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THE INTEGRATION OF A LIE ALGEBRA REPRESENTATION

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Let $u: G \rightarrow A$ be a differentiable representation of a Lie group into a b-algebra. The differential $u_0 = du_e$ of u at the neutral element e of G is a representation of the Lie algebra $\mathfrak{g}$ of G into A. Because a Lie group is locally the union of one-parameter subgroups and since the infinitesimal generator of a differentiable (multiplicative) sub-semi-group of A determines this sub-semi-group, the representation u_0 determines u if G is connected.

We shall be concerned with the converse: given a representation u_0 of $\mathfrak{g}$, when can it be obtained by differentiating a representation u of G? We shall assume G connected and simply connected, which means that we are only interested in the local aspect of the problem.

Call $a \in A$ *integrable* if a differentiable $r: \boldsymbol{R} \rightarrow A$ can be found such that $r(s + t) = r(s)r(t)$ and $r'(0) = a$. We can only hope to integrate $u_0: \mathfrak{g} \rightarrow A$ to a differentiable $u: G \rightarrow A$ if u_0x is integrable for all $x \in \mathfrak{g}$. We shall prove the

THEOREM. *The set $\mathfrak{h}$ of all elements $x \in \mathfrak{g}$ such that u_0x is integrable, is a Lie subalgebra of $\mathfrak{g}$; the representation u_0 can be integrated to a representation $u: G \rightarrow A$ of the simply connected group G if and only if $\mathfrak{h} = \mathfrak{g}$.*

This result is "best possible" in the following sense:

PROPOSITION 1. *Given a real Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$, there exists a representation $u_0: \mathfrak{g} \rightarrow A$ of $\mathfrak{g}$ in a b-algebra A, so that*

$$\mathfrak{h} = \{x \in \mathfrak{g} \mid u_0x \textit{ is integrable}\} .$$

As a consequence of the theorem, we have the following result: Let x, y be two integrable elements of a b-algebra, and assume that the Lie algebra $\mathfrak{g}$ they generate is finite-dimensional. Then all elements of $\mathfrak{g}$ are integrable.

We cannot drop the assumption that $\mathfrak{g}$ is finite-dimensional. There exists a b-algebra which contains integrable elements x, y such that neither $x + y$ nor $xy - yx$ is integrable.

Elementary properties of b-spaces and b-algebras can be found in [2] or [3]. Differentiable mappings into such spaces are investigated in [4]. The results we need about differentiable semi-groups are established in [5], [6]. Our results are related to, but different from, those of R. T. Moore [1].

2. We first prove Proposition 1. Let G be a Lie group having $\mathfrak{g}$ as Lie algebra and let H be the subgroup of G "generated" by $\mathfrak{h}$. Call A the ring of distributions on G whose support is compact and contained in H. The product in A is the convolution. A subset B of A is bounded if B is a bounded set of distributions with compact support, the union of the supports being relatively compact in H. Then, it is easily seen that the elements of $\mathfrak{g}$ whose image by the natural inclusion $u_0: \mathfrak{g} \to A$ are integrable, are precisely the elements of $\mathfrak{h}$. This completes the proof.

REMARK. If H is simply connected, the algebra A described above is the solution of a universal problem: every representation $u: \mathfrak{g} \to A'$ of $\mathfrak{g}$ in a b-algebra A' such that $u\mathfrak{h}$ is integrable can be factorized in a unique way as $u = v \circ u_0$, where $v: A \to A'$ is a morphism of b-algebras. An easy but somewhat technical modification of our definition of A would provide a solution of this problem in general (for an arbitrary H); the reader will have no difficulty to figure it out.

3. Let u be a differentiable mapping of a manifold D into another manifold D' or into a b-space E. We denote by $du(x; \cdot)$ the derivative of u at x, so that $du(x, \xi)$ is a tangent vector to D' at ux or an element of E when ξ is a tangent vector at $x \in D$. The chain rule says that if D, D', D'' are manifolds, if E is a b-space and if $u: D \to D'$, $v: D' \to D''$ or $D' \to E$ are differentiable mappings, then

$$d(v \circ u)(x; \xi) = dv(ux; du(x; \xi)) \ . \tag{1}$$

Let G be a Lie group whose neutral element will be denoted by e and let $\mathfrak{g}$ be its Lie algebra. If $x, y \in G$ and if ξ is a tangent vector at x, then $y\xi$ and ξy will be the tangent vectors at yx, xy respectively obtained by translating ξ to the left or to the right. We shall denote by $\pi: G \times G \to G$ the product mapping ($\pi(x, y) = xy$), by $i: G \to G$ the inverse mapping ($i(x) = x^{-1}$), by $Ad: G \to \mathrm{Aut}\,\mathfrak{g}$ the adjoint representation ($\mathrm{Ad}\, x \cdot \xi = x\xi x^{-1}$) and by ad the derivative of Ad at e ($ad\xi \cdot \eta = [\xi, \eta]$). We have

$$d\pi(x, y; \xi, \eta) = x\eta + \xi y \ ; \tag{2}$$

$$di(x; \xi) = -x^{-1} \cdot \xi \cdot x^{-1} \ . \tag{3}$$

Let H be a Lie group, let A be a b-algebra and let u denote either a Lie group homomorphism $G \to H$ or a differentiable mapping $G \to A$ which is a homomorphism of G in the multiplicative group of A. Finally, set $u_0 = du(e; \cdot)\colon \mathfrak{g} \to \mathfrak{h} = \operatorname{Lie} H$ or $\mathfrak{g} \to A$ accordingly. Then

$$du(x; \xi) = u(x)u_0(x^{-1}\xi) = u_0(\xi x^{-1})u(x) . \tag{4}$$

In particular

$$dAd(x; \xi) = Ad\, x \cdot ad(x^{-1}\xi) = ad(\xi x^{-1}) \cdot Ad\, x . \tag{5}$$

4. Let A be a b-algebra and A^* be the set of its invertible elements. A mapping $u\colon D \to A^*$ will be called *differentiable* if both $x \to u(x)$ and $x \to u(x)^{-1}$ are differentiable mappings.

It is not difficult to construct differentiable A-valued mappings which are A^*-valued but are not differentiable A^*-valued mappings.

Consideration of the resolvent identity

$$a^{-1} - b^{-1} = -a^{-1}(a - b)b^{-1}$$

and standard proofs show that a differentiable mapping $u\colon D \to A^*$ with values in A^* is a differentiable A^*-valued mapping in the above sense if and only if $u^{-1}\colon D \to A$ is locally bounded. It turns out that

$$du^{-1}(x; \xi) = -u^{-1}(x) \cdot du(x; \xi) \cdot u^{-1}(x) . \tag{6}$$

5. From now on, G will be a connected, simply connected Lie group, $\mathfrak{g}$ will be its Lie algebra, A a b-algebra and $u_0\colon \mathfrak{g} \to A$ a representation. A differentiable submanifold D of G is called *right* (resp. *left*) integrable for u_0 if a differentiable $u\colon D \to A^*$ exists such that the equation (7) (resp. (8)) holds:

$$du(x; \xi) = u_0(\xi \cdot x^{-1})u(x) ; \tag{7}$$

$$du(x; \xi) = u(x)u_0(x^{-1} \cdot \xi) . \tag{8}$$

It will follow from Proposition 2 that the representation u_0 is integrable in the sense of § 1 if and only if the manifold G itself is right or left integrable; therefore the terminology. We note that, if u satisfies (7), then

$$du^{-1}(x; \xi) = -u^{-1}(x)u_0(\xi \cdot x^{-1}) . \tag{9}$$

A right translate of a right integrable manifold is right integrable. If u satisfies (7), so does au for every $a \in A^*$.

LEMMA 1. *Let D be connected, right integrable, containing e, and let u be a solution of* (7) *such that* $u(e) = 1$. *Then*

(10) $$u_0(x\xi x^{-1}) = u(x)u_0(\xi)u(x)^{-1}$$

for all $x \in D$ *and* $\xi \in \mathfrak{g}$.

It suffices to show that if $\varphi: D \to A$ is defined by

$$\varphi(x) = u(x)^{-1}u_0(x\xi x^{-1})u(x) ,$$

then $d\varphi = 0$, and this follows from a straightforward computation using (7), (9), (5) and the fact that $u_0: \mathfrak{g} \to A$ is a homomorphism of Lie algebras.

LEMMA 2. *If D is connected, right integrable and contains e, it is also left integrable. Furthermore, the solution u of* (7) *such that $u(e) = 1$ is also a solution of* (8).

This is clear since, by (10),

$$u(x)u_0(x^{-1}\xi) = u_0(x \cdot x^{-1}\xi \cdot x^{-1})u(x) = u_0(\xi x^{-1})u(x) .$$

In view of Lemma 2, it is now meaningful to say that a manifold containing e is integrable.

6. Let D, D' be two differentiable manifolds. The *rank* r_x of a differentiable mapping $u: D \to D'$ at a point $x \in D$ is the dimension of the image of the derivative $du(x; \cdot)$. We recall that r_x is upper semi-continuous as a function of x. The mapping u is said to be *regular* at x if r_x is constant in a neighborhood of x; in that case, there exists a neighborhood U of x, a submanifold D'' of D', a manifold E and a diffeomorphism $u': U \to D'' \times E$, so that $u|_U = p_{D''} \circ u'$ where $p_{D''}$ denotes the projection of $D'' \times E$ of its first factor.

LEMMA 3. *For $i = 1, 2$, let D_i be an integrable submanifold of G containing e, and let $u_i: D_i \to A$ be a solution of* (7) *mapping e on 1. Assume that the product mapping $D_1 \times D_2 \to G$ is regular at (e, e). Then, one can find neighborhoods D'_1, D'_2 of e in D_1, D_2 respectively, so that $D = D'_1 \cdot D'_2$ is an integrable manifold and the relation*

(11) $$u(x_1 \cdot x_2) = u_1(x_1) \cdot u_2(x_2) \quad (x_i \in D'_i)$$

defines a mapping $u: D \to A$ which is a solution of (7).

Put $v(x_1, x_2) = u_1(x_1)u_2(x_2)$, differentiate and apply (7), (10) and (2). This yields

(12) $$dv(x_1, x_2; \xi_1, \xi_2) = u_0(d\pi(x_1, x_2; \xi_1, \xi_2)x_2^{-1}x_1^{-1})v(x_1, x_2) .$$

In particular, $dv = 0$ whenever $d\pi = 0$. This, the regularity assumption and the implicit function theorem imply the existence of a function u satisfying (11) locally. In view of (12), this function is locally a solution of (7).

7. Our main theorem is an immediate consequence of the

PROPOSITION 2. *Let D be an integrable submanifold of G of maximum dimension containing e and let $u: D \to A$ be the solution of (7) with $u(e) = 1$. Then D is a local subgroup, u is a local homomorphism of D into A^* and D contains locally every integrable submanifold of G containing e.*

We first show that

(*) if D' is any integrable submanifold of G containing e, the tangent space to D' at e is contained in that of D.

Assume the contrary. Then there exists a neighborhood U of (e, e) in $D \times D'$ such that, for every $(x, x') \in U$, the tangent space to $x^{-1}D$ at e does not contain that to $D'x'^{-1}$. Let $(f, f') \in U$ be a point where the product mapping $D \times D' \to D \cdot D'$ is regular (one knows that the set of those points is dense). Then, by Lemma 3, there exist neighborhoods E of f in D and E' of f' in D' such that $f^{-1}EE'f'^{-1}$ is an integrable manifold, which is obviously of dimension greater than that of D, in contradiction to the maximality assumