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ON WITT'S THEOREM FOR UNIMODULAR QUADRATIC FORMS

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In this paper we give an integral generalization of Witt's theorem for quadratic forms. If J and K are sublattices of a unimodular lattice L, we investigate conditions under which an isometry from J to K will extend to an isometry of L.

Let L be a free $\boldsymbol{Z}$-module (that is a lattice) of finite rank and $\Phi: L \times L \rightarrow \boldsymbol{Z}$ a unimodular symmetric bilinear form on L. We denote $\Phi(\alpha, \beta)$ by $\alpha \cdot \beta$, so that $\alpha \cdot \beta = \beta \cdot \alpha$. A bijective linear mapping $\varphi: J \rightarrow K$, where J and K are sublattices of L, is called an *isometry* if $\varphi(\alpha) \cdot \varphi(\beta) = \alpha \cdot \beta$ for $\alpha, \beta \in J$. Witt's theorem concerns the extension of such an isometry to an isometry of L (onto L). The set of isometries of L form the *orthogonal group* $O(L, \boldsymbol{Z})$ of L.

Vectors α and β in L are called *orthogonal* if $\alpha \cdot \beta = 0$; α^2 denotes $\alpha \cdot \alpha$, the *norm* of α. Any nonzero vector $\alpha \in L$ may be written as $\alpha = d\beta$ with $\beta \in L, d \in \boldsymbol{Z}$ maximal. If $d = 1$, α is called *primitive*; d is the *divisor* of α. It is clear that an isometry φ of L must leave invariant the divisors of all vectors; that is, α and $\varphi(\alpha)$ have the same divisor.

A sublattice U of L is called *primitive* if all the vectors of U which are "primitive in U" are also "primitive in L". In particular the basis vectors of U must be primitive (in L). In considering the extension of an isometry $\varphi: J \rightarrow K$ to an isometry of L, it clearly suffices to consider the case where J and K are primitive sublattices.

A primitive vector $\alpha \in L$ is called *characteristic* if $\alpha \cdot \beta \equiv \beta^2 \pmod 2$ for all $\beta \in L$. Again it is clear that an isometry must map a characteristic vector into a characteristic vector.

Let $r(L)$ and $s(L)$ denote the rank and signature of L. Then we shall prove the following.

THEOREM. *Let $\varphi: J \rightarrow K$ be an isometry between the primitive sublattices J and K of L, where*

$$(1) \qquad r(L) - |s(L)| \geqq 2(r(J) + 1) .$$

Then φ extends to an isometry of L if and only if:

α a characteristic vector $\Leftrightarrow \varphi(\alpha)$ a characteristic vector (for each α in J).

This result is a generalization of Wall [1]; in fact we shall use similar arguments and many of the results contained in Wall's paper.

1. Let $\langle\alpha_1, \alpha_2, \cdots, \alpha_m\rangle$ denote the lattice spanned by the vectors $\alpha_1, \alpha_2, \cdots, \alpha_m$. If L is the orthogonal direct sum of the sublattices U and V we write $L = U \oplus V$. In this case we say U (or V) *splits* L. $U^\perp$ will denote the orthogonal complement of U.

We show first how to reduce the proof to the case where $s(L) = 0$. Let $s(L) = s$. We consider the case $s > 0$ ($s < 0$ is similar). Enlarge the lattice L to

$$L' = L \oplus \langle\zeta_1\rangle \oplus \cdots \oplus \langle\zeta_s\rangle$$

where $\zeta_i^2 = -1$, $1 \leqq i \leqq s$, so that $s(L') = 0$. Let

$$J' = J \oplus \langle\zeta_1\rangle \oplus \cdots \oplus \langle\zeta_s\rangle$$

and

$$K' = K \oplus \langle\zeta_1\rangle \oplus \cdots \oplus \langle\zeta_s\rangle .$$

J' and K' are primitive sublattices of L'. Furthermore if L satisfies (1)

$$r(L') - s(L') = r(L) + s \geqq 2(r(J') + 1) .$$

Also, extending φ to J' by $\varphi(\zeta_i) = \zeta_i$, we see immediately that $\alpha \in J'$ is characteristic if and only if $\varphi(\alpha) \in K'$ is characteristic. (Notice that if $\alpha \in L'$ is characteristic, all the coefficients of the ζ_i in α must be odd.) If, therefore, we establish the theorem when the signature is zero, we know φ extends to an isometry of L'. Restricting back to L will establish the general result.

From now on we assume $s(L) = 0$. Let H denote a *hyperbolic plane* of the form $\langle\lambda, \mu\rangle$ where $\lambda^2 = \mu^2 = 0$ and $\lambda \cdot \mu = 1$; and let I denote a sublattice of the form $\langle\xi, \rho\rangle = \langle\xi\rangle \oplus \langle\xi - \rho\rangle$ where $\xi^2 = \xi \cdot \rho = 1$ and $\rho^2 = 0$. Then it is well known that any unimodular lattice of zero signature is either an orthogonal direct sum of H's (if *improper*) or an orthogonal direct sum of I's (if *proper*); see Wall [1, Th. 5]. We might also mention that if L is improper there are no primitive characteristic vectors.

Before proving the theorem we give an example to show the necessity of the restriction (1) we have placed on the ranks of L and J.

