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THE SUPPORT OF REPRESENTING MEASURES FOR R(X)

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There are a couple of recent results about algebras of rational functions in the plane with essentially the same method of proof. One result states that the nontrivial Gleason parts of the function algebra $R(X)$ have positive Lebesgue planar measure. A second asserts the lack of completely singular annihilating measures. In this note it is shown how with little extra effort the same method of proof provides even more information about $R(X)$. Specifically it is shown that representing measures for $R(X)$ actually represent for uniform limits of rational functions whose poles lie off the closure of a part. The most noteworthy corollary establishes that the closure of a part must be connected.

Also included is a brief summary of the proofs of the two previously known results mentioned above.

2. Notation, basic lemmas. The starting points for all of the proofs contained in this note are two lemmas about measures with compact support in the plane (Lemmas 2.1 and 2.2). Bishop [1] noted the importance of these lemmas in working with the algebra $R(X)$ and the proofs of all of the results described in the introduction rely to a great extent on his initial work.

Let X be a compact set in the complex plane C. By $R(X)$ we mean the function algebra which consists of functions uniformly approximable on X by rational functions whose poles lie outside of X. Let λ denote Lebesgue planar measure. If μ is any measure on X set

$$\tilde{\mu}(z) = \int \frac{d|\mu|(w)}{|w - z|}, z \in C ,$$

$$\hat{\mu}(z) = \int \frac{d\mu(w)}{w - z}, z \in C .$$

LEMMA 2.1. $\tilde{\mu}(z) < \infty$ a.e. $- d\lambda$.

For the proof it suffices to note that $\tilde{\mu}$ is a convolution of a locally integrable function and a measure with compact support.

LEMMA 2.2. *Let U be an open set in C. If $\hat{\mu}(z) = 0$ a.e. $- d\lambda$ on U, then $|\mu|(U) = 0$.*

An elegant proof due to Beurling [7] merely relies on integration around squares and an application of Fubini's theorem.

For an arbitrary function algebra A let M denote its space of maximal ideals.

DEFINITION 2.3. A *representing measure* for a point x in M is a *positive* measure on M which satisfies $(*)\, f(x) = \int f dm$, all $f \in A$. A *complex representing measure* is a complex measure which satisfies (*).

LEMMA 2.4. *Let μ be a complex representing measure for a point x in M. Then x has a (positive) representing measure m, absolutely continuous with respect to $|\mu|$.*

Proof. Let H be the closure of the algebra A in the space $L^2(|\mu|)$, and let H_0 be the closure of $A_0 = \{f \in A : f(x) = 0\}$ in $L^2(|\mu|)$. If h is closen in $L^2(|\mu|)$ so that h is orthogonal to H_0 and with norm 1, it is easy to check that $m = |h|^2 |\mu|$ is the desired measure.

The lemma was first stated in the above form by Hoffman and Rossi [5], although the above proof is credited to Sarason. It was found earlier by Konig [6].

Gleason introduced an equivalence relation on M which, for our purposes when applied to $R(X)$, can best be described by $x \sim y$ if x and y have representing measures which are not singular.

The equivalence classes under this relation are called the (Gleason) parts of M. For the original ideas about parts see [4]. For a general treatment of all the notions discussed above refer to the expository paper [7]. The connection between parts and representing measures is contained in [2]. For specific applications of Lemmas 2.1 and 2.2 see [3], [1], [8] and for more about Lemma 2.4 see [5].

Now let μ be a measure on X which annihilates $R(X)$, i.e. $\int f d\mu = 0$ for all $f \in R(X)$. The measure μ is said to be *completely singular* if $|\mu|$ is singular with respect to every representing measure for $R(X)$. For an arbitrary annihilating measure we have the following lemmas.

LEMMA 2.5. *If $\tilde{\mu}(y) < \infty$ and $\hat{\mu}(y) \neq 0$, then $\nu_y = 1/\hat{\mu}(y)\, \mu/(z - y)$ is a complex representing measure for y.*

Proof. Let f be a rational function in $R(X)$. Then the function $z \rightarrow \{f(z) - f(y)\}/(z - y)$ is in $R(X)$. Thus

$$0 = \int \frac{f(z) - f(y)}{z - y} d\mu(z) = \int \frac{f(z)}{z - y} d\mu(z) - f(y)\hat{\mu}(y) ,$$

or

$$f(y) = \int f(z) d\nu_y(z) .$$

Taking uniform limits we obtain the lemma.

LEMMA 2.6. *If* $\tilde{\mu}(y) < \infty$ *and* $\hat{\mu}(y) = 0$ a.e. $- d\lambda$ *on an open set* U *in* C, *then* μ *is an annihilating measure for* $R(X \backslash U)$.

Proof. By Lemma 2.2 μ is supported on $X \backslash U$. Any function in $R(X \backslash U)$ is uniformly approximable on $X \backslash U$ by functions of the form $\sum c_n/(z - y_n)$ with $y_n \notin X \backslash U$, $\tilde{\mu}(y_n) < \infty$, $\hat{\mu}(y_n) = 0$ and each c_n a complex constant. But $\int \sum c_n/(z - y_n) d\mu = \sum c_n \hat{\mu}(y_n) = 0$. Hence taking uniform limits μ annihilates $R(X \backslash U)$.

3. Something old, someting new. We now are in a position to quickly prove the three results referred to in the introduction, two of which, as we have mentioned above, are already known and appear in [8] and [9] respectively.

