. Then rank and char are complete invariants for Grothendieck equivalence. Rank is a homomorphism of $[\mathscr{M}]$ onto Z^+ with kernel equal to $[\mathscr{M}']$ and char is an isomorphism of $[\mathscr{M}']$ onto the multiplicative semigroup of nonzero members of R modulo units.*

2. **Grothendieck groups.** In the last section Grothendieck equivalence was used to construct semigroups. In the present section we pass from semigroups by a standard method. Several examples are then discussed.

If S is an abelian semigroup its *Grothendieck group* $G(S)$ is an abelian group which is universal with respect to homomorphisms from S to abelian groups. That is, there is a canonical homomorphism $\gamma: S \to G(S)$ such that for each homomorphism φ of S to an abelian group G there is a unique homomorphism $\varphi': G(S) \to G$ with $\varphi = \varphi' \circ \gamma$. It is easy to see that $G(S)$ and γ, if they exist, are essentially unique; we show existence by an explicit construction.

Define an equivalence relation $\equiv$ in $S \times S$ by the condition $(a, b) \equiv (a', b')$ if and only if there is $c \in S$ with

$$a + b' + c = a' + b + c .$$

Let $\{(a, b)\}$ denote the equivalence class of (a, b). There is a well-defined operation $+$ in the set of equivalence classes such that

$$\{(a, b)\} + \{(a', b')\} = \{(a + a', b + b')\} .$$

The set of equivalence classes with this operation forms an abelian group $G(S)$, and a homomorphism

$$\gamma: S \longrightarrow G(S)$$

is defined by $\gamma(a) = \{(a, 0)\}$. This has the desired universal property because, given a homomorphism $\varphi; S \to G$, where G is an abelian group, the unique homomorphism $\varphi': G(S) \to G$ such that $\varphi' \circ \gamma = \varphi$ is characterized by the equation

$$\varphi'\{(a, b)\} = \varphi(a) - \varphi(b) .$$

In the above, in case S is a semigroup with cancellation, γ is an imbedding of S in $G(S)$, and the construction is merely a generalization of the usual construction of Z from Z^+.

We can apply this construction to obtain Grothendieck groups $G[\mathscr{M}, \mathscr{S}]$ corresponding to the semigroups $[\mathscr{M}, \mathscr{S}]$ defined by Grothendieck equivalence. Then $G[\mathscr{M}, \mathscr{S}]$ is generated, as a group, by the collection $\gamma[A]$ for A in $\mathscr{M}$. Furthermore, if $A \to B \to C$ is in $\mathscr{S}$ then $\gamma[B] = \gamma[A] + \gamma[C]$. Also either $\gamma[A] + \gamma[B] = \gamma[A \vee B]$ or $\gamma[A] + \gamma[B] = \gamma[A \wedge B]$ depending on which of the two types of Grothendieck equivalence we start with.

There is a different way of describing the Grothendieck groups for the categories $\mathscr{M}$ and collections $\mathscr{S}$ which concern us. Suppose $\mathscr{M}$ is a category with sums and a zero object satisfying conditions (1) and (2) of § 1, and further satisfying

(3) If X and Y belong to $\mathscr{M}$ then there is a sequence $X \to X \vee Y \to Y$ in $\mathscr{S}$.

Let G' be the free abelian group with generators the objects of $\mathscr{M}$ modulo the subgroup generated by all $B - (A + C)$ for $A \to B \to C$ in $\mathscr{S}$, and let $\psi(M)$ for $M \in \mathscr{M}$, be the canonical image of M in G'. Then (3) implies that ψ is a monoid homomorphism of $\mathscr{M}$ with $\vee$ into G', and it can be verified that if A is Grothendieck equivalent to B then $\psi(A) = \psi(B)$. Consequently ψ induces a homomorphism ψ' of $[\mathscr{M}]$ into G' and, in view of the universal property of $G(\mathscr{M})$, we see that there is a canonical homomorphism φ of $G(\mathscr{M})$ into G' so that the diagram

$$\begin{array}{ccc} & & G(\mathscr{M}) \\ & \overset{\gamma}{\nearrow} & \downarrow \varphi \\ [\mathscr{M}] & \xrightarrow[\psi']{} & G' \end{array}$$

is commutative. It follows easily that φ is, in fact, an isomorphism of the Grothendieck group onto G'. Similar considerations apply to the Grothendieck group of a category with products.

EXAMPLE 2.1. Let $\mathscr{T}$ be the category of compact polyhedra with base points. Any one-point object in $\mathscr{T}$ is an initial object and $\mathscr{T}$ has finite sums. Let $\mathscr{S}$ be the collection of sequences

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

where α is an imbedding of A as a subpolyhedron in B and $\beta \circ \alpha(A) =$ base point of C and $\beta \mid B - \alpha(A)$ is a bijection from $B - \alpha(A)$ to C $-$ basepoint of C. The corresponding Grothendieck group $G[\mathscr{T}, \mathscr{S}]$ has been considered by Watts [4] who showed that there is a well-defined function $\chi: G[\mathscr{T}, \mathscr{S}] \to \boldsymbol{Z}$ which assigns to $\gamma[A]$ the Euler characteristic of A and this is an isomorphism between the two groups.

EXAMPLE 2.2. Let $\mathscr{T}'$ be the category of path-connected topological spaces having abelian fundamental groups and finitely generated homotopy groups. Any one-point object in $\mathscr{T}'$ is a terminal object and $\mathscr{T}'$ has finite products. Let $\mathscr{S}'$ be the collection of sequences

$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in $\mathscr{T}'$ such that there is a homomorphism $\partial: \pi_*(C) \to \pi_*(A)$ of degree -1 for which there is an exact triangle

$$\begin{array}{ccc} \pi_*(A) & \xrightarrow{\alpha_*} & \pi_*(B) \\ & {\scriptstyle\partial}\nwarrow \quad \swarrow{\scriptstyle\beta_*} & \\ & \pi_*(C) & \end{array} .$$

There is a well-defined homomorphism $\chi': G[\mathscr{T}', \mathscr{S}'] \to \boldsymbol{Z}$ such that

$$\chi'(\gamma[A]) = \sum(-1)^j \operatorname{rank} \pi_j(A) ,$$

and this homomorphism can be seen to be an isomorphism (by using Postnikov sequences to see that$G$ $[\mathscr{T}', \mathscr{S}']$ is generated by $\gamma[A]$ where A is a polyhedron having at most one nonzero homotopy group, and then observing that if A is a polyhedron having one nonzero homotopy group equal to $\boldsymbol{Z}$ in dimension $n \geqq 1$ and B has one nonzero homotopy group equal to $\boldsymbol{Z}_p$ in dimention n, there is a sequence $A \xrightarrow{\alpha} A \xrightarrow{\beta} B$ in $\mathscr{S}$ whence $G[\mathscr{T}', \mathscr{S}']$ is cyclic generated by $\gamma[A]$).

EXAMPLE 2.3. Let X be a topological space and let $\mathscr{M}$ be the category of all real (or complex) vector bundles over X. The unique 0-dimensional bundle is an initial object and Whitney sum is a sum in the category $\mathscr{M}$. Let $\mathscr{S}$ be the collection of all short exact sequences $0 \to A \to B \to C \to 0$ in $\mathscr{M}$. Then $G[\mathscr{M}, \mathscr{S}]$ is denoted by $KO(X)$ (or $K(X)$) and plays an important role in algebraic topology. These K groups were the first Grothendieck groups extensively studied from this point of view [1].

EXAMPLE 2.4. Let R be an associative ring with a unit and let $\mathscr{M}$ be the category of all finitely generated projective R modules (a module is projective if and only if it is isomorphic to a direct summand of a free module). Let $\mathscr{S}$ be the class of all $A \to B \to C$ such that $0 \to A \to B \to C \to 0$ is exact. Then the group $G[\mathscr{M}]$ is usually denoted by $K_0(R)$. It contains a subgroup H generated by $\gamma[R]$ and the quotient group $K_0(R)/H$ is denoted by $\tilde{K}_0(R)$ and called the *projective class group* of R. It can be shown that $\tilde{K}_0(R)$ is isomorphic to the group of stable equivalence classes of finitely generated projective modules (two such modules A and A' being *stably equivalent* if and only if there are free modules F and F' such that $A \oplus F \approx A' \oplus F'$) with addition induced by direct sum. For a finitely generated projective module A we see that $\gamma[A]$ maps to zero in $\tilde{K}_0(R)$ if and only if A is stably equivalent to a free module (in which case A is said to be *stably free*).

EXAMPLE 2.5. Let R be a principal ideal domain. Then the Grothendieck group $G[\mathscr{M}]$ of all finitely generated R modules is isomorphic to $\boldsymbol{Z}$ by

$$\text{rank}: G[\mathscr{M}] \longrightarrow \boldsymbol{Z}$$

where $\text{rank}\, \gamma[A] = \text{rank}\, A$. This follows from Lemma 1.1 and the observation that $\gamma[A] = 0$ if A is a torsion module (because in this case $\gamma[R] + \gamma[A] = \gamma[R]$).

EXAMPLE 2.6. Let R be a principal ideal domain and let $\mathscr{M}'$ be the class of finitely generated torsion R modules. Let Q' be the multiplicative group of nonzero elements of the field of quotients of R modulo the group U of units of R. Then Q' is easily seen to be the Grothendieck group of the semigroup of nonzero elements of R modulo units. It follows from Theorem 1.5 that there is an isomorphism

$$\text{char}: G[\mathscr{M}'] \approx Q'$$

such that $\text{char}\, \gamma[A] = \text{char}\, A$.

3. The Grothendieck group of linear operators.

Throughout this section F will denote a field. Given a finite dimensional vector space A over F and a linear operator $T: A \to A$ we can regard A as a finitely generated module over the principal ideal domain $F[\lambda]$ by defining $p(\lambda)a = p(T)(a)$ for $p(\lambda) \in F[\lambda]$ and for $a \in A$. Because A is finite dimentional it is easy to see that A is a torsion module over $F[\lambda]$.

On the other hand, if A is a finitely generated torsion module over $F[\lambda]$, then it is easy to see that A is a finite dimensional vector space over F, and if we define $T: A \to A$ by $T(a) = \lambda a$ for $a \in A$, then A is just the $F[\lambda]$ module constructed from T. Thus there is a natural equivalence of the category $\mathscr{L}$ of linear operators on finite dimentional vector spaces over F (a morphism $A \to B$ being required to commute with the respective operators) and the category $\mathscr{M}'$ of finitely generated torsion modules over $F[\lambda]$. Under this equivalence of categories the Grothendieck equivalence relation on $\mathscr{L}$, with $\mathscr{S}$ the class of sequences $A \to B \to C$ with $0 \to A \to B \to C \to 0$ exact, corresponds to Grothendieck equivalence on $\mathscr{M}'$. We have a complete invariant, char, for the latter, and we now identify the corresponding invariant for $\mathscr{L}$.