EXAMPLE. Let

$$L = H_1 \oplus H_2 \oplus \cdots \oplus H_n$$

where $H_i = \langle\lambda_i, \mu_i\rangle$, $1 \leqq i \leqq n$. Take

$$J = \langle \lambda_1, \cdots, \lambda_{n-1}, \lambda_n + uv\mu_n \rangle$$

and

$$K = \langle \lambda_1, \cdots, \lambda_{n-1}, u\lambda_n + v\mu_n \rangle$$

where u and v are integers ($\neq \pm 1$) such that $(u, v) = 1$. We shall show that the isometry $\varphi: J \to K$ defined by

$$(2)\qquad \begin{aligned} \varphi(\lambda_i) &= \lambda_i , \qquad 1 \leqq i \leqq n-1 , \\ \varphi(\lambda_n + uv\mu_n) &= u\lambda_n + v\mu_n , \end{aligned}$$

does not extend to an isometry of L. For if it did, (2) and the conditions $\lambda_i \cdot \varphi(\mu_n) = \varphi(\lambda_i) \cdot \varphi(\mu_n) = \lambda_i \cdot \mu_n = 0$, $1 \leqq i \leqq n-1$, would force

$$\varphi(\mu_n) = x_1\lambda_1 + x_2\lambda_2 + \cdots + x_{n-1}\lambda_{n-1} + x\lambda_n + y\mu_n$$

and

$$\begin{aligned} \varphi(\lambda_n) = & -uvx_1\lambda_1 - uvx_2\lambda_2 - \cdots - uvx_{n-1}\lambda_{n-1} \\ & + u(1 - vx)\lambda_n + v(1 - uy)\mu_n \end{aligned}$$

for some integers $x_1, \cdots, x_{n-1}, x, y$ as yet undetermined. But $\varphi(\mu_n)^2 = \mu_n^2 = 0$ implies that $xy = 0$; while $\varphi(\lambda_n) \cdot \varphi(\mu_n) = 1$ implies $xv + yu = 1$. These two conditions are incompatible with our choice $u, v \neq \pm 1$. Thus we need, at least, $r(L) > 2r(J)$.

We shall now proceed with the proof of the theorem. There will be three stages in the proof.

(i) First we establish the result when L is improper. In this case there are no characteristic vectors to consider.

(ii) Secondly, we consider L proper, but with J and K containing no characteristic vectors.

(iii) Finally, we treat the general proper case.

NOTATION. The following notation will be used for an isometry. Let

$$L = \langle \alpha_1, \alpha_2, \cdots, \alpha_m \rangle \oplus U = \langle \beta_1, \beta_2, \cdots, \beta_m \rangle \oplus U$$

where $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j$, $1 \leqq i, j \leqq m$. Then

$$\theta: \langle \alpha_1, \alpha_2, \cdots, \alpha_m \rangle \to \langle \beta_1, \beta_2, \cdots, \beta_m \rangle$$

is the isometry of L defined by $\theta(\alpha_i) = \beta_i$, $1 \leqq i \leqq m$, with θ restricted to U being the identity map.

Many of the isometries will be used repeatedly. We will label them $\theta_1, \theta_2, \cdots$ as they are defined so that we may refer back to them.

2. Throughout this section we let L be of the form

$$L = \langle \lambda_1, \mu_1 \rangle \oplus \cdots \oplus \langle \lambda_n, \mu_n \rangle$$

where each $\langle \lambda_i, \mu_i \rangle$ is a hyperbolic plane. The following lemma follows immediately from Wall [1, Th. 1].

LEMMA 1. *Let $r(L) \geqq 4$. For each primitive vector $\alpha \in L$ there exists an isometry $\psi \in o(L, \boldsymbol{Z})$ such that*

$$\psi(\alpha) = \lambda_1 + \frac{1}{2}\alpha^2 \mu_1 .$$

As a first step in the proof of the theorem we show there exists an isometry $\psi \in o(L, \boldsymbol{Z})$ such that $\psi(J) = \langle \alpha_1, \cdots, \alpha_m \rangle$, where

$$(3) \qquad \begin{cases} \alpha_1 = \lambda_1 + c_1\mu_1 \\ \alpha_2 = a_{12}\mu_1 + \lambda_2 + c_2\mu_2 \\ \cdots\cdots\cdots\cdots\cdots \\ \alpha_m = a_{1m}\mu_1 + a_{2m}\mu_2 + \cdots + a_{m-1m}\mu_{m-1} + \lambda_m + c_m\mu_m . \end{cases}$$

We use induction on m. The case $m = 1$ is Lemma 1. Assume now $\alpha_1, \alpha_2, \cdots, \alpha_h$ have been constructed using an isometry ψ_1; that is $\psi_1(J) = \langle \alpha_1, \cdots, \alpha_h, \beta, \gamma, \cdots \rangle$. Adding to β linear combinations of $\alpha_1, \cdots, \alpha_h$ (if necessary) we may assume β has the form

$$\beta = \sum_{i=1}^{h} b_i\mu_i + \sum_{i=h+1}^{n} (a_i\lambda_i + b_i\mu_i) .$$

By applying Lemma 1 on $E = \langle \lambda_{h+1}, \mu_{h+1} \rangle \oplus \cdots \oplus \langle \lambda_n, \mu_n \rangle$ to the component of β in E ($r(E) \geqq 4$ by (1)), we may assume

$$(4) \qquad \beta = \sum_{i=1}^{h} b_i\mu_i + a\lambda_{h+1} + b\mu_{h+1} .$$

If $(a, b) = 1$ we may obtain α_{h+1} by using Lemma 1 on the component $a\lambda_{h+1} + b\mu_{h+1}$ in E. Otherwise we proceed as follows. We may assume β primitive, so that $(b_1, \cdots, b_h, a, b) = 1$. Apply the isometry (writing k for $h + 2$);

$$\begin{aligned} \theta_1\colon & \langle \lambda_1, \mu_1 \rangle \oplus \langle \lambda_2, \mu_2 \rangle \oplus \cdots \oplus \langle \lambda_h, \mu_h \rangle \oplus \langle \lambda_k, \mu_k \rangle \longrightarrow \\ & \langle \lambda_1 - c_1\mu_k, \mu_1 + \mu_k \rangle \oplus \langle \lambda_2 - a_{12}\mu_k, \mu_2 \rangle \oplus \cdots \oplus \langle \lambda_h - a_{1h}\mu_k, \mu_h \rangle \\ & \quad \oplus \langle \lambda_k - \lambda_1 + c_1\mu_1 + a_{12}\mu_2 + \cdots + a_{1h}\mu_h + c_1\mu_k, \mu_k \rangle . \end{aligned}$$

