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EXTENSIONS OF OPIAL'S INEQUALITY

P. R. BEESACK AND K. M. DAS

In this paper certain inequalities involving integrals of powers of a function and of its derivative are proved. The prototype of such inequalities is Opial's Inequality which states that $2\int_0^X |yy'|\,dx \leqq X\int_0^X y'^2 dx$ whenever y is absolutely continuous on $[0, X]$ with $y(0) = 0$. The extensions dealt with here are all integral inequalities of the form

$$\int_a^b s\,|y|^p\,|y'|^q\,dx \leqq K(p, q)\int_a^b r\,|y'|^{p+q}\,dx\ ,$$

(or with $\leqq$ replaced by $\geqq$), where r, s are nonnegative, measurable functions on $I = [a, b]$, and y is absolutely continuous on I with either $y(a) = 0$, or $y(b) = 0$, or both. In some cases y may be complex-valued, while in other cases y' must not change sign on I. The inequality (as stated) is obtained in case $pq > 0$ and either $p + q \geqq 1$ or $p + q < 0$, while the opposite inequality is obtained in case $p < 0, q \geqq 1, p + q < 0$, or $p > 0, p + q < 0$. In all cases, necessary and sufficient conditions are obtained for equality to hold.

1. In a recent paper [11], G. S. Yang proved the following generalization of an inequality of Z. Opial [7]:

If y *is absolutely continuous on* $[a, X]$ *with* $y(a) = 0$, *and if* $p, q \geqq 1$, *then*

$$(1)\qquad \int_a^X |y|^p\,|y'|^q\,dx \leqq \frac{q}{p+q}(X-a)^p\int_a^X |y'|^{p+q}\,dx\ .$$

Yang's proof is actually valid for $p \geqq 0, q \geqq 1$. For $p = q = 1, a = 0$, (1) is Opial's result. (See also Olech [6], Beesack [1], Levinson [4], Mallows [5], and Pederson [8] for successively simpler proofs of Opial's inequality; as well as Redheffer [9] for other generalizations of this inequality.) The case $q = 1, p$ a positive integer, was proved by Hua [3], and the result for $q = 1, p \geqq 0$ is included in a generalization of Calvert [2]; a short, direct proof of the latter case was also given by Wong [10]. If $q = 1$ the inequality (1) is sharp, but it is not sharp for $q > 1$.

2. The purpose of this paper is to obtain sharp generalizations of (1), and to consider other values of the parameters p, q; the method of proof is a modification of that of Yang [11]. To this end, *we suppose first that* y *is absolutely continuous on* $[a, X]$, *where* $-\infty \leqq a < X \leqq \infty$, *and that* y' *does not change sign on* (a, X), *so that*

$$(2) \qquad |y(x)| = \int_a^x |y'(t)|\,dt\,, \qquad a \leqq x \leqq X\,.$$

If r is nonnegative on (a, X) and the integrals exist, then it follows from Hölder's inequality that

$$(3) \qquad \int_a^x |y'|\,dt \leqq \left(\int_a^x r^{-\{1/(p+q-1)\}}dt\right)^{(p+q-1)/(p+q)}\left(\int_a^x r\,|y'|^{p+q}\,dt\right)^{1/(p+q)}$$

if $p + q > 1$, while

$$(4) \qquad \int_a^x |y'|\,dt \geqq \left(\int_a^x r^{-\{1/(p+q-1)\}}dt\right)^{(p+q-1)/(p+q)}\left(\int_a^x r\,|y'|^{p+q}\,dt\right)^{1/(p+q)}$$

if either $p + q < 0$ or $0 < p + q < 1$. Taking the case $p + q > 1$, we suppose first that $p > 0, q > 0$. Then,

$$(5) \qquad |y|^p \leqq \left(\int_a^x r^{-\{1(p+q-1)\}}dt\right)^{p(p+q-1)/(p+q)}\left(\int_a^x r\,|y'|^{p+q}\,dt\right)^{p/(p+q)} \qquad a \leqq x \leqq X\,.$$

Now, set $z(x) = \int_a^x r\,|y'|^{p+q}\,dt$. So $z' = r\,|y'|^{p+q}$, and

$$|y'|^q = r^{-\{q/(p+q)\}}(z')^{q/(p+q)}\,.$$

Thus, *if s is nonnegative on* (a, X),

$$s\,|y|^p\,|y'|^q \leqq s r^{-\{q/(p+q)\}}\left(\int_a^x r^{-\{1/(p+q-1)\}}dt\right)^{p(p+q-1)/(p+q)} z^{p/(p+q)}(z')^{q/(p+q)}\,.$$

If we assume the existence of the following integrals, then applying Hölder's inequality again, with indices $(p + q)/p$ and $(p + q)/q$, we obtain

$$(6) \qquad \begin{aligned}\int_a^X s\,|y|^p\,|y'|^q\,dx &\leqq K_1(X, p, q)\left(\int_a^X z^{p/q}z'dx\right)^{q/(p+q)}\\ &= K_1(X, p, q)\int_a^X r\,|y'|^{p+q}\,dx\,,\end{aligned}$$

since $z(a) = 0$ and $(p + q)/q > 0$. Here,

$$(7) \qquad \begin{aligned}&K_1(X, p, q)\\ &\quad = \left(\frac{q}{p+q}\right)^{q/(p+q)}\left\{\int_a^X s^{(p+q)/p} r^{-(q/p)}\left(\int_a^x r^{-\{1/(p+q-1)\}}dt\right)^{p+q-1}dx\right\}^{p/(p+q)}\,.\end{aligned}$$

Similarly, if $p < 0$ and $q < 0$, then (5) again follows from (2) and (4). As above, since $(p + q)/p > 1$ and $(p + q)/q > 1$ again, we obtain inequality (6). This proves the main part of

THEOREM 1. *Let* p, q *be real numbers such that* $pq > 0$, *and* *either* $p + q > 1$, *or* $p + q < 0$, *and let* r, s *be nonnegative, measurable functions on* (a, X) *such that* $\int_a^X r^{-1/(p+q-1)}dx < \infty$, *and the constant* $K_1(X, p, q)$ *defined by* (7) *is finite, where* $-\infty \leqq a < X \leqq \infty$. *If* y *is absolutely continuous on* $[a, X]$, $y(a) = 0$, *and* y' *does not change sign on* (a, X), *then*

$$\int_a^X s\,|\,y\,|^p\,|\,y'\,|^q\,dx \leqq K_1(X, p, q)\int_a^X r\,|\,y'\,|^{p+q}\,dx\ . \tag{8}$$

Equality holds in (8) *if and only if either* $q > 0$ *and* $y \equiv 0$, *or*

$$s = k_1 r^{(q-1)/(p+q-1)}\left(\int_a^x r^{-\{1/(p+q-1)\}}dt\right)^{p(1-q)/q}, \tag{9}$$

and

$$y = k_2\int_a^x r^{-\{1/(p+q-1)\}}dt\ ,$$

for some constants $k_1(\geqq 0)$, k_2 *real.*

It only remains to prove the assertion concerning (9). Now, equality holds in (8) only if it holds in (3)—or (4)— and in Hölder's inequality leading to (6); that is, only if both

$$r\,|\,y'\,|^{p+q} = Ar^{-\{1/(p+q-1)\}} \quad \text{or} \quad y' = k_2 r^{-\{1/(p+q-1)\}}\ ,$$

and

$$z^{p/q}z' = Bs^{(p+q)/p}r^{-(q/p)}\left(\int_a^x r^{-\{1/(p+q-1)\}}dt\right)^{p+q-1}\ .$$

The first of these conditions is equivalent to the second of equations (9) since $y(a) = 0$. Using this condition and the definition of z, the second reduces to

$$R^{(p+q)(1-q)/q} = Cs^{(p+q)/p}(R')^{(p+q)(q-1)/q}\ , \qquad \left(R \equiv \int_a^x r^{-\{1/(p+q-1)\}}dt\right),$$

which is equivalent to the first of equations (9). Finally, if s is given by (9), it is easy to verify that the corresponding value of K_1 in (7) is

$$k_1\frac{q}{p+q}\left(\int_a^X r^{-\{1/(p+q-1)\}}dt\right)^{p/q}\ ,$$

and hence is finite. Similarly, choosing y as in (9),

$$\int_a^X r\,|\,y'\,|^{p+q}\,dx = |\,k_2\,|^{p+q}\int_a^X r^{-\{1/(p+q-1)\}}dx < \infty\ ,$$

completing the proof of the theorem.

COROLLARY 1. *If $pq > 0$, $p + q > 1$, (8) holds even if y is complex-valued. Equality holds if and only if s and y are given by (9) with $k_1 \geqq 0$, k_2 complex.*

Proof. The inequality (8) follows as above but in place of (2) we have

$$|y(x)| \leqq \int_a^x |y'(t)|\,dt\,, \qquad a \leqq x \leqq X\,.$$

Equality holds in (8) only if, in addition to

$$|y'| = Ar^{-\{1/(p+q-1)\}},\ z^{p/q}z' = Bs^{(p+q)/p}r^{-(q/p)}\left(\int_a^x r^{-\{1/(p+q-1)\}}dt\right)^{p+q-1}\,,$$

we also have

$$|y(x)| = \int_a^x |y'(t)|\,dt\,;$$

thus only if

$$y(x) = \left(A\int_a^x r^{-\{1/(p+q-1)\}}dt\right)e^{i\theta(x)}\,,$$

which, in view of the condition on $|y'|$, leads to $\theta'(x) \equiv 0$ and, therefore, only if

$$y = Ae^{i\alpha}\int_a^x r^{-\{1/(p+q-1)\}}dt = k_2\int_a^x r^{-\{1/(p+q-1)\}}dt\,.$$

The rest follows as before.

REMARK 1. If $pq > 0$ and $p + q = 1$, then in place of (5) we have

$$|y|^p \leqq M^p\left(\int_a^x r\,|y'|\,dt\right)^p\,,$$

where $M(x) = \text{ess. sup}_{t\in[a,x]}\, r^{-1}(t)$ and r is a positive, measurable function on (a, X). Therefore, if

$$\tilde{K}_1(X, p, q) = q^q\left\{\int_a^X Ms^{1/p}r^{-(q/p)}dx\right\}^p < \infty\,,$$

then

$$\int_a^X s\,|y|^p\,|y'|^q\,dx \leqq \tilde{K}_1(X, p, q)\int_a^X r\,|y'|\,dx\,. \tag{10}$$

As in the corollary above, equality holds in (10) if and only if $y \equiv 0$, or

$$r = \text{const.} > 0 \quad \text{and} \quad y = k\left(\int_a^x s^{1/p} dt\right)^q ,$$

k complex.

We only state the next theorem, since its proof is the same as that of Theorem 1, with $[a, x]$ replaced by $[x, b]$ throughout.

THEOREM 2. *Let p,q be real numbers satisfying the same conditions as in Theorem 1, and let r, s be nonnegative measurable functions on (X, b), where $-\infty \leqq X < b \leqq \infty$, such that $\int_X^b r^{-1/(p+q-1)} dx < \infty$, and*

$$K_2(X, p, q) \tag{11}$$
$$= \left(\frac{q}{p+q}\right)^{q/(p+q)} \left\{\int_X^b s^{(p+q)/p} r^{-(q/p)} \left(\int_x^b r^{-\{1/(p+q-1)\}} dt\right)^{p+q-1} dx\right\}^{p/(p+q)}$$

is finite. If y is absolutely continuous on $[X, b]$, $y(b) = 0$, (and y' does not change sign on (X, b) in case $q < 0$), then

$$\int_X^b s \mid y \mid^p \mid y' \mid^q dx \leqq K_2(X, p, q) \int_X^b r \mid y' \mid^{p+q} dx . \tag{12}$$

Equality holds in (12) if and only if either $q > 0$ and $y \equiv 0$, or

$$s = k_3 r^{(q-1)/(p+q-1)} \left(\int_x^b r^{-\{1/(p+q-1)\}} dt\right)^{p(1-q)/q} ,$$

and

$$y = k_4 \int_x^b r^{-\{1/(p+q-1)\}} dt ,$$

for some constants $k_3(\geqq 0)$, k_4 real.

REMARK 2. As above, if $pq > 0$ and $p + q > 1$, then (12) holds even if y is complex-valued. Also, if $p + q = 1$, r is a positive, measurable function on (X, b), $\hat{M}(x) = \text{ess. sup}_{t \in [x,b]} r^{-1}(t)$ and

$$\widetilde{K}_2(X, p, q) = q^q \left\{\int_X^b \hat{M} s^{1/p} r^{-(q/p)} dx\right\}^p < \infty ,$$

then

$$\int_X^b s \mid y \mid^p \mid y' \mid^q dx \leqq \widetilde{K}_2(X, p, q) \int_X^b r \mid y' \mid dx , \tag{13}$$

where y is again complex-valued. Equality holds if and only if $r =$ const. >0 and $y = \hat{k}\left(\int_x^b s^{1/p} dt\right)^q$.

COROLLARY 2. *Let $pq > 0$ with $p + q > 1$, and let r, s be nonnegative, measurable functions on (a, b), where $-\infty \leqq a < b \leqq \infty$, such that $\int_a^b r^{-\{1/(p+q-1)\}}dx < \infty$, and*

$$(K(p, q) \equiv)\, K_1(X_1, p, q) = K_2(X, p, q) < \infty \ , \tag{14}$$

where K_1, K_2 are defined by (7), (11) respectively, and $X (a < X < b)$ is the (unique) solution of equation (14). If y is complex-valued, absolutely continuous on $[a, b]$, with $y(a) = y(b) = 0$, then

$$\int_a^b s\,|\,y\,|^p\,|\,y'\,|^q\,dx \leqq K(p, q)\int_a^b r\,|\,y'\,|^{p+q}\,dx \ . \tag{15}$$

Moreover, equality holds if and only if either $y \equiv 0$, or

$$s = \begin{cases} \alpha_1 r^{(q-1)/(p+q-1)}\left(\int_a^x r^{-\{1/(p+q-1)\}}dt\right)^{p(1-q)/q} , & a \leqq x < X \ , \\ \alpha_2 r^{(q-1)/(p+q-1)}\left(\int_x^b r^{-\{1/(p+q-1)\}}dt\right)^{p(1-q)/q} , & X < x \leqq b \ , \end{cases}$$

and

$$y = \begin{cases} \beta_1\int_a^x r^{-\{1/(p+q-1)\}}dt \ , & a \leqq x \leqq X \ , \\ \beta_2\int_x^b r^{-\{1/(p+q-1)\}}dt \ , & X \leqq x \leqq b \ , \end{cases}$$

where α_1, α_2 are nonnegative constants, and β_1, β_2 are complex constants such that

$$\beta_1\int_a^X r^{-\{1/(p+q-1)\}}dt = \beta_2\int_X^b r^{-\{1/(p+q-1)\}}dt \ .$$

Proof. The conclusion follows from Corollary 1 and Theorem 2 since, on choosing X to be the unique solution of equation (14), we have

$$\begin{aligned} \int_a^b s\,|\,y\,|^p\,|\,y'\,|^q\,dx &= \int_a^X s\,|\,y\,|^p\,|\,y'\,|^q\,dx + \int_X^b s\,|\,y\,|^p\,|\,y'\,|^q\,dx \\ &\leqq K_1(X, p, q)\int_a^X r\,|\,y'\,|^{p+q}\,dx + K_2(X, p, q)\int_X^b r\,|\,y'\,|^{p+q}\,dx \\ &= K(p, q)\int_a^b r\,|\,y'\,|^{p+q}\,dx \ . \end{aligned}$$

Moreover, equality holds in (15) if and only if it holds in both (8) and (12).

REMARK 3. As before, if $pq > 0$ and $p + q = 1$, then for r a positive, measurable function on (a, b),

$$\int_a^b s\,|\,y\,|^p\,|\,y'\,|^q\,dx \leqq \tilde{K}(p, q)\int_a^b r\,|\,y'\,|\,dx\,, \tag{16}$$

where

$$(\tilde{K}(p, q) \equiv)\ \tilde{K}_1(X, p, q) = \tilde{K}_2(X, p, q)\,.$$

Equality holds in (16) if and only if either $y \equiv 0$, or

$$r(x) = \begin{cases} c_1(>0), & a \leqq x < X\,, \\ c_2(>0), & X < x \leqq b, \end{cases} \quad \text{and} \quad y = \begin{cases} \gamma_1\Big(\int_a^x s^{1/p}dt\Big)^q\,, & a \leqq x \leqq X\,, \\ \gamma_2\Big(\int_x^b s^{1/p}dt\Big)^q\,, & X \leqq x \leqq b\,, \end{cases}$$

where

$$\gamma_1\Big(\int_a^X s^{1/p}dt\Big)^q = \gamma_2\Big(\int_X^b s^{1/p}dt\Big)^q\,.$$

EXAMPLES

1. Setting $r = s \equiv 1$ in (8) or (10), we obtain as an improvement of (1),

$$\int_a^X |\,y\,|^p\,|\,y'\,|^q\,dx \leqq \frac{q^{q/(p+q)}}{p+q}(X-a)^p\int_a^X |\,y'\,|^{p+q}\,dx \tag{17}$$

if $pq > 0, p + q \geqq 1$. It may be remarked that (17) is also true if $p = 0$. Equality holds in (17) in case $p + q > 1$ if and only if either $p = 0$, or else $y \equiv 0$, or else $q = 1$ and $y = A(x - a)$; if $p + q = 1$, equality holds if and only if $y = A(x - a)$. In case $q = 1$, (17) reduces to the results of Hua, Yang, Calvert and Wong, while Opial's original inequality is obtained for $p = q = 1$. (Note that if $p < 0$ and $q < 0$, $K_1(X, p, q) = \infty$.)