For each linear operator T on a finite dimensional vector space A let $[A, T]$ be the Grothendieck equivalence class of the corresponding torsion module over $F[\lambda]$. Then $\text{char}\,[A, T]$ is a nonzero member of $F[\lambda]$ modulo the nonzero members of F. We show:

PROPOSITION 3.1. *If T is a linear operator on a finite dimensional vector space over a field F, then* char $[A, T]$ *is the set of nonzero scalar multiples of the characteristic polynomial of T.*

Proof. As a finitely generated torsion module over $F[\lambda]$ we have $A \approx F[\lambda]/p_1(\lambda)F[\lambda] \oplus \cdots \oplus F[\lambda]/p_m(\lambda)F[\lambda]$ where the polynomials $p_j(\lambda)$ may be assumed to have leading coefficient 1. Under this isomorphism $T \approx T_1 \oplus \cdots \oplus T_m$ where $T_i: F(\lambda)/p_i(\lambda)F[\lambda] \rightarrow F[\lambda]/p_i(\lambda)F[\lambda]$ is the operator multiplying each element by λ. If degree $p_i(\lambda) = n_i$, there is a basis $1, \lambda, \cdots, \lambda^{n_i-1}$ of $F[\lambda]/p_i(\lambda)F[\lambda]$ over F and relative to this basis T_i is represented by the matrix $M_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \ddots & 1 \\ \cdot & \cdots & 0 \end{pmatrix}$ where the bottom row has entry in jth column equal to minus the coefficient of λ^j in p_i for $0 \leqq j < n_i$. It is then clear that

$$\det(\lambda I - M_i) = \det\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \ddots & -1 \\ \cdot & \cdots & \lambda \end{pmatrix} = \det\begin{pmatrix} 0 & -1 & 0 \\ & \ddots & -1 \\ p_i\lambda & \cdots & \lambda \end{pmatrix} = p_i(\lambda)$$

and so $p_i(\lambda)$ is the characteristic polynomial of T_i. Therefore, $p_1(\lambda) \cdots p_m(\lambda)$ is the characteristic polynomial of T, and from the definition of $\mathrm{char}_{F[\lambda]} A$ we obtain the result.

COROLLARY 3.2. *The function $(A, T) \mapsto$ characteristic polynomial (T) is a complete invariant for Grothendieck equivalence on the category $\mathscr{L}$ of linear operators on finite dimentional vector spaces over a field F. The Grothendieck group of $\mathscr{L}$ is isomorphic to the multiplicative group of quotients of monic polynomials.*

It is not hard to see that the trace function, which assigns to a linear operator T on a finite dimensional vector space A the trace of T, is invariant under Grothendieck equivalence on the category $\mathscr{L}$ of such operators. (This amounts to noticing that if A' is a subspace of A which is invariant under T, then the trace of T is the sum of the trace of the restriction of T to A' and the trace of the operator induced by T on A/A'.) In fact, if k is a nonnegative integer then Tr_k, where $\mathrm{Tr}_k(A, T)$ is trace A^k, is Grothendieck invariant, and induces a homomorphism of $[\mathscr{L}]$ into F. In view of the foregoing the function Tr_k must be itself a function of the characteristic polynomial. We now establish the explicit relationship between the sequence $\{\mathrm{Tr}_k(A, T)\}$ and the characteristic polynomial of T.

Let $F[[\lambda]]$ denote the integral domain of formal power series over F with indeterminate λ. The formal derivative $'$ is a derivation in

$F[[\lambda]]$ and can be extended via the usual formula for the derivative of a quotient to a derivation on the field of quotients $Q[\lambda]$. Each element $r \in Q[\lambda]$ has a unique *canonical representation* of the form $r = \lambda^m \sum_0^\infty a_j \lambda^j$ with $a_0 \neq 0$ for some integer m. (It is easy to see that r is in the field of quotients of $F[\lambda]$ if and only if $\{r, \lambda r, \lambda^2 r, \cdots\}$ spans a finite dimensional subspace of $Q[\lambda]/F[\lambda]$.)

THEOREM 3.3. *Let $f(\lambda) =$ characteristic polynomial of a linear operator $T\colon A \to A$ and let $r(\lambda) = f(1/\lambda) \in Q[\lambda]$. Then*

$$r'/r = -\sum_{j=0}^{\infty} \operatorname{Tr}(T^j)\lambda^{j-1} .$$

Proof. Note that

$$r(\lambda) = f\left(\frac{1}{\lambda}\right) = \left|\left(\frac{1}{\lambda}\right)I - T\right| = \lambda^{-\dim A}|\, I - \lambda T\,| .$$

Extending F if necessary, we may assume that $|\, I - \lambda T\,|$ splits into linear factors, say $|\, I - \lambda T\,| = \Pi_i(1 - \lambda e_i)$ for $e_1, \cdots, e_m \in F$, where $m = \dim A$. Then

$$r'/r = (-\dim A)\lambda^{-1} - \sum_{i=1}^{m} e_j/(1 - \lambda e_j) .$$

Since $e_j/(1 - \lambda e_j) = e_j + \lambda e_j^2 + \lambda^2 e_j^3 + \cdots$ we see that

$$r'/r = -\dim A\lambda^{-1} - \sum_{j=1}^{\infty}\left(\sum_{i=1}^{m} e_i^j\right)\lambda^{j-1} .$$

The proof of the result is thus reduced to showing that $\sum_{i=1}^m e_i^j = \operatorname{Tr}(T^j)$, but this follows easily on using the Jordan normal form of the matrix of T.

It is worthwhile to notice the algebraic nature of the relationship just established. Let $Q'[\lambda]$ be the multiplicative group of quotients of monic polynomials (an isomorph of the Grothendieck group $G[\mathscr{L}]$). Let φ be the function which assigns to $[A, T] \in [\mathscr{L}]$ the formal series, $\lambda^{-1}\sum_{j=0}^\infty \operatorname{Tr}(T^j)\lambda^j$. Then φ is a homomorphism of $[\mathscr{L}]$ into the additive abelian group S of formal series $\lambda^{-1}\sum_{j=0}^\infty a_j\lambda^j$, and consequently there is a homomorphism φ' of $Q'[A]$ into S so that the following diagram is commutative.

$$\begin{array}{ccccc}
[\mathscr{L}] & \xrightarrow{\gamma} & G[\mathscr{L}] & \xrightarrow[\approx]{} & Q'[\lambda] \subset Q[\lambda] . \\
& \searrow^{\varphi} & & \swarrow_{\varphi'} & \\
& & S & &
\end{array}$$

The preceding theorem yields the following description of φ': To obtain $\varphi'(r)$, where $r \in Q'[\lambda]$, (a) apply the multiplicative automorphism $p(\lambda) \mapsto p(1/\lambda)$ of $Q[\lambda]$, then (b) the formal logarithmic derivative $r(\lambda) \mapsto r'(\lambda)/r(\lambda)$, which is a homomorphism carrying multiplication into addition, and then (c) apply the isomorphism carrying each member of $Q[\lambda]$ into its canonical series. Thus φ' is exhibited as a multiplication-to-addition homomorphism.

In case F has characteristic zero, the homomorphism φ' is an isomorphism, as we now show. If F has characteristic zero, $\exp \lambda = \sum_{j=0}^{\infty} \lambda^j/j!$ is a well-defined element of $F[[\lambda]]$. If $s(\lambda) \in F[[\lambda]]$ has zero constant term, say $s = \sum_{j=1}^{\infty} a_j \lambda^j$, then the "composed series" $\exp s = \sum_{k=0}^{\infty} (\sum_{j=1}^{\infty} a_j \lambda^j)^k/k!$ is well-defined since each coefficient of a fixed power of λ is a finite sum. Furthermore, $(\exp s)' = (\exp s)s'$. The following shows how the sequence $\{\mathrm{Tr}\,(T^j)\}$ determines the characteristic polynomial of T.

THEOREM 3.4. *Let $T: A \to A$ be a linear operator on a finite dimensional vector space A over a field of characteristic zero and let $f(\lambda)$ be the characteristic polynomial of λ. If $r(\lambda) = f(1/\lambda)$, then*

$$r = \lambda^{-\dim A} \exp \left[-\sum_{j=1}^{\infty} (\mathrm{Tr}\,(T^j)/j)\lambda^j \right] .$$

Proof. Let $s = \lambda^{-\dim A} \exp [-\sum_{j=1}^{\infty} (\mathrm{Tr}\,(T^j)/j)\lambda^j]$. Then it is easily seen that $r'/r = s'/s$ whence $(r/s)' = 0$. Since r and s both have the form $\lambda^{-\dim A}$ times a power series with constant term 1, it follows that $r = s$.

It follows from the last theorem that if F has characteristic zero the sequence $\{\mathrm{Tr}\,(T^j)\}$ is a complete set of invariants for Grothendieck equivalence of the corresponding $F[\lambda]$ modules. This is not the case if F has nonzero characteristic p. For example, if A is a p-dimensional vector space and I is the identity operator on A, then the characteristic polynomial of A is the polynomial $(\lambda - 1)^p$ but $\mathrm{Tr}\,(I^j) = 0$ for all j. Thus, the zero operator and I are distinguished by their characteristic polynomials but not by the sequence of traces.

If $f(\lambda)$ is the characteristic polynomial of $T: A \to A$ and $r(\lambda) = f(1/\lambda)$, then we can expand $r(\lambda)$ in canonical representation

$$r = \lambda^{-\dim A}[1 + a_1\lambda + a_2\lambda^2 + \cdots] .$$

The coefficients a_j are called the *canonical coefficients* of T.

COROLLARY 3.5. *Let $\{a_j\}_{j \geqq 1}$ be the canonical coefficients of T. If $a_j = 0$ for $1 \leqq j < s$, then $\mathrm{Tr}\,(T^j) = 0$ for $1 \leqq j < s$ and $sa_s = -\mathrm{Tr}\,(T^s)$.*

If F has characteristic 0 *and* $\operatorname{Tr}(T^j) = 0$ *for* $1 \leqq j < s$, *then* $a_j = 0$ *for* $1 \leqq j < s$.

Proof. Since $r = \lambda^{-\dim A}(1 + \sum_{j=1}^{\infty} a_j\lambda^j)$ we see that

$$r'/r = -\dim A/\lambda + \left(\sum_{j=1}^{\infty} ja_j\lambda^{j-1}\right)\left(1 + \sum_{j=1}^{\infty} a_j\lambda^j\right)^{-1} .$$

Combining this with Theorem 3.3 we have

$$\sum_{j=1}^{\infty} ja_j\lambda^{j-1} = \left(1 + \sum_{j=1}^{\infty} a_j\lambda^j\right)\left(-\sum_{j=1}^{\infty} \operatorname{Tr}(T^j)\lambda^{j-1}\right)$$

and the corollary follows.

4. Euler equivalence. In this section we consider equivalence relations on a category $\mathscr{C}$ of finitely nonzero chain complexes. First we consider Grothendieck equivalence, and it turns out that two chain complexes are Grothendieck equivalent if and only if all their corresponding chain modules are Grothendieck equivalent as modules. Next we consider the more interesting Euler equivalence, generated by adjoining to Grothendieck equivalence the addition or delection of a split acyclic chain complex. The Euler equivalence classes form an abelian group and, in case the modules used in $\mathscr{C}$ are closed with respect to submodules and quotient modules, we give an alternate description of Euler equivalence.

Let $\mathscr{M}$ be a category of modules over an associative ring R with unit and assume $\mathscr{M}$ contains the trivial module and is closed with respect to finite sums. Let $\mathscr{C}$ be the category of finitely nonzero chain complexes with chain modules in $\mathscr{M}$. Thus, an object C of $\mathscr{C}$ is a collection $C = \{C_n, \partial_n\}$ indexed by the set of integers such that:

(1) C_n is an object of $\mathscr{M}$ and $C_n = 0$ except for finitely many n's.
(2) $\partial_n\colon C_n \to C_{n-1}$
(3) $\partial_n \circ \partial_{n+1} = 0$.

The morphisms in $\mathscr{C}$ are the chain maps. Thus, a morphism $\tau\colon C \to C'$ is a collection $\tau = \{\tau_n\}$ indexed by the set of integers such that:

(4) $\tau_n\colon C_n \to C'_n$ is in $\mathscr{M}$
(5) $\partial'_n \circ \tau_n = \tau_{n-1} \circ \partial_n$.

Note that $\mathscr{C}$ contains a zero chain complex and is closed with respect to finite sums.

The Grothendieck equivalence relation on $\mathscr{C}$ is generated by

$$\mathscr{R} = \{C, C' \oplus C'') \mid 0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0 \text{ is exact in } \mathscr{C}\} .$$

Denoting equivalence by $\sim$, note that $C \sim C'$ implies $C \oplus C'' \sim C' \oplus C''$ for any C''.

LEMMA 4.1. *Given C in $\mathscr{C}$ and an integer q, let C^q be the chain complex in $\mathscr{C}$ with*

$$(C^q)_n = \begin{cases} 0, & q \neq n \\ C_n, & q = n \end{cases}$$

and with $(\partial^q)_n$ the trivial map for all n. Then C^q is the trivial chain complex except for finitely many q's and C is Grothendieck equivalent to $\bigoplus_q C^q$.

Proof. We prove the lemma by induction on the number of nonzero chain modules in C. If C has no nonzero chain module, then C^q also has no nonzero chain module, whence, C^q is trivial for all q and $C = \bigoplus_q C^q$ so the lemma is valid in this case. If C has one nonzero chain module, say $C_p \neq 0$, then C^q is trivial unless $q = p$ and $C^p = C$ so the lemma is valid in this case also.

Assume the lemma valid for chain complexes having fewer than m nonzero chain modules where $m > 1$ and let C be a chain complex with m nonzero chain modules. Let p be the smallest integer such that $C_p \neq 0$. Then C^p is a chain subcomplex of C and the quotient complex C/C^p has the same chain modules as C except in degree p. Since there is a short exact sequence in $\mathscr{C}$

$$0 \longrightarrow C^p \longrightarrow C \longrightarrow C/C^p \longrightarrow 0$$

we have $C \sim C^p \oplus C/C^p$. Since C/C^p has $m - 1$ nonzero chain modules, the lemma holds for it and so

$$C/C^p \sim \bigoplus_q (C/C^p)^q = \bigoplus_{q \neq p} C^q$$

the last equality because $(C/C^p)^q = C^q$ if $q \neq p$ and $(C/C^p)^p = 0$. Therefore, we have

$$C \sim C^p \oplus C/C^p \sim \bigoplus_q C^q$$

and the proof is complete.

COROLLARY 4.2. *If C, C' are chain complexes in $\mathscr{C}$ such that $C_n \approx C'_n$ for all n, then C is Grothendieck equivalent to C'.*

Proof. The hypothesis implies that $C^q \approx C'^q$ for all q. Therefore, by the lemma we have $C \sim \bigoplus_q C^q \sim \bigoplus_q C'^q \sim C'$, which is the desired result.

The preceding results show that the Grothendieck equivalence class of a chain complex C is entirely independent of the boundary operator of C. Thus C is Grothendieck equivalent to C' if and only if C_n is Grothendieck equivalent to C'_n in the category $\mathscr{M}$ for each

n. We now concern ourselves with an equivalence relation which contains that of Grothendieck. The suspension $S(C)$ of a chain complex C is defined by $S(C)_n = C_{n-1}$ and ∂_n on $S(C)$ equal to ∂_{n-1} on C, for each integer n. The *Euler equivalence relation* is generated by $\mathscr{R} \cup \mathscr{R}'$ where

$$\mathscr{R} = \{(C, C' \oplus C'') \mid 0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0 \text{ is exact in } \mathscr{C}\}$$

and

$$\mathscr{R}' = \{(C, C \oplus C' \oplus S(C') \mid C, C' \text{ in } \mathscr{C}\} .$$

Notice that $C' \oplus S(C')$ is always of very special form. By redefining the boundary maps, which according to the lemma does not change the equivalence class, $C' \oplus S(C')$ is always exact, and in fact a sequence which splits (i.e. each boundary map is the direct sum of an isomorphism and a zero map). It is also easy to see that split exact complexes are isomorphic to $C \oplus S(C)$ for some C. Thus members of Euler equivalence classes are obtained from members of Grothendieck equivalence classes by "adjoining or removing as direct summands the split exact chain complexes."

We shall, for the rest of this section, use $\sim$ to denote Euler equivalence. For C in $\mathscr{C}$ let $[C]$ denote its equivalence class. There is a well-defined operation $+$ on equivalence classes such that

$$[C] + [C'] = [C \oplus C'] .$$

This is clearly commutative and associative and has the equivalence class of the trivial complex as unit element. Furthermore,

$$[C] + [S(C)] = [C \oplus S(C)] = 0$$

so that inverses exist. Thus, the set of equivalence classes is an abelian group with respect to $+$. This group is called the *Euler group* of $\mathscr{C}$ and denoted by $E(\mathscr{C})$.

If C is a chain complex its *homology module* $H(C)$ is the sequence $\{H_n(C)\}$ where $H_n(C) = \ker \partial_n / \operatorname{im} \partial_{n+1}$. C is said to be *acyclic* if $H_n(C) = 0$ for all n. Note that $H(C)$ need not be a collection of modules of $\mathscr{M}$. We shall sometimes refer to the *homology complex* $H(C)$, it being understood that the boundary homomorphisms are the zero homomorphisms. A chain complex is a *special* chain complex if and only if $\ker \partial_n$, $\operatorname{im} \partial_n$ and $H_n(C)$ all belong to $\mathscr{M}$ for every n.

LEMMA 4.3. *Each special chain complex is Euler equivalent to its homology complex.*

Proof. Let C be a special chain complex, $Z = \{Z_n\}$ where $Z_n =$

$\ker \partial_n$, and let $B = \{B_n\}$ where $B_n = \operatorname{im} \partial_{n+1}$. We consider Z and B to be chain complexes with trivial boundary maps. Since there is a short exact sequence $0 \to Z_n \to C_n \xrightarrow{\partial_n} B_{n-1} \to 0$ for every n it follows that there is an exact sequence (in $\mathscr{C}$ because C is special)

$$0 \longrightarrow Z \longrightarrow C \longrightarrow S(B) \longrightarrow 0 .$$

Therefore, $C \sim Z \oplus S(B)$.

Since there is also an exact sequence $0 \to B_n \to Z_n \to H_n(C) \to 0$ for every n it follows that there is an exact sequence (in $\mathscr{C}$ because C is special)

$$0 \longrightarrow B \longrightarrow Z \longrightarrow H(C) \longrightarrow 0 .$$

Therefore, $Z \sim B \oplus H(C)$. Combining these we have

$$C \sim B \oplus H(C) \oplus S(B) \sim H(C) .$$

It follows from the preceding lemma that for special complexes Euler equivalence is a homology invariant, in the sense that if C and C' are special and $H(C)$ is isomorphic (or even Euler equivalent) to $H(C')$ then C is Euler equivalent to C'. It seems unlikely that isomorphism of the homology complexes always implies Euler equivalence. However

PROPOSITION 4.4. *If $\mathscr{C}$ has the property that each acyclic complex is special, and if τ is a chain map of C into C' such that $\tau_*: H(C) \approx H(C')$, then C and C' are Euler equivalent.*

Proof. Let C'' be the mapping cone of τ [2, p. 166]. Then $C''_n = C_{n-1} \oplus C'_n$ and $\partial''(x, y) = (-\partial(x), \tau(x) + \partial'(y))$ for $x \in C_{n-1}$ and $y \in C'_n$. Let $\alpha: C' \to C''$ be the chain map $\alpha(y) = (0, y)$. Then α imbeds C' as a subcomplex of C'' and the quotient complex $C''/\alpha(C')$ has the property that $(C''/\alpha C')_n \approx C_{n-1} \approx (S(C))_n$ for all n. Therefore, $C''/\alpha(C')$ is Euler equivalent to $S(C)$. From the short exact sequence

$$0 \longrightarrow C' \xrightarrow{\alpha} C'' \longrightarrow C''/\alpha(C') \longrightarrow 0$$

we see that C'' is Euler equivalent to $C' \oplus C''/\alpha(C')$. Because $\tau_*: H(C) \approx H(C')$ it follows that $H(C'') = 0$. By the lemma, C'' is Euler equivalent to $H(C'') = 0$. Therefore, $C' \oplus S(C)$ is Euler equivalent to $C' \oplus C''/\alpha(C')$, which is Euler equivalent to 0. Since $C \oplus S(C)$ is also Euler equivalent to 0, it follows that C is Euler equivalent to C'.

The preceding proof actually establishes a slightly stronger result: if $\mathscr{C}$ has the property that each acyclic complex is Euler equivalent to 0 and if a chain map $\tau: C \to C'$ induces an isomorphism of homology, then C and C' are Euler equivalent.

A category $\mathscr{M}$ of modules is *closed* if every submodule and every quotient module of a module in $\mathscr{M}$ is also in $\mathscr{M}$. If $\mathscr{M}$ is closed, every chain complex in $\mathscr{C}$ is special, and we have the following description of Euler equivalence.

THEOREM 4.5. *If $\mathscr{M}$ is closed, Euler equivalence is generated by $\mathscr{R}'' \cup \mathscr{R}'''$ where*

$$\mathscr{R}'' = \{(C, C') \mid C_n \approx C'_n \text{ all } n\}$$
$$\mathscr{R}''' = \{(C, H(C)) \mid C \text{ in } \mathscr{C}\} .$$

Proof. By the lemma above $\mathscr{R}'''$ is contained in the Euler equivalence relation and by the corollary previously, $\mathscr{R}''$ is contained in the Euler equivalence relation. To complete the proof it suffices to show that the equivalence relation generated by $\mathscr{R}'' \cup \mathscr{R}'''$ (which we denote by $\equiv$) contains $\mathscr{R} \cup \mathscr{R}'$.

Given any C, C' let C'' be the chain complex with

$$C''_n = C_n \oplus C'_n \oplus C'_{n-1}$$

and with $\partial''_n(x, y, z) = (0, z, 0)$. Then

$$H_n(C'') = (C_n \oplus C'_n)/(0 \oplus C'_n) \approx C_n$$

and so

$$C'' \equiv H(C''') \equiv C .$$

Since $C'' \equiv C \oplus C' \oplus S(C')$ we see that $C \equiv C \oplus C' \oplus S(C)$. Therefore, $\equiv$ contains $\mathscr{R}'$.

Given a short exact sequence $0 \longrightarrow C' \xrightarrow{\alpha} C \longrightarrow C'' \longrightarrow 0$ let $\bar{C}$ be the chain complex with $\bar{C}_n = C_n \oplus C'_n \oplus C'_{n-1}$ and with

$$\bar{\partial}_n(x, y, z) = (\alpha(z), 0, 0) .$$

Then

$$H_n(\bar{C}) = (C_n \oplus C'_n)/(\alpha(C'_n) \oplus 0) \approx C''_n \oplus C'_n .$$

Therefore, $\bar{C} \equiv H(\bar{C}) \equiv C' \oplus C''$. Since it is clear that $\bar{C} \equiv C \oplus C' \oplus S(C')$, by the fact that $\equiv$ contains $\mathscr{R}'$ we have

$$C \equiv C \oplus C' \oplus S(C') \equiv \bar{C} \equiv C' \oplus C''$$

therefore, $\equiv$ contains $\mathscr{R}$.

5. The Euler characteristic.

In this section we define the Euler characteristic and prove that it is an isomorphism of the Euler group with a corresponding Grothendieck group. We deduce several consequences from this.

Given a chain complex C of $\mathscr{C}$ defined with respect to a category $\mathscr{M}$ of modules, its Euler characteristic $\chi(C)$ is the member of the Grothendieck group $G(\mathscr{M})$ defined by

$$\chi(C) = \sum_n (-1)^n \gamma[C_n]$$

where γ is the Grothendieck mapping. It is easy to verify that if C, C' are Euler equivalent then $\chi(C) = \chi(C')$. Hence, there is a well-defined function

$$\chi\colon E(\mathscr{C}) \longrightarrow G(\mathscr{M})$$

such that $\chi[C] = \chi(C)$, and this function is easily seen to be a homomorphism.

THEOREM 5.1. *The Euler characteristic induces an isomorphism*

$$\chi\colon E(\mathscr{C}) \approx G(\mathscr{M})$$

of the Euler group of $\mathscr{C}$ with the Grothendieck group of $\mathscr{M}$.

Proof. Any element of $G(\mathscr{M})$ can be expressed in the form $\gamma[A] - \gamma[B]$ where A, B are modules in $\mathscr{M}$. Let C be any chain complex with $C_0 = A$, $C_1 = B$ and $C_j = 0$ for $j \neq 0, 1$. Then $\chi(C) = \gamma[A] - \gamma[B]$ showing that χ is an epimorphism.

We show that χ is a monomorphism. Note that for any chain complex C in $\mathscr{C}$ we have

$$C \sim C \oplus S(C) \oplus S(S(C)) \sim S(S(C)) .$$

From this it follows that if C has exactly one nonzero chain module in degree q, then C is Euler equivalent to a chain complex having the same nonzero chain module but occuring in degree 0 if q is even and in degree 1 if q is odd. Thus, if C is any chain complex in $\mathscr{C}$ and C' is any chain complex with $(C')_0 = \bigoplus_j C_{2j}$, $(C')_1 = \bigoplus_j C_{2j+1}$ and $C'_q = 0$ if $q \neq 0, 1$ we see that

$$C \sim \bigoplus_q C^q \sim \bigoplus_j C^{2j} \oplus \bigoplus_j C^{2j+1} \sim (C')^0 \oplus (C')^1 \sim C' .$$

Therefore, to prove χ is a monomorphism it suffices to assume that $\chi(C) = 0$, and that C is such that $C_n = 0$ if $n \neq 0, 1$, and then to show that $[C] = 0$.

Thus, we have $\gamma(C_0) - \gamma(C_1) = 0$. From the explicit construction of $G(\mathscr{M})$ it follows that there is a module A in $\mathscr{M}$ such that $C_0 \oplus A$ is Grothendieck equivalent to $C_1 \oplus A$. If C' is any chain complex with $C'_n = \begin{cases} 0 & n \neq 0, 1 \\ C_n \oplus A, & n = 0, 1 \end{cases}$ then $C \sim C'$ (in fact (C, C') belongs to $\mathscr{R}'$). By using equivalences in $\mathscr{R}$ we can replace any chain module in C' by one which is Grothendieck equivalent to it without changing the Euler equivalence class. Thus, $C' \sim C''$ where

$$C''_n = \begin{cases} 0 & \text{if } n \neq 0, 1 \\ C_0 \oplus A & \text{if } n = 0, 1 \end{cases}.$$

Since $C''_0 = C''_1$ it follows that $C'' \sim 0$ and so we see that $C \sim 0$ and χ is a monomorphism.

If $\mathscr{M}$ is the category of finitely generated modules over a principal ideal domain, then $\mathscr{M}$ is closed and so Theorem 4.5 applies. In this case, since there is an isomorphism, rank: $G[\mathscr{M}] \approx \boldsymbol{Z}$, we can interpret the Euler characteristic in the usual way as the function $\chi(C) = \Sigma(-1)^j \operatorname{rank}(C_j)$. Combining these remarks we have the following

COROLLARY 5.2. *Let $\mathscr{C}$ be the category of finitely generated chain complexes over a principal ideal domain. If Φ is a function on $\mathscr{C}$ such that:*

(a) *If $C_n \approx C'_n$ for all n, then $\Phi(C) = \Phi(C')$.*

(b) *$\Phi(C) = \Phi(H(C))$ for all C.*

Then Φ is a function of the Euler characteristic in the sense that $\Phi = \varphi \circ \chi$ where φ is some function on $\boldsymbol{Z}$.

Let F be a field and let (C, T) be a finitely generated chain complex C over F with a chain map $T: C \to C$. Then (C, T) is equivalent to a finitely generated chain complex over the principal ideal domain $F[\lambda]$. By Theorem 4.5, Euler equivalence of two such complexes (C, T) and (C', T'), regarded as finitely generated chain complexes over $F[\lambda]$, is generated by the following (wherein C, C' are regarded as chain complexes over F with respective chain maps T, T'):

(a) C is equivalent to C' if $T_n: C_n \to C_n$ is isomorphic to $T'_n: C'_n \to C'_n$ for all n (i.e. there is an isomorphism $S_n: C_n \approx C'_n$ such that $T'_n \circ S_n = S'_n \circ T_n$).

(b) C with T is equivalent to $H(C)$ with T_* (where T_* is the linear map on homology induced by T).

The *characteristic rational form* χ' of (C, T) is defined by

$$\chi'(C, T) = \prod_q \operatorname{charpoly}(T_{2q}) / \prod_q \operatorname{charpoly}(T_{2q+1}) .$$

It follows from § 3 that the Euler characteristic of (C, T) is equivalent to χ'. Combining all of these remarks we obtain the following.

COROLLARY 5.3. *Let $\mathscr{C}$ be the category of (C, T) where C is a finitely generated chain complex over a field F and $T: C \to C$ is a chain map. If Φ is a function on $\mathscr{C}$ such that:*

(a) *If* $T_n: C_n \to C_n$ *is isomorphic to* $T'_n: C'_n \to C'_n$ *for all* n, *then* $\Phi'(C, T) = \Phi(C', T')$.

(b) $\Phi(C, T) = \Phi(H(C), T_*)$ *for any* (C, T).

Then Φ' *is a function of the characteristic rational form in the sense that* $\Phi' = \varphi' \circ \chi$ *where* φ' *is some function.*

Note that for each n the *Lefschetz number* $\Lambda(T^n)$ defined by $\Lambda(T^n) = \Sigma(-1)^j \operatorname{Tr}((T_j)^n)$ is a function satisfying (a) and (b) of the last theorem. The form of the relationship between the sequence $\{\Lambda(T^n)\}$ and $\chi'(C, T)$ is the same as the relationship between the sequence $\{\operatorname{Tr}(S^n)\}$ and the characteristic polynomial of S (where S is a linear operator on a finite dimensional vector space over F) given in § 3. Similarly, in case F has characteristic zero, two objects (C, T) and (C', T') in $\mathscr{C}$ are Euler equivalent if and only if $\Lambda(T^n) = \Lambda(T'^n)$ for all n.

We now turn to the case where $\mathscr{M}$ is the category of all finitely generated projective modules over an associative ring R with a unit. Let $\mathscr{C}$ be the class of finitely generated projective chain complexes over R. Recall that $\widetilde{K}_0(R)$ is the quotient of the Grothendieck group $G[\mathscr{M}]$ modulo the subgroup generated by $\gamma[R]$. We define the *Wall characteristic* $\theta(C) \in \widetilde{K}_0(R)$ to be the image of $\chi(C)$ under the quotient homomorphism. We shall give an interpretation of the Wall chacteristic.

Note that any direct summand of a projective module is itself projective and any direct summand of a finitely generated module is itself finitely generated. Therefore, any direct summand of an object in $\mathscr{M}$ is itself in $\mathscr{M}$.

LEMMA 5.4. *Each acyclic projective complex* C *is a special chain complex.*

Proof. Since $H_n(C) = 0$, we have $\ker \partial_n = \operatorname{im} \partial_{n+1}$ for all n. Thus it suffices to show that $\ker \partial_n$ is in $\mathscr{M}$ for all n. This is true for n small enough because C is finitely nonzero. We show that if $\ker \partial_n$ is in $\mathscr{M}$ then $\ker \partial_{n+1}$ is also in $\mathscr{M}$ and this will complete the proof.

If $\ker \partial_n$ is in $\mathscr{M}$, then $\operatorname{im} \partial_{n+1} = \ker \partial_n$ is also in $\mathscr{M}$. Since there is a short exact sequence

$$0 \longrightarrow \ker \partial_{n+1} \longrightarrow C_{n+1} \longrightarrow \operatorname{im} \partial_{n+1} \longrightarrow 0$$

and $\operatorname{im} \partial_{n+1}$ is projective, it follows that

$$C_{n+1} \approx \ker \partial_{n+1} \oplus \operatorname{im} \partial_{n+1} .$$

Therefore, $\ker \partial_{n+1}$ being a direct summand of C_{n+1}, belongs to $\mathscr{M}$.

As an immediate corollary to the preceding and Proposition 4.4 we have:

COROLLARY 5.5. *If* $\tau: C \rightarrow C'$ *is a chain map of a member of* $\mathscr{M}$ *into another such that* $\tau_*: H(C) \approx H(C')$, *then* C *is Euler equivalent to* C'.

In particular, chain equivalent complexes are Euler equivalent.

THEOREM 5.6. *Let* C *be a finitely generated projective chain complex over* R. *Then* C *is chain equivalent to a finitely generated free chain complex over* R *if and only if* $\theta(C) = 0$.

Proof. It follows from the corollary above that if C is chain equivalent to a finitely generated free chain complex C', then $\chi(C) = \chi(C')$ and so $\theta(C) = \theta(C') = 0$.

If C is a finitely generated projective chain complex, then for any n there is a finitely generated projective module A such that $C_n \oplus A$ is free. If C' is the chain complex with

$$C'_j = \begin{cases} 0 & j \neq n, \ n+1 \\ A & j = n, \ n+1 \end{cases}$$

and $\partial'_{n+1} = 1_A$, then the inclusion map $C \subset C \oplus C'$ is a chain equivalence and $(C \oplus C')_n = C_n \oplus A$ is free. By repeated use of this technique we see that C is chain equivalent to a finitely generated projective chain complex in which every chain module is free except possibly for the nonzero one of highest degree.

Thus, to complete the proof it suffices to show that if C is a finitely generated projective chain complex in which C_j is free if $j \neq m$, for some fixed m, and $\theta(C) = 0$, then C is chain equivalent to a free chain complex. The condition $\theta(C) = 0$ is equivalent to the condition that C_m be stably free, and thus there are free modules, F, F' such that $C_m \oplus F \approx F'$. Let C' be the chain complex with

$$C'_j = \begin{cases} 0, & j \neq m, \ m+1 \\ F, & j = m, \ m+1 \end{cases}$$

and with $\partial'_{j+1} = 1_F$. Then C is chain equivalent to $C \oplus C'$ and $C \oplus C'$ is a finitely generated free chain complex.

This last result shows that $\theta(C)$ is an obstruction to realizing C by a finitely generated free complex. This is the way that Wall [3] was led to introduce $\theta(C)$.

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