Then, we see, $\theta_1(\alpha_i) = \alpha_i$ for $1 \leqq i \leqq h$, and $\theta_1(\beta) = \beta + b_1\mu_k$. Applying Lemma 1 to the component of $\theta_1(\beta)$ in E, namely $a\lambda_{h+1} + b\mu_{h+1} + b_1\mu_k$, we can transform it back to the form of (4), but now with

$$(b_2, b_3, \cdots, b_h, a, b) = 1 .$$

Repeating this process, this time in $\langle\lambda_1, \mu_1\rangle^\perp$, we may obtain a new β this time with $(b_3, \cdots, b_h, a, b) = 1$. Ultimately, we obtain a β with $(a, b) = 1$, so that we may finish by using lemma 1 as before.

It now suffices to prove the theorem with $J = \langle\alpha_1, \cdots, \alpha_m\rangle$. We shall prove the theorem by induction on $r(J)$. When $r(J) = 1$, the result follows from Wall (our Lemma 1). For the general case we may assume K has the form $\langle\alpha_1, \cdots, \alpha_{m-1}, \alpha\rangle$, with $\varphi\colon J \to K$ being the mapping defined by $\varphi(\alpha_i) = \alpha_i$ for $1 \geqq i \leqq m-1$, and

$$\varphi(\alpha_m) = \alpha = \sum_{i=1}^{m-1}(x_i\lambda_i + y_i\mu_i) + u\lambda_m + v\mu_m \, . \tag{5}$$

(It suffices to consider $u\lambda_m + v\mu_m$ by Lemma 1). It remains to find an isometry $\psi \in o(L, Z)$ such that $\psi(\alpha_i) = \alpha_i$ for $1 \leqq i \leqq m-1$, and $\psi(\alpha_m) = \alpha$.

We show first that we may take $u = 1$. Using Lemma 1, we may assume u divides v. Now $\alpha - \sum_{i=1}^{m-1} x_i\alpha_i$ is primitive (since K is a primitive lattice), so that

$$(u, z_{m-1}, \cdots, z_2, z_1) = 1 \tag{6}$$

where

$$\left\{\begin{aligned} z_{m-1} &= y_{m-1} - x_{m-1}c_{m-1} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ z_2 &= y_2 - x_2c_2 - x_3a_{23} - \cdots - x_{m-1}a_{2m-1} \\ z_1 &= y_1 - x_1c_1 - x_2a_{12} - \cdots - x_{m-1}a_{1m-1} \, . \end{aligned}\right. \tag{7}$$

We apply the isometry θ_1 again, but with h replaced by $m-1$ and $k(=h+2)$ by $m+1$. As before $\theta_1(\alpha_i) = \alpha_i$ for $1 \leqq i \leqq m-1$, but now

$$\theta_1(\alpha) = \alpha + z_1\mu_{m+1} \, .$$

Using Lemma 1 on $u\lambda_m + v\mu_m + z_1\mu_{m+1}$ in $\langle\lambda_m, \mu_m\rangle \oplus \langle\lambda_{m+1}, \mu_{m+1}\rangle$, we may replace α by a new α in which u divides z_1. By repeating this argument, now in $\langle\lambda_1, \mu_1\rangle^\perp$, we can get a new u again, this time also dividing z_2. Eventually, from (6), we may reduce u to 1.

Finally, we reduce the $x_1, \cdots, x_{m-1}$ in (5), in turn to zero. Apply the isometry

$$\begin{aligned} \theta_2\colon &\langle\lambda_1, \mu_1\rangle \oplus \langle\lambda_2, \mu_2\rangle \oplus \cdots \oplus \langle\lambda_{m-1}, \mu_{m-1}\rangle \oplus \langle\lambda_m, \mu_m\rangle \to \\ &\langle\lambda_1 - x_1c_1\mu_m, \mu_1 + x_1\mu_m\rangle \oplus \langle\lambda_2 - x_1a_{12}\mu_m, \mu_2\rangle \\ &\quad \oplus \cdots \oplus \langle\lambda_{m-1} - x_1a_{1m-1}\mu_m, \mu_{m-1}\rangle \\ &\quad \oplus \langle\lambda_m - x_1\lambda_1 + x_1c_1\mu_1 + x_1a_{12}\mu_2 + \cdots + x_1a_{1m-1}\mu_{n-1} + x_1^2c_1\mu_m, \mu_m\rangle \, . \end{aligned}$$

Then we have $\theta_2(\alpha_i) = \alpha_i$ for $1 \leqq i \leqq m-1$, and

$$\theta_2(\alpha) = (y_1 + x_1c_1)\mu_1 + x_2\lambda_2 + \cdots + x_{m-1}\lambda_{m-1} + (y_{m-1} + x_1a_{m-1})\mu_{m-1} + \lambda_m + w\mu_m ,$$

so that the coefficient of λ_1 is now zero. By repeating this process all the coefficients of $\lambda_1, \cdots, \lambda_{m-1}$ may be reduced to zero. But then, using the conditions $\alpha_i \cdot \alpha = \alpha_i \cdot \alpha_m$ for $1 \leqq i \leqq m-1$, and $\alpha^2 = \alpha_m^2$, we find that we have succeeded in mapping α into α_m, while leaving α_i, $1 \leqq i \leqq m-1$, invariant. This completes the proof of the theorem when L is improper.