THEOREM 3.1. *If* x *is not a peak point of* $R(X)$ *then the part* P *containing* x *has positive* λ*-measure.*

Proof. Let m be a representing measure for x, distinct from the unit point mass at x. Then $\mu = (z - x)m$ is an annihilating measure for $R(X)$ and $Q = \{y \in X : \tilde{\mu}(y) < \infty \text{ and } \hat{\mu}(y) \neq 0\}$ has positive measure —an immediate consequence of Lemmas 2.1 and 2.2. But then Lemma 2.5 provides a complex representing measure ν_y for y and therefore Lemma 2.4 a (positive) representing measure, say σ_y, for y which is absolutely continuous with respect to $|\mu|$, hence with respect to m. That is, $y \sim x$ and $Q \subset P$ so that $\lambda(P) \geqq \lambda(Q) > 0$.

THEOREM 3.2. *There are no nonzero completely singular annihilating measures for* $R(X)$.

Proof. Let $\mu \neq 0$ annihilate $R(X)$. As in Theorem 3.1 each y in the set $Q = \{y : \tilde{\mu}(y) < \infty, \hat{\mu}(y) \neq 0\}$ has a complex representing measure and hence a (positive) representing measure σ_y absolutely continuous with resprect to $|\mu|$. Clearly then μ is not completely singular.

We would like to point out that although Bishop has indicated how to avoid using Lemma 2.4 in Theorem 3.1 (cf. [8]), nevertheless it appears to be essential in the proof of Theorem 3.2.

THEOREM 3.3. *Let m represent x for $R(X)$ and have closed support S. Let P be the part of $R(X)$ containing x. Then $S \subset \bar{P}$ and m represents x for $R(\bar{P})$.*

Proof[1]. Let $\mu = (z - x)m$ so that μ annihilates $R(X)$. As in Theorem 3.1 the set $Q = \{y : \tilde{\mu}(y) < \infty, \hat{\mu}(y) \neq 0\} \subset P$. Hence if $y \in U = C \backslash \bar{P}$ and $\tilde{\mu}(y) < \infty$, then $\hat{\mu}(y) = 0$. Lemmas 2.1 and 2.6 then yield μ annihilates $R(\bar{P})$. But if f is any rational function in $R(\bar{P})$, we have the function $z \to \{f(z) - f(x)\}/(z - x)$ is in $R(\bar{P})$ and

$$0 = \int \frac{f(z) - f(x)}{z - x} d\mu = \int [f(z) - f(x)] dm(z)$$

so

$$f(x) = \int f(z) dm(z) .$$

Taking uniform limits shows m represents for $R(\bar{P})$. Clearly the closed support S of M satisfies $S \subset \bar{P}$.

COROLLARY 3.4. *If P is a part of $R(X)$ then $\bar{P}$ is connected.*

Proof. If $\bar{P} = A \cup B$ with A and B closed and disjoint, then the characteristic functions X_A and X_B of A and B respectively lie in $R(\bar{P})$. Hence if $x \in P \cap A$ and $y \in P \cap B$, then each representing measure for x and for y on $R(X)$, and therefore also for $R(\bar{P})$ by the theorem, are supported respectively on A and B. That is, each pair of representing measures is mutually singular so that x and y lie in different parts. This is only possible if either $A = \emptyset$ or $B = \emptyset$ and $\bar{P}$ is connected.

COROLLARY 3.5. *If V is an open connected subset of X contained in a part P, then each point x in the topological boundary of V in X is either a peak point or in P.*

Proof. Let Q be the part containing x. If x is not in P and m represents x for $R(X)$, then m is supported on $\bar{Q}$ and represents for $R(\bar{Q})$. But $\bar{Q} \subset X \backslash V$ so that x lies in the boundary of a component of the complement of $\bar{Q}$. This makes x a peak point of $R(\bar{Q})$ so that m must a unit point mass at x. Since this must be the case for every representing measure m for x on $R(X)$, x is also a peak point of $R(X)$.

There is a fourth theorem which belongs in any discussion about the parts of $R(X)$ and which can be viewed as a strengthening of

[1] The proof as given is mostly due to T. W. Gamelin and is much simpler than that originally constructed by the author.

Theorem 3.1. It says, roughly, that a point which is not a peak point is a point of density for the part containing it. The strongest version of the theorem is due to A. Browder ([3]) and is expressed in terms of the norm topology on the dual space of A. For a point x in a part P, let $P_\varepsilon = \{y \in P: \|y - x\| < \varepsilon\}$. Let

$$\Delta_n = \left\{z \in C: |z - x| \leqq \frac{1}{n}\right\}.$$

Then if $0 < \varepsilon \leqq 2$,

THEOREM 3.6. (*Browder*). $P = \{x\}$ *if and only if*

$$\limsup \frac{\lambda(\Delta_n \backslash P_\varepsilon)}{\lambda(\Delta_n)} > 0 .$$

Of course if $P = \{x\}$ the conclusion is obvious; the theorem only says something for nontrivial parts. In particular when $P \neq \{x\}$,

$$\lim \frac{\lambda(\Delta_n \cap P_\varepsilon)}{\lambda(\Delta_n)} = 1$$

and x is a point of density for P_ε. The author established the same conclusion independently for $\varepsilon = 2$, in which case $P_\varepsilon = P$ and

$$\lim \frac{\lambda(\Delta_n \cap P)}{\lambda(\Delta_n)} = 1 .$$

4. Concluding remarks. As we mentioned in the introduction there is a link between Theorem 3.3 and its corollaries, and two of the out-standing conjectures about $R(X)$. One conjecture is that a part is always connected. Corollary 3.4 eliminates certain types of disconnectedness, most specifically isolated points and isolated components. The second conjecture is that parts are always "separated" by peak points. Put another way it says that if P is a part, then $\bar{P}\backslash P$ consists entirely of peak points. Corollary 3.5 provides for many points in $\bar{P}\backslash P$ to be peak points and actually proves the conjecture for parts which have a dense open connected interior. In cases where X has a connected dense interior, the only nontrivial part is that which contains the interior.

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