2. Taking $q = 1, s \equiv 1$ in (15), we obtain

$$\int_a^b |\,y^p y'\,|\,dx \leqq \frac{1}{p+1}\Big(\int_a^X r^{-(1/p)}dx\Big)^p\int_a^b r\,|\,y'\,|^{p+1}\,dx\,, \tag{18}$$

if $p \geqq 0$, and y is complex-valued, absolutely continuous on $[a, b]$ with $y(a) = y(b) = 0$. Here, X is the unique solution of

$$\int_a^X r^{-(1/p)}dx = \int_X^b r^{-(1/p)}dx,\ \int_a^b r^{-(1/p)}dx < \infty\,.$$

Equality holds in (18) if and only if $y = A\int_a^x r^{-(1/p)}dt$ for $a \leqq x \leqq X$ and $y = B\int_x^b r^{-(1/p)}dt$ for $X \leqq x \leqq b$. In case $p = 1$, (18) reduces to a result of Beesack [2].

3. Taking $r \equiv 1, s \equiv (x - a)^{p(1-q)/q}$ in Theorem 1,

$$(19)\quad \int_a^X (x-a)^{p(1-q)/q}\,|y|^p\,|y'|^q\,dx \leqq \frac{q}{p+q}(X-a)^{p/q}\int_a^X |y'|^{p+q}\,dx\,.$$

Equality holds if and only if either $q > 0$ and $y \equiv 0$, or $y = A(x-a)$. As a special case of (19), let $y = u^{1/2}$, $p = q = -1$, $a = 0$. Then

$$\int_0^X \frac{x^2}{|u'|}dx < X\int_0^X \frac{|u|}{|u'|^2}dx \qquad \text{unless } u = Ax^2\,.$$

4. Taking $r \equiv (x-a)^{p(p+q-1)/(p+q)}$, $s \equiv 1$ in Theorem 1,

$$(20)\quad \begin{aligned}&\int_a^X |y|^p\,|y'|^q\,dx\\ &\leqq \left(\frac{q}{p+q}\right)^{1-p}(X-a)^{p/(p+q)}\int_a^X (x-a)^{p(p+q-1)/(p+q)}\,|y'|^{p+q}\,dx\,.\end{aligned}$$

Equality holds if and only if either $q > 0$ and $y \equiv 0$, or $y = A(x-a)^{q/p+q}$. As a special case of (20), let $y = u^{1/2}$, $p = q = -1$, $a = 0$. Then

$$\int_0^X \frac{dx}{|u'|} < \frac{1}{2}X^{1/2}\int_0^X \frac{x^{-3/2}\,|u|}{|u'|^2}dx \qquad \text{unless } u = Ax\,.$$

3. To obtain lower bounds for $\int_a^X s\,|y|^p\,|y'|^q\,dx$ $\left(\text{or } \int_a^b s\,|y|^p\,|y'|^q\,dx\right)$ consider first the case when $p + q > 1$. If, in addition, $p < 0$, (3) yields

$$(21)\quad |y|^p \geqq \left(\int_a^x r^{-\{1/(p+q-1)\}}dt\right)^{p(p+q-1)/(p+q)}\left(\int_a^x r\,|y'|^{p+q}\,dt\right)^{p/(p+q)}\,.$$

If s is non-negative on (a, X), then

$$s\,|y|^p\,|y'|^q \geqq s r^{-\{q/(p+q)\}}\left(\int_a^x r^{-\{1/(p+q-1)\}}dt\right)^{p(p+q-1)/(p+q)} z^{p/(p+q)}(z')^{q/(p+q)}\,,$$

where $z(x) = \int_a^x r\,|y'|^{p+q}\,dt$.

Thus, Hölder's inequality with indices $(p+q)/p$ and $(p+q)/q$—note that the latter lies between 0 and 1—gives

$$(22)\quad \int_a^X s\,|y|^p\,|y'|^q\,dx \geqq K_1(X, p, q)\int_a^X r\,|y'|^{p+q}\,dx\,,$$

where $K_1(X, p, q)$ is defined by (7).

Similarly, if $p > 0$ and $p + q < 0$, then (4) yields (21). Again, if s is non-negative on (a, X), Hölder's inequality with indices $(p+q)/p$ and $(p+q)/q$—note that $0 < (p+q)/q < 1$ still holds—leads to (22). Equality holds in (22) if and only if it holds in (3)—or (4)—and in Hölder's inequality leading to (22); that is, if and only if s, y are given by (9). This proves

Theorem 3. *Let p, q be real numbers such that either $p < 0$ and*

$p+q>1$, or $p>0$ and $p+q<0$. Let r, s be nonnegative measurable functions on (a, X) such that $\int_a^X r^{-1/(p+q-1)}dx < \infty$, and the constant $K_1(X, p, q)$ defined by (7) is finite, where $-\infty \leqq a < X \leqq \infty$. If y is absolutely continuous on $[a, X]$, $y(a) = 0$, and y' does not change sign on (a, X), then (22) holds. There is equality in (22) if and only s and y are as defined in (9).

COROLLARY 3. *If $p<0$ and $p+q>1$, (22) holds even if y is complex-valued. Equality holds if and only if s and y are given by (9) with $k_1 \geqq 0$, k_2 complex.*

The proof of this is essentially the same as that of Corollary 1.

REMARK 4. If $p<0$ and $p+q=1$, then in place of (21) we have

$$|y|^p \geqq M^p\left(\int_a^x r\,|y'|\,dt\right)^p ,$$

where $M(x) = \text{ess sup}_{t\in[a,x]}\, r^{-1}(t)$ and r is a positive, measurable function on (a, X).

Thus, if

$$\tilde{K}_1(X, p, q) = q^q\left\{\int_a^X M s^{1/p} r^{-(q/p)} dx\right\}^p < \infty ,$$

then

$$\int_a^X s\,|y|^p\,|y'|^q\,dx \geqq \tilde{K}_1(X, p, q)\int_a^X r\,|y'|\,dx . \tag{23}$$

As in the corollary above, equality holds in (23) if and only if

$$r = \text{const.} > 0 \quad \text{and} \quad y = k\left(\int_a^x s^{1/p}dt\right)^q ,$$

k complex.

Replacing $[a, x]$ by $[x, b]$ throughout Theorem 3, we obtain

THEOREM 4. *Let p, q be real numbers satisfying the same conditions as in Theorem 3, and let r, s be non-negative measurable functions on (X, b), where $-\infty \leqq X < b \leqq \infty$, such that $\int_X^b r^{-1/(p+q-1)}dx < \infty$, and $K_2(X, p, q)$ defined by (11) is finite. If y is absolutely continuous on $[X, b]$, $y(b) = 0$, (and y' does not change sign on (X, b) in case $p>0$), then*

$$\int_X^b s\,|y|^p\,|y'|^q\,dx \geqq K_2(X, p, q)\int_X^b r\,|y'|^{p+q}\,dx . \tag{24}$$

Equality holds in (24) if and only if

$$
(25)\qquad \begin{aligned} s &= k_3 r^{(q-1)/(p+q-1)}\left(\int_x^b r^{-\{1/(p+q-1)\}}dt\right)^{p(1-q)/q}, \textit{ and} \\ y &= k_4 \int_x^b r^{-\{1/(p+q-1)\}}dt , \end{aligned}
$$

for some constants $k_3(\geqq 0)$, k_4 *real.*

REMARK 5. If $p < 0$ and $p+q > 1$, then (24) holds even if y is complex-valued. Also, if $p < 0$, $p+q = 1$ and r is a positive, measurable function on (X, b), and

$$
\hat{M}(x) = \operatorname*{ess\,sup}_{t\in[x,b]} r^{-1}(t),\ \tilde{K}_2(X, p, q) = q^q\int_X^b \hat{M} s^{1/p} r^{-(q/p)}dx\Big\} < \infty ,
$$

then

$$
(26)\qquad \int_X^b s\,|\,y\,|^p\,|\,y'\,|^q\,dx \geqq \tilde{K}_2(X, p, q)\int_X^b r\,|\,y'\,|\,dx ,
$$

where y is again complex-valued. Equality holds if and only if $r = \text{const.} > 0$ and $y = \hat{k}\left(\int_x^b s^{1/p}dt\right)^q$.

COROLLARY 4. *Let* $p < 0$ *and* $p+q > 1$. *Let* r, s *be nonnegative, measurable functions on* (a, b), $-\infty \leqq a < b \leqq \infty$, *such that* $\int_a^b r^{-\{1/(p+q-1)\}}dx$ *is finite. Let* y *be complex-valued, absolutely continuous on* $[a, b]$ *with* $y(a) = y(b) = 0$. *Then,*

$$
(27)\qquad \int_a^b s\,|\,y\,|^p\,|\,y'\,|^q \geqq K(p, q)\int_a^b r\,|\,y'\,|^{p+q}\,dx ,
$$

where $K(p, q)$ *is defined by* (14). *Moreover, equality holds if and only if* s *and* y *are defined as in theorem* 2.

The proof is immediate in view of Theorems 3 and 4, Corollary 3 and Remark 5.

REMARK 6. Again if $p < 0$ and $p+q = 1$, then for $r(x)$ positive, measurable on (a, b),

$$
(28)\qquad \int_a^b s\,|\,y\,|^p\,|\,y'\,|^q\,dx \geqq \tilde{K}(p, q)\int_a^b r\,|\,y'\,|\,dx ,
$$

where $\hat{K}(p, q)$ *is defined as in Remark* 3. Further, equality holds in (28) if and only if r and y are defined as in Remark 3.

Our next result is an extension of Theorem 3 to the case when $0 < p+q < 1$ and $q > 1$. (Note that in Theorem 3 the restriction $q > 1$ is implicit since $p+q > 1$ and $p < 0$ imply $q > 1$.)

THEOREM 5. *Let $p < 0, q > 1$ and $0 < p + q < 1$. Let r, s be non-negative, measurable functions on (a, X) such that $\int_a^X r^{-\{1/(p+q-1)\}}dx$ and $\int_a^X s^{-\{1/(q-1)\}}dx$ are finite. If y is complex-valued, absolutely continuous on $[a, X]$, $y(a) = 0$, then*

$$\int_a^X s\,|\,y\,|^p\,|\,y'\,|^q\,dx \geqq \hat{K}_1(X, p, q)\int_a^X r\,|\,y'\,|^{p+q}\,dx\,, \tag{29}$$

where

$$\hat{K}_1(X, p, q) = \left(\frac{q}{p+q}\right)^q\left(\int_a^X s^{-\{1/(q-1)\}}dx\right)^{1-q}\left(\int_a^X r^{-\{1/(p+q-1)\}}dx\right)^{p+q-1}. \tag{30}$$

Equality holds in (29) *if and only if s and y are as defined by* (9) *with k_2 complex.*

Proof. Since $p/q < 0$,

$$|\,y\,|^{p/q} \geqq \left(\int_a^x |\,y'\,|\,dt\right)^{p/q}, \qquad a \leqq x \leqq X\,.$$

Therefore,

$$\int_a^X |\,y\,|^{p/q}\,|\,y'\,|\,dx \geqq \frac{q}{p+q}\left(\int_a^X |\,y'\,|\,dx\right)^{(p+q)/q}. \tag{31}$$

From Hölder's inequality with indices q and its conjugate, it follows that

$$\int_a^X |\,y\,|^{p/q}\,|\,y'\,|\,dx \leqq \left(\int_a^X s^{-\{1/(q-1)\}}dx\right)^{(q-1)/q}\left(\int_a^X s\,|\,y\,|^p\,|\,y'\,|^q\,dx\right)^{1/q};$$

and also with indices $p + q$ and its conjugate, that

$$\int_a^X |\,y'\,|\,dx \geqq \left(\int_a^X r^{-\{1/(p+q-1)\}}dx\right)^{(p+q-1)/(p+q)}\left(\int_a^X r\,|\,y'\,|^{p+q}\,dx\right)^{1/(p+q)}$$

In view of the above inequalities, (29) follows from (31).

Again, equality holds in (29) if and only if

$$|\,y\,| = \int_a^x |\,y'\,|\,dt\,, \quad A_1 s^{-\{1/(q-1)\}} = s\,|\,y\,|^p\,|\,y'\,|^q\,,$$

and

$$A_2 r^{-\{1/(p+q-1)\}} = r\,|\,y'\,|^{p+q}\,;$$

that is, if and only if

$$|\,y'\,| = a_2 r^{-\{1/(p+q-1)\}}\,, \qquad |\,y\,| = a_2\int_a^x r^{-\{1/(p+q-1)\}}dt\,,$$

and

$$s = k_3 r^{(q-1)/(p+q-1)}\left(\int_a^x r^{-\{1/(p+q-1)\}}dt\right)^{p(1-q)/q} ;$$

thus, as in Corollary 1, if and only if s and y are as defined by (9) with k_4 complex.

REMARK 7. If $p < 0, 0 < p + q < 1$ and $q = 1, s(x)$ positive and measurable on (a, X), then in place of (29) the following holds:

$$(32) \qquad \int_a^X s \mid y \mid^p \mid y' \mid dx \geqq \frac{M^{*-1}}{p+1}\left(\int_a^X r^{-(1/p)}dx\right)^p \int_a^X r \mid y' \mid^{p+1} dx ,$$

where $M^* = M^*(X) = \text{ess sup}_{x \in [a,X]} s^{-1}(x)$. Equality holds in (32) if and only if $s = \text{const.} > 0$ and $y = k^* \int_a^X r^{-(1/p)}dt$, k^* complex.

Replacing $[a, x]$ by $[x, b]$ throughout Theorem 5, we obtain

THEOREM 6. *Let p, q be real numbers satisfying the same conditions as in Theorem 5. Let r, s nonnegative, measurable functions on (X, b) such that $\int_X^b r^{-\{1/(p+q-1)\}}dx$ and $\int_X^b s^{-\{1/(q-1)\}}dx$ are finite. If y is complex-valued, absolutely continuous on $[X, b]$, $y(b) = 0$, then*

$$(33) \qquad \int_X^b s \mid y \mid^p \mid y' \mid^q dx \geqq \hat{K}_2(X, p, q) \int_X^b r \mid y' \mid^{p+q} dx ,$$

where

$$\hat{K}_2(X, p, q) = \left(\frac{q}{p+q}\right)^q \left(\int_X^b s^{-\{1/(q-1)\}}dx\right)^{1-q} \left(\int_X^b r^{-\{1/(p+q-1)\}}dx\right)^{p+q-1} .$$

Equality holds in (33) if and only if s and y are defined by (25) with k_4 complex.

As a direct consequence of Theorem 5 and 6 we have

COROLLARY 5. *Let p, q be real numbers satisfying the same conditions as in Theorem 5. Let r, s be nonnegative measurable functions on (a, b) such that $\int_a^b r^{-1/(p+q-1)}dx$ and $\int_a^b s^{-1/(q-1)}$ are finite. If y is complex-valued, absolutely continuous on $[a,b]$ with $y(a) = y(b) = 0$, then,*

$$(34) \qquad \int_a^b s \mid y \mid^p \mid y' \mid^q dx \geqq \hat{K}(p, q) \int_a^b r \mid y' \mid^{p+q} dx ,$$

where $\hat{K}(p, q) = \hat{K}_1(X, p, q) = \hat{K}_2(X, p, q)$, with X the unique solution $(a < X < b)$ of the latter equation. Moreover equality holds in (34) if and only if s and y are defined as in corollary 1.

REMARK 8. Let $p < 0, 0 < p + q < 1$ and $q = 1$; $s(x)$ positive and measurable on (X, b). Then, for complex-valued, absolutely continuous y on $[X, b]$ such that $y(b) = 0$,

$$\int_X^b s\,|\,y\,|^p\,|\,y'\,|\,dx \geqq \frac{\hat{M}^{*-1}}{p+1}\Big(\int_X^b r^{-1/p}dx\Big)^p \int_X^b r\,|\,y'\,|^{p+1}\,dx\ , \tag{35}$$

where $\hat{M}^* = \hat{M}^*(X) = \text{ess sup}_{x\in[X,b]}\, s^{-1}(x)$.

Finally, if y is complex-valued, absolutely continuous on $[a, b]$ such that $y(a) = y(b) = 0$, and if s is positive and continuous on (a, b), then (32) and (35) yield

$$\int_a^b s\,|\,y\,|^p\,|\,y'\,|\,dx \geqq \frac{\bar{M}^{-1}}{p+1}\Big(\int_a^x r^{-1/p}dx\Big)^p \int_a^b r\,|\,y'\,|^{p+1}\,dx\ , \tag{36}$$

where $\bar{M} = M^*(X)$ and X is the unique solution $(a < X < b)$ of the equation $\hat{M}^*(X)\Big(\int_a^X r^{-(1/p)}dx\Big)^p = M^*(X)\Big(\int_a^X r^{-(1/p)}dx\Big)^p$. Equality holds in (36) if and only if $s = \text{const.} > 0$ and

$$y = k_1^*\Big(\int_a^x r^{-(1/p)}dt\Big)^p\Big(k_2^*\Big(\int_x^b r^{-(1/p)}dt\Big)^p\Big)$$

according as $a \leqq x \leqq X (X \leqq x \leqq b)$.