3. For the rest of this paper L will be considered to be a proper lattice with zero signature. Thus we have

$$L = \langle \xi_1, \rho_1 \rangle \oplus \cdots \oplus \langle \xi_n, \rho_n \rangle$$

where $\xi_i^2 = \xi_i \cdot \rho_i = 1$ and $\rho_i^2 = 0$ for $1 \leqq i \leqq n$. By (1) we must have $n \geqq 2$. A primitive vector $\alpha = \sum_{i=1}^n (a_i\xi_i + b_i\rho_i)$ is characteristic if and only if $a_i \equiv 0 \pmod 2$ and $b_i \equiv 1 \pmod 2$ for each i. (We see this by applying the condition $\alpha \cdot \beta \equiv \beta^2 \pmod 2$ with β ranging through the basis vectors ξ_i, ρ_i).

LEMMA 2. *A primitive vector $\alpha \in L$ may be embedded in a binary sublattice B which splits L. If α is characteristic then B is proper and $B^\perp$ is improper. If α is not characteristic, then B is proper if α^2 is odd, and B is improper if α^2 is even.*

Proof. From Wall [1, p. 333], if $\alpha^2 = 2a + 1$ (and hence α is not characteristic), we can map α into $\xi_1 + a\rho_1$. Thus an isometric image of α is contained in $\langle \xi_1, \rho_1 \rangle$. Apply the inverse isometry to L. This will embed α in the inverse image of $\langle \xi_1, \rho_1 \rangle$. If α is not characteristic and $\alpha^2 = 2a$, then we may map α into

$$\beta = (a-1)\rho_1 + \xi_1 + \xi_2 .$$

Then $\beta \cdot \rho_2 = 1$. Put $\zeta = \beta - a\rho_2$, so that $\zeta^2 = 0$ and $\zeta \cdot \rho_2 = 1$. Then $\beta \in H = \langle \zeta, \rho_2 \rangle$, a binary sublattice splitting L. Thus α may similarly be embedded in an improper binary sublattice which splits L.

Finally, we consider the case where α is characteristic with norm $8b$. Take a splitting of L of the form

$$L = \langle \xi \rangle \oplus \langle \eta \rangle \oplus H_2 \oplus \cdots \oplus H_n$$

where $\xi^2 = -\eta^2 = 1$. The vector $\beta = (2b+1)\xi + (2b-1)\eta$ is characteristic with norm $8b$. Therefore α may be mapped by an isometry into $\beta \in \langle \xi \rangle \oplus \langle \eta \rangle$, and the result follows as before. This completes the proof of the lemma.

We will now consider the case where J and K do not contain characteristic vectors. We obtain an embedding of an isometric image of J as close as possible to that obtained in § 2. Suppose we have already obtained $\psi(J) = \langle \alpha_1, \cdots, \alpha_h, \beta_1, \cdots, \beta_k \rangle$ where $\alpha_1, \cdots, \alpha_h$ are of the form given in (3) and thus embedded in a sublattice

$$L_h = \langle \lambda_1, \mu_1 \rangle \oplus \cdots \oplus \langle \lambda_h, \mu_h \rangle$$

which splits L. Assuming that $k \geqq 3$, we now show how to obtain α_{h+1} (and as a special case α_1, to start the construction).

At least one of the three vectors $\beta_1, \beta_3, \beta_1 + \beta_3$ must have even norm. We may therefore assume, changing the basis of $\psi(J)$ if necessary, that β_1^2 and β_2^2 are even. Write

$$\beta_i = \sigma_i + d_i \tau_i \ , \qquad 1 \leqq i \leqq k \ ,$$

where $\tau_i \in L_h^\perp$ is primitive and $\sigma_i \in L_h$. It is possible that the τ_i, while not characteristic vectors in L, may be characteristic vectors in $L_h^\perp$. However, replacing β_1 by a linear combination of β_1 and β_2 if necessary, we may assume τ_1 at least is not characteristic in $L_h^\perp$. (We may achieve this by eliminating a suitable basis vector ρ between τ_1 and τ_2). There are two cases to consider.

Case 1. τ_1^2 even. Then by Lemma 2, τ_1 may be embedded in an improper binary sublattice H_1 of $L_h^\perp$. Since $k \geqq 2$, we have from (1) that the rank of $(L_h \oplus H_1)^\perp$ is at least 4. Therefore, there exists another hyperbolic plane H_2 such that

$$L = L_h \oplus H_1 \oplus H_2 \oplus U \ .$$

But now $\langle \alpha_1, \cdots, \alpha_h, \beta_1 \rangle \subseteqq L_h \oplus H_1 \oplus H_2$ and we may transform β_1 into the form α_{h+1} using the results already established for improper lattices in § 2.

Case 2. $\tau_1^2 = 2a + 1$ odd. Then since β_1^2 is even, $d_1^2 \tau_1^2$ is also even. As in the proof of Lemma 2, τ_1 may be embedded in a sublattice $I = \langle \xi, \rho \rangle$ with $\tau_1 = \xi + a\rho$. Again, from (1), we know the rank of $(L_h \oplus I)^\perp$ is at least 4, so that we may write L in the form

$$L = L_h \oplus I \oplus H \oplus U$$

where $H = \langle \lambda, \mu \rangle$ is a hyperbolic plane. Adding a linear combination of $\alpha_1, \cdots, \alpha_h$ to β_1, we may assume β_1 has the form

$$\beta_1 = \sum_{i=1}^{h} b_i \mu_i + d_1(\xi + a\rho)$$

where $(b_1, \cdots, b_h, d_1) = 1$. The next step is to apply isometries to

$L_h \oplus I \oplus H$ that leave $\alpha_1, \cdots, \alpha_h$ invariant, but change β_1 into a form as above with $d_1 = 1$. As in § 2, we may use θ_1 on $L_h \oplus H$ and Lemma 2 to achieve this. Applying θ_1 on $L_h \oplus H$, we transform β_1 into $\beta_1 + b_1\mu$, so that $d_1\tau_1$ becomes $d_1\tau_1 + b_1\mu = d'\tau'$ (say), where $d' = (d_1, b_1)$. If now τ'^2 is even we use case 1. Otherwise, as in Lemma 2, we transform τ' into $\xi + a'\rho$, and repeat the argument, this time introducing $b_2\mu$ by working in $\langle \lambda_1, \mu_1 \rangle^\perp$. Ultimately, since we may reduce d_1 to 1, we must get a form with τ_1^2 even, so that we can use Case 1.

In this manner we may apply a succession of isometries to J until we obtain $\psi(J) = \langle \alpha_1, \cdots, \alpha_{m-2}, \beta, \gamma \rangle$ where $\alpha_1, \cdots, \alpha_{m-2}$ are embedded in an improper sublattice L_{m-2} of L. Furthermore, we may assume β^2 is even. Write $\beta = \sigma + d\tau$ where $\tau \in L_{m-2}^\perp$ is primitive, and $\sigma \in L_{m-2}$. By adding a linear combination of $\alpha_1, \cdots, \alpha_{m-2}$ to β, we may assume

$$\beta = \sum_{i=1}^{m-2} b_i\mu_i + d\tau \tag{8}$$

and since J is primitive, we have $(b_1, \cdots, b_{m-2}, d) = 1$. τ may or may not be a characteristic vector in $L_{m-2}^\perp$. We show first how to reduce d to unity. By Lemma 2 τ may be embedded in a binary lattice B. Again by (1), the rank of $(L_{m-2} \oplus B)^\perp$ is at least 4, so that we may write

$$L = L_{m-2} \oplus B \oplus H \oplus U$$

where $H = \langle \lambda, \mu \rangle$ is a hyperbolic plane. Using θ_1 on $L_{m-2} \oplus H$ and Lemma 2, we reduce d to 1 as before. Then τ^2 is even.

If τ is not characteristic in $L_{m-2}^\perp$ we may use the argument of case 1 above to transform β into α_{m-1}. Suppose therefore τ is characteristic in $L_{m-2}^\perp$. But we know β is not characteristic in L. In (8), with $d = 1$, it therefore follows that at least one of the coefficients b_i must be odd. For if they were all even, β would be characteristic in L. Say b_s is odd. We apply an isometry of type θ_1 to

$$\langle \lambda_s, \mu_s \rangle \oplus \langle \lambda_{s+1}, \mu_{s+1} \rangle \oplus \cdots \oplus \langle \lambda_{m-2}, \mu_{m-2} \rangle \oplus H .$$

Then $\theta_1(\alpha_i) = \alpha_i$ for $1 \leqq i \leqq m-2$, and $\theta_1(\beta) = \beta + b_s\mu$. Then τ becomes $\tau + b_s\lambda$ which is no longer characteristic in $L_{m-2}^\perp$. Therefore β may always be transformed into the form α_{m-1} as before.

It therefore suffices to consider the case $J = \langle \alpha_1, \cdots, \alpha_{m-1}, \gamma \rangle$. We treat $K = \varphi(J)$ in a similar manner. Since the norms of the vectors $\varphi(\alpha_1), \cdots, \varphi(\alpha_{m-1})$ are even, and they are not characteristic vectors, they may be embedded in an improper sublattice L'_{m-1} which splits L. Adding hyperbolic planes to L_{m-1} and L'_{m-1} (they exist since the rank of $L_{m-1}^{\perp}$ is at least 4) and applying our theorem, already established for the improper case, we may assume $\varphi(\alpha_i) = \alpha_i$ for $1 \leqq i \leqq m-1$. Thus it suffices to consider K of the form $\langle \alpha_1, \cdots, \alpha_{m-1}, \delta \rangle$. There are now two cases depending on whether $\gamma^2 = \delta^2$ is odd or even.

Case 1. $\gamma^2 = \delta^2$ odd. Using Lemma 2 and $\alpha_1, \cdots, \alpha_{m-1}$ to eliminate the coefficients of $\lambda_1, \cdots, \lambda_{m-1}, \gamma$ may be written as

$$\gamma = \sum_{i=1}^{m-1} u_i \mu_i + d(\xi' + a\rho') \tag{9}$$

where $(u_1, \cdots, u_{m-1}, d) = 1$. L may be split thus

$$L = L_{m-1} \oplus \langle \xi', \rho' \rangle \oplus \langle \xi, \rho \rangle \oplus U .$$

We show first how to reduce d to unity. Apply the isometry

$$\begin{aligned} \theta_3 \colon & \langle \lambda_1, \mu_1 \rangle \oplus \langle \lambda_2, \mu_2 \rangle \oplus \cdots \oplus \langle \lambda_{m-1}, \mu_{m-1} \rangle \oplus \langle \xi, \rho \rangle \rightarrow \\ & \langle \lambda_1 - c_1\rho, \mu_1 + \rho \rangle \oplus \langle \lambda_2 - a_{12}\rho, \mu_2 \rangle \oplus \cdots \oplus \langle \lambda_{m-1} - a_{1m-1}\rho, \mu_{m-1} \rangle \\ & \oplus \langle \xi - \lambda_1 + c_1\mu_1 + a_{12}\mu_2 + \cdots + a_{1m-1}\mu_{m-1} + c_1\rho, \rho \rangle . \end{aligned}$$

We may easily check that $\theta_3(\alpha_i) = \alpha_i$ for $1 \leqq i \leqq m-1$. Furthermore $\theta_3(\gamma) = \gamma + u_1\rho$. Mapping $d(\xi' + a\rho') + u_1\rho$ back into $\langle \xi', \rho' \rangle$ we may restore γ to the form (9), but now with d dividing u_1. Now repeating this process in $\langle \lambda_1, \mu_1 \rangle^{\perp}$, we may obtain a new γ with d also dividing u_2. Since $(u_1, \cdots, u_{m-1}, d) = 1$ we ultimately reach a form with $d = 1$.

Using again Lemma 2, we may arrange for δ to have the form

$$\delta = \sum_{i=1}^{m-1} (x_i\lambda_i + y_i\mu_i) + f(\xi' + e\rho') . \tag{10}$$

We may assume $\delta - \sum_{i=1}^{m-1} x_i\alpha_i$ is primitive (since K is primitive) and therefore, using the notation of (7)

$$(f, z_1, z_2, \cdots, z_{m-1}) = 1 .$$

Applying θ_3, we find $\theta_3(\delta) = \delta + z_1\rho$. By the usual chain of arguments we may assume $f = 1$ in (10).

Finally we apply isometries that reduce $x_1, \cdots, x_{m-1}$ in turn to zero. Define

$$\begin{aligned} \theta_4 & \langle \lambda_1, \mu_1 \rangle \oplus \langle \lambda_2, \mu_2 \rangle \oplus \cdots \oplus \langle \lambda_{m-1}, \mu_{m-1} \rangle \\ & \oplus \langle \xi', \rho' \rangle \rightarrow \langle \lambda_1 - c_1x_1\rho', \mu_1 + x_1\rho' \rangle \\ & \oplus \langle \lambda_2 - x_1a_{12}\rho', \mu_2 \rangle \oplus \cdots \oplus \langle \lambda_{m-1} - x_1a_{1m-1}\rho', \mu_{m-1} \rangle \\ & \oplus \langle \xi' - x_1\lambda_1 + x_1c_1\mu_1 + x_1a_{12}\mu_2 + \cdots + x_1a_{1m-1}\mu_{m-1} + x_1^2c_1\rho', \rho' \rangle . \end{aligned}$$

Then $\theta_4(\alpha_i) = \alpha_i$ for $1 \leqq i \leqq m-1$, and

$$\theta_4(\delta) = (y_1 + x_1 c_1)\mu_1 + x_2\lambda_2 + \cdots + x_{m-1}\lambda_{m-1} + (y_{m-1} + x_1 a_{1m-1})\mu_{m-1} + \xi' + e'\rho' .$$

We have thus reduced the coefficient of λ_1 to zero. Proceeding in this manner we may reduce all the coefficients of $\lambda_1, \cdots, \lambda_{m-1}$ to zero. Using the conditions $\alpha_i \cdot \gamma = \alpha_i \cdot \delta$ and $\gamma^2 = \delta^2$, we find we have mapped δ into γ, and hence K into J, by an isometry of L. This completes the proof in this case.

Case 2. $\gamma^2 = \delta^2$ even. Write $\gamma = \sigma + d\tau$ where $\tau \in L_{m-1}^{\perp}$ is primitive and $\sigma \in L_{m-1}$. We first show that we may take $d = 1$. We use a combination of the previous methods. We may assume γ has the form (compare (8))

$$\gamma = \sum_{i=1}^{m-1} u_i \mu_i + d\tau$$

where $(u_1, \cdots, u_{m-1}, d) = 1$. If τ is characteristic in $L_{m-1}^{\perp}$, we may embed τ in a proper binary lattice B such that

$$L_{m-1}^{\perp} = B \oplus H_1 \oplus \cdots \oplus H_t .$$

Applying the isometry θ_1 on $L_{m-1} \oplus H_1$, as before, we may assume d divides u_1. If τ is not characteristic in $L_{m-1}^{\perp}$, we embed τ in a binary lattice B so that L splits thus

$$L = L_{m-1} \oplus B \oplus \langle \xi, \rho \rangle \oplus U .$$

Applying θ_3 on $L_{m-1} \oplus \langle \xi, \rho \rangle$, as before, we may assume d divides u_1. Proceeding in this manner we reduce d to unity. Then τ^2 is even and may be embedded in a hyperbolic plane H(after another isometry if τ is characteristic in $L_{m-1}^{\perp}$), so that, in fact, γ takes the form α_m given in (3).

By similar reasoning δ may be written

$$\delta = \sum_{i=1}^{m-1} (x_i \lambda_i + y_i \mu_i) + d\tau ,$$

d reduced to unity, and τ embedded in H. Finally we reduce the coefficients $x_1, \cdots, x_{m-1}$ to zero by applying θ_2, exactly as at the end of § 2.

This completes the proof of the theorem when J and K contain no characteristic vectors.

4. It remains for us to consider the case where J and K contain characteristic vectors. As in § 3, L has the form

$$L = \langle \xi_1, \rho_1 \rangle \oplus \cdots \oplus \langle \xi_n, \rho_n \rangle$$

where $\xi_i^2 = \xi_i \cdot \rho_i = 1$ and $\rho_i^2 = 0$, $1 \leqq i \leqq n$.

We may choose a basis for J that contains only one characteristic vector; for example, eliminate the coefficients of ρ_1 in all but one of the basis vectors. Applying the results of the previous section, it therefore suffices to consider the special case where

$$J = \langle \alpha_1, \cdots, \alpha_{m-2}, \beta, \gamma \rangle \quad \text{and} \quad K = \langle \alpha_1, \cdots, \alpha_{m-2}, \beta, \delta \rangle$$

with the α_i as in (3), $\delta = \varphi(\gamma)$ is characteristic, and with β either α_{m-1} (if β^2 is even) or of the form given in (9) with $d = 1$. There are therefore two cases to consider depending on whether the norm of β is even or odd.

Case 1. β^2 even; so that $\beta = \alpha_{m-1}$ and $J = \langle \alpha_1, \cdots, \alpha_{m-1}, \gamma \rangle$. γ may be assumed to have the form

$$\gamma = \sum_{i=1}^{m-1} u_i \mu_i + d(2\xi + (2e - 1)\rho)$$

(after using the α_i to eliminate the coefficients of the λ_i, and Lemma 2 to simplify the component of γ in $L_{m-1}^{\perp}$). L may now be written

$$L = L_{m-1} \oplus \langle \xi, \rho \rangle \oplus U$$

where U is an orthogonal sum of hyperbolic planes. By the usual argument we may reduce d to unity. Similarly, we can transform δ into

$$\delta = \sum_{i=1}^{m-1} (x_i \lambda_i + y_i \mu_i) + 2\xi + (2f - 1)\rho \ .$$

It therefore remains to transform δ into a form where the coefficients of λ_i are zero. Since δ is characteristic

$$x_i = \delta \cdot \mu_i \equiv \mu_i^2 \equiv 0 \pmod 2 \ , \qquad 1 \leqq i \leqq m - 1 \ .$$

Now apply the isometry

$$\begin{aligned}
\theta_5 \colon \langle \lambda_1, \mu_1 \rangle \oplus \langle \lambda_2, \mu_2 \rangle \oplus \cdots \oplus \langle \lambda_{m-1}, \mu_{m-1} \rangle \\
\oplus \langle \xi, \rho \rangle \to \left\langle \lambda_1 - \frac{1}{2} x_1 c_1 \rho, \mu_1 + \frac{1}{2} x_1 \rho \right\rangle \\
\oplus \left\langle \lambda_2 - \frac{1}{2} x_1 a_{12} \rho, \mu_2 \right\rangle \oplus \cdots \oplus \left\langle \lambda_{m-1} - \frac{1}{2} x_1 a_{1m-1} \rho, \mu_{m-1} \right\rangle \\
\oplus \left\langle \xi - \frac{1}{2} x_1 \lambda_1 + \frac{1}{2} x_1 c_1 \mu_1 + \frac{1}{2} x_1 a_{12} \mu_2 + \cdots \right. \\
\left. + \frac{1}{2} x_1 a_{1m-1} \mu_{m-1} + \frac{1}{4} x_1^2 c_1 \rho, \rho \right\rangle .
\end{aligned}$$

Then $\theta_5(\alpha_i) = \alpha_i$ for $1 \leqq i \leqq m - 1$, and

$$\theta_5(\delta) = (y_1 + x_1c_1)\mu_1 + x_2\lambda_2 + \cdots + x_{m-1}\lambda_{m-1} + (y_{m-1} + x_1a_{1m-1})\mu_{m-1} + 2\xi + (2f' - 1)\rho .$$

We have thus reduced the coefficient of λ_1 to zero. Proceeding in this manner, we may reduce all the coefficients of the λ_i in turn to zero. Finally, since $\alpha_i \cdot \gamma = \alpha_i \cdot \delta$ and $\gamma^2 = \delta^2$, the coefficients of δ now match those in γ, so that we have mapped δ into γ, and so K into J. This completes the proof in this case.

Case 2. β^2 odd. Then β may be chosen as

$$\beta = \sum_{i=1}^{m-2} b_i\mu_i + \xi + b\rho .$$

Using $\alpha_1, \cdots, \alpha_{m-2}$ and β to eliminate the coefficients of $\lambda_1, \cdots, \lambda_{m-2}$ and ξ, we may write γ as

$$\gamma = \sum_{i=1}^{m-2} u_i\mu_i + u\rho + d(2\xi' + (2e - 1)\rho') .$$

L is now split into the form

$$L = L_{m-2} \oplus \langle \xi, \rho \rangle \oplus \langle \xi', \rho' \rangle \oplus H \oplus U$$

where $H = \langle \lambda, \mu \rangle$ and U is an improper lattice (see Lemma 2). We now reduce the coefficient d to unity. Isometries on $L_{m-2} \oplus \langle \xi, \rho \rangle \oplus H$ of the type

$$\begin{aligned} \theta_6\colon & \langle \lambda_1, \mu_1 \rangle \oplus \langle \lambda_2, \mu_2 \rangle \oplus \cdots \oplus \langle \lambda_{m-2}, \mu_{m-2} \rangle \oplus \langle \xi, \rho \rangle \\ & \oplus \langle \lambda, \mu \rangle \to \langle \lambda_1 - c_1\mu, \mu_1 + \mu \rangle \oplus \langle \lambda_2 - a_{12}\mu, \mu_2 \rangle \oplus \cdots \\ & \oplus \langle \lambda_{m-2} - a_{1m-2}\mu, \mu_{m-2} \rangle \oplus \langle \xi - b^1\mu, \rho \rangle \\ & \oplus \langle \lambda - \lambda_1 + c_1\mu_1 + a_{12}\mu_2 + \cdots + a_{1m-2}\mu_{m-2} + b_1\rho + c_1\mu, \mu \rangle \end{aligned}$$

leave $\alpha_1, \cdots, \alpha_{m-2}$ and β invariant. γ is transformed into $\gamma + u_1\mu$, so that with the usual argument we may assume d divides u_1. We may transform γ in this manner into a form where $(u, d) = 1$.

Since γ is characteristic we know $\gamma \cdot \xi' \equiv 1 \pmod 2$, and hence that d is odd. Now apply the isometry

$$\begin{aligned} \theta_7\colon \langle \xi, \rho \rangle \oplus \langle \lambda, \mu \rangle \to & \langle \xi - 2b\mu, \rho + 2\mu \rangle \\ & \oplus \langle \lambda - 2\xi + 2(1 + b)\rho + 2(2b + 1)\mu, \mu \rangle . \end{aligned}$$

This leaves $\alpha_1, \cdots, \alpha_{m-2}$ and β invariant and transforms γ into $\gamma + 2u\mu$. Since $(2u, d) = 1$, we may reduce d to 1 in γ.

As above we may also put $\delta = \varphi(\gamma)$ into the form

$$\delta = \sum_{i=1}^{m-2} (x_i\lambda_i + y_i\mu_i) + v\xi + w\rho + 2\xi' + (2f - 1)\rho' .$$

Since δ is characteristic, we have $x_i \equiv y_i \equiv 0 \pmod 2$ for each i, $v \equiv 0 \pmod 2$ and $w \equiv 1 \pmod 2$.

It now remains to reduce the coefficients $x_1, \cdots, x_{m-2}, v$ to zero. First apply the isometry

$$\begin{aligned}\theta_8\colon &\langle\lambda_1, \mu_1\rangle \oplus \langle\lambda_2, \mu_2\rangle \oplus \cdots \oplus \langle\lambda_{m-2}, \mu_{m-2}\rangle \oplus \langle\xi, \rho\rangle \oplus \langle\xi', \rho'\rangle \to \\ &\left\langle \lambda_1 - \frac{1}{2}x_1c_1\rho', \mu_1 + \frac{1}{2}x_1\rho' \right\rangle \oplus \left\langle \lambda_2 - \frac{1}{2}x_1a_{12}\rho', \mu_2 \right\rangle \oplus \cdots \\ &\oplus \left\langle \lambda_{m-2} - \frac{1}{2}x_1a_{1m-2}\rho', \mu_{m-2} \right\rangle \oplus \left\langle \xi - \frac{1}{2}x_1b_1\rho', \rho \right\rangle \\ &\oplus \left\langle \xi' - \frac{1}{2}x_1\lambda_1 + \frac{1}{2}x_1c_1\mu_1 + \frac{1}{2}x_1a_{12}\mu_2 + \cdots \right. \\ &\qquad \left. + \frac{1}{2}x_1a_{1m-2}\mu_{m-2} + \frac{1}{2}x_1b_1\rho + \frac{1}{4}x_1^2c_1\rho', \rho' \right\rangle .\end{aligned}$$

Then $\theta_8(\alpha_i) = \alpha_i$ for $1 \leqq i \leqq m-2$, $\theta_8(\beta) = \beta$, and in $\theta_8(\delta)$ the coefficient of λ_1 is zero. Working now in $\langle\lambda_1, \mu_1\rangle^\perp$ we reduce the coefficient of λ_2 to zero. We may therefore assume

$$x_1 = x_2 = \cdots = x_{m-2} = 0 .$$

The final step, the reduction of v to zero appears to be more difficult. If $v \equiv 0 \pmod 4$ we may apply the isometry

$$\begin{aligned}\theta_9\colon \langle\xi, \rho\rangle \oplus \langle\xi', \rho'\rangle \to &\left\langle \xi - \frac{1}{2}vb\rho', \rho + \frac{1}{2}v\rho' \right\rangle \\ &\oplus \left\langle \xi' - \frac{1}{2}v\xi + \frac{1}{2}v(1+b)\rho + t\rho', \rho' \right\rangle\end{aligned}$$

where $2t = (1/4)v^2(1+2b)$. (If $v \equiv 2 \pmod 4$ then $t \notin \boldsymbol{Z}$). Then θ_9 leaves $\alpha_1, \cdots, \alpha_{m-2}$ and β invariant, while the coefficient of ξ in $\theta_9(\delta)$ is reduced to zero. From the various products $\delta\cdot\alpha_i = \gamma\cdot\alpha_i$, $1 \leqq i \leqq m-2$, $\delta\cdot\beta = \gamma\cdot\beta$ and $\delta^2 = \gamma^2$ we see that all the coefficients of δ (actually an isometric image of our original δ) now match those of γ. Thus we have mapped δ into γ and so K into J.

If, however, $v \equiv 2 \pmod 4$ we must modify the above argument. We first change the basis of L so that $G = \langle\xi', \rho'\rangle \oplus \langle\lambda, \mu\rangle$ becomes $G = \langle\xi_1, \rho_1\rangle \oplus \langle\xi_2, \rho_2\rangle$ where $\xi_i^2 = \xi_i\cdot\rho_i = 1$ and $\rho_i^2 = 0$ for $i = 1, 2$. Since the characteristic vector $2\xi' + (2f-1)\rho'$ in G can be mapped into any other characteristic vector of G by an isometry, we may assume δ has the form

$$\delta = \sum_{i=1}^{m-2} y_i\mu_i + v\xi + w\rho + 2\xi_1 + (2e_1 - 1)\rho_1 + 2\xi_2 + (2e_2 - 1)\rho_2$$

where e_1 is chosen such that

$$2e_1 + w - 1 \equiv 0 \pmod 4 ,$$

(recall that $w \equiv 1 \pmod 2$ since δ is characteristic).

We now apply the isometry

$$\begin{aligned}\theta_{10}\colon \langle \xi, \rho\rangle \oplus \langle \xi_1, \rho_1\rangle \to \\ &\langle (1-b)\xi + b(1+b)\rho + b\xi_1 + b(b-1)\rho_1, \\ &\quad \xi - b\rho - \xi_1 + (1-b)\rho_1\rangle \\ &\oplus \langle -b\xi + b(1+b)\rho + (1+b)\xi_1 + b(b-1)\rho_1, \\ &\quad -\xi + (1+b)\rho + \xi_1 + b\rho_1\rangle .\end{aligned}$$

Again $\alpha_1, \cdots, \alpha_{m-2}$ and β are left invariant by θ_{10}. But the coefficient of ξ is changed from v to $v' = v - vb + w - 2b - (2e_1 - 1)$. But now

$$\begin{aligned} v' &\equiv 2 - 2b + w - 2b - 2e_1 + 1 \\ &\equiv 2e_1 - 1 + w \equiv 0 \pmod 4 .\end{aligned}$$

After restoring G to the form $\langle \xi', \rho'\rangle \oplus \langle \lambda, \mu\rangle$ we are in a position to finish the proof by means of the isometry θ_9 as above.

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