Examples can be constructed for special cases of r and s as before. However, we content ourselves with noting that if $s(x) \equiv 1$, (32) reduces to the following inequality of Calvert's paper [2, p. 75],

$$\int_a^X |\,u^{p-1}u'\,| \geqq \frac{1}{p}\Big(\int_a^X r^{1-q}\Big)^{p-1}\int_a^X r\,|\,u'\,|^p\ ,\quad 0 < p < 1 \quad \text{and} \quad \frac{1}{q} + \frac{1}{p} = 1\ .$$

4. Let u be a given function and let

$$y = u^{q/(p+q)} \qquad (p + q \neq 0)\ .$$

If p and q are such that $q/(p + q) > 0$, then it is obvious that y is absolutely continuous on an interval if and only if u is, and that y vanishes at a point if and only if u does. A simple computation gives

$$|\,y\,|^p\,|\,y'\,|^q = \Big(\frac{q}{p+q}\Big)^q\,|\,u'\,|^q \quad \text{and} \quad |\,y'\,|^{p+q} = \Big(\frac{q}{p+q}\Big)^{p+q}\,|\,u\,|^{-p}\,|\,u'\,|^{p+q}\ ,$$

that is,

$$|\,y\,|^p\,|\,y'\,|^q = \Big(\frac{P+Q}{Q}\Big)^{P+Q}\,|\,u'\,|^{P+Q} \quad \text{and} \quad |\,y'\,|^{p+q} = \Big(\frac{P+Q}{Q}\Big)^Q\,|\,u\,|^P\,|\,u'\,|^Q\ , \tag{37}$$

where $p = -P, p + q = Q$.

In view of (37) and Theorem 1 we have

THEOREM 7. *Let P, Q be real numbers such that either $P < 0$, $Q > 1$ and $P + Q > 0$ or $P > 0$ and $P + Q < 0$. Let r,s be nonnegative, measurable functions on (a, X) such that* $\int_a^X s^{-1/(Q-1)}dx < \infty$. *Let the constant*

(38)

$$K_1^*(X, P, Q) = \left(\frac{Q}{P+Q}\right)^{\{(P+Q)/Q\}-P}\left\{\int_a^X r^{-(Q/P)}s^{(P+Q)/P}\left(\int_a^x s^{-\{1/(Q-1)\}}dt\right)^{Q-1}dx\right\}^{P/Q}$$

be finite. If u is absolutely continuous on $[a, X]$, $u(a) = 0$, and u' does not change sign on (a, X), then

$$\int_a^X s\,|\,u\,|^P\,|\,u'\,|^Q\,dx \geqq K_1^*(X, P, Q)\int_a^X r\,|\,u'\,|^{P+Q}\,dx\,. \tag{39}$$

Equality holds in (38) *if and only if*

$$r = k_1^* s^{(P+Q-1)/(Q-1)}\left(\int_a^x s^{-\{1/(Q-1)\}}dt\right)^{P-\{P/(P+Q)\}}, \quad and$$

$$u = k_2^*\left(\int_a^x s^{-\{1/(Q-1)\}}dt\right)^{Q/(P+Q)},$$

for some constants $k_1^*(\geqq 0)$, k_2^* real.

Theorems 3 and 7 lead to

COROLLARY 6. *Let p, q be real numbers as in Theorem 3. Let r, s be nonnegative measurable functions on (a, X) such that $K_1(X, p, q)$, $K_1^*(X, p, q)$ defined by (7), (38) respectively are finite. If y is absolutely continuous on $[a, X]$, $y(a) = 0$, and y' does not change sign on (a, X), then*

$$\int_a^X s\,|\,y\,|^p\,|\,y'\,|^q\,dx \geqq \max\,(K_1, K_1^*)\int_a^X r\,|\,y'\,|^{p+q}\,dx\,.$$

Moreover, equality holds if and only if s and y are defined by (9) *or*

$$\begin{aligned} r &= k_1^* s^{(p+q-1)/(q-1)}\left(\int_a^x s^{-\{1/(q-1)\}}dt\right)^{p-\{p/(p+q)\}}, \quad and \\ y &= k_2^*\left(\int_a^x s^{-\{1/(q-1)\}}dt\right)^{q/(p+q)}, \end{aligned} \tag{40}$$

for some constants $k_1^*(\geqq 0)$, k_2^* real.

Proof. The inequality is immediate in view of (22) and (39) and the fact that $q > 1$ is implicit if $p < 0$. Again, a straight-forward computation shows that (9) holds if and only if (40) holds. Thus, equality holds in (22) if and only if it holds in (39). Also, then $K_1 = K_1^*$. This completes the proof.

REMARK 9. If $r = s \equiv 1$, K_1^* are meaningful constants when $p + q > 0$ and $q > 0$ respectively. Therefore, in Corollary 6 if $r = s \equiv 1$ and $p < 0, p + q > 1$,

$$K_1 = \frac{q^{q/(p+q)}}{p+q}(X-a)^p \ , \qquad K_1^* = \frac{q^{1-p}}{(p+q)^{(p/q)+1-p}}(X-a)^p \ .$$

It is easy to verify that $\ln x/(1 - x^{-1})$ is an increasing function of x for $x > 1$. Thus,

$$\frac{1}{1-\dfrac{1}{q}}\ln q > \frac{1}{1-\dfrac{1}{p+q}}\ln(p+q) \ ,$$

whence

$$q^{p-\{p/(p+q)\}} < (p+q)^{p-(p/q)} \ .$$

Consequently, in this case $K_1^* > K_1$.

Another example where $K_1^* \geqq K_1$ is when $r = (x-a)^{p(p+q-1)/(p+q)}$, $s = (x-a)^{p(1-q)/q}$, $p < 0$ and $p + q > 1$. Then,

$$K_1 = \left(\frac{q}{p+q}\right)^{1-p}\left(\frac{q}{q+(p+q)(1-q)}\right)^{p/(p+q)}(X-a)^{\{p/(p+q)\}+\{(1-q)p/q\}} \ ,$$

and

$$K_1^* = \frac{q}{p+q}\left(\frac{p+q}{q+(p+q)(1-q)}\right)^{p/q}(X-a)^{\{p/(p+q)\}+(1-q)p/q} \ .$$

If $q \leqq 2$, $q + (p+q)(1-q) > 0$ and therefore, in view of

$$0 < -p/(p+q)(q-1) < 1$$

and $-\ln x$ convex if $x > 0$, we have

$$(q+(p+q)(1-q))^{-\{p/(p+q)(q-1)\}} \cdot q^{1+\{p/(p+q)(q-1)\}} \leqq p+q \ ,$$

whence

$$\left(\frac{p+q}{q+(p+q)(1-q)}\right)^{p+q} \leqq \left(\frac{q}{p+q}\right)^{-q(p+q)}\left(\frac{q}{q+(p+q)(1-q)}\right)^{q} \ ,$$

that is,

$$\left(\frac{p+q}{q+(p+q)(1-q)}\right)^{p/q} \geqq \left(\frac{q}{p+q}\right)^{-p}\left(\frac{q}{q+(p+q)(1-q)}\right)^{p/(p+q)} \qquad \text{if } 2 \geqq q > p+q > 1 \ ,$$

proving that $K_1^* \geqq K_1$ in this case.

As above, in view of (37) and Theorem 3 we have

THEOREM 8. *Let P, Q be real numbers such that $PQ > 0$, and either $Q > 1$ or $Q < 0$. Let r, s be nonnegative, measurable functions on (a, X) such that $\int_a^X s^{-1/(Q-1)}dx < \infty$, and the constant K_1^* defined by* (38) *is finite. If y is absolutely continuous on $[a, X]$, $y(a) = 0$, and y' does not change sign on (a, X), then*

$$\int_a^X s\,|\,u\,|^P\,|\,u'\,|^Q\,dx \leqq K_1^* \int_a^X r\,|\,u'\,|^{P+Q}\,dx\ . \tag{40}$$

Equality holds in (40) *if and only if r and u are as defined in Theorem* 7.

REMARK 10. If P and Q above satisfy

$$P > 0, P + Q > 1 \quad \text{and} \quad 0 < Q < 1\ ,$$

then (37) and Theorem 5 yield

$$\int_a^X s\,|\,u\,|^P\,|\,u'\,|^Q\,dx \leqq \hat{K}_1(X, P, Q) \int_a^X r\,|\,u'\,|^{P+Q}\,dx\ , \tag{41}$$

where $\hat{K}_1$ is defined by (30). Here u can be taken as complex-valued. Equality holds if and only if it holds in (29), that is if and only if s and $u(=y)$ are as defined by (9) with k_2 complex.

If $P > 0$ and $Q = 1$, then (37) and (23) yield

$$\int_a^X s\,|\,u\,|^P\,|\,u'\,|\,dx \leqq \hat{K} \int_a^X r\,|\,u'\,|^{P+1}\,dx \tag{42}$$

where s is a positive, measurable function on (a, X) and

$$\hat{K}(P) = \frac{1}{P+1}\left(\int_a^X M^* s^{(P+1)/P} r^{-(1/P)}dx\right)^P,\ M^*(x) = \operatorname*{ess\,sup}_{t\in[a,x]} s^{-1}(t)\ . \tag{43}$$

Equality holds in (42) if and only if $s = \text{const.} > 0$ and $u = k\left(\int_a^x r^{-(1/P)}dt\right)$, k complex.

Combining Theorems 1 and 8 and Remark 10 we have

COROLLARY 7. *Let p, q be real numbers such that $pq > 0$. Let r, s be nonnegative, measurable functions on (a, X) such that*

$$\int_a^X r^{-\{1/(p+q-1)\}}dx,\ \int_a^X s^{-\{1/q-1)\}}dx$$

(or $M^*(x)$ if $p > 0, q = 1$) exist, and the constants $K_1, K_1^*, \hat{K}_1$ and $\hat{K}(p)$ are finite. If y is absolutely continuous on $[a, X]$, $y(a) = 0$,

and y' does not change sign on (a, X), then

$$\int_a^X s\,|\,y\,|^p\,|\,y'\,|^q\,dx \leqq K\int_a^X r\,|\,y'\,|^{p+q}\,dx\ ,$$

where $K = \min(K_1, K_1^*)$ *if* α) $q > 1$ *or* $q < 0$, $= \min(K_1, \hat{K}_1)$ *if* β) $0 < q < 1$ *and* $p + q > 1$, $= \min(K_1, \hat{K})$ *if* γ) $q = 1$. *Moreover, equality holds if and only if it holhs in both* (8) *and* (40), (8) *and* (41), (8) *and* (42) *according as* α), β), γ) *is the case.*

REMARK 11. If $r = s \equiv 1$ and $q > 1$ (so $p > 0$) in Corollary 7, the fact that $\ln x/(1 - x^{-1})$ is an increasing function of x for $x > 1$ leads to $K_1^* > K_1$ and thus $K = K_1$. Again, if $r = s \equiv 1$ and $q = 1$ above, $K = K_1 = \hat{K}$. Also, if $r = s \equiv 1$ and $0 < q < 1 < p + q$ then

$$K_1 = \frac{q^{q/(p+q)}}{p+q}(X - a)^p\ , \qquad \hat{K}_1 = \Big(\frac{q}{p+q}\Big)^q (X - a)^p\ .$$

That $\hat{K}_1 > K_1$ follows from the fact that for $0 < q < 1 < p + q$,

$$\frac{q}{q-1}\ln q < 1 < \frac{p+q}{p+q-1}\ln(p+q)\ ,$$

whence

$$\Big(1 - \frac{1}{q}\Big)\ln(p+q) < \Big(1 - \frac{1}{p+q}\Big)\ln q\ .$$

Similar results could be stated on $[X, b]$ and $[a, b]$.

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CARLETON UNIVERSITY, OTTAWA
AND
MICHIGAN STATE UNIVERSITY.
(Now at Iowa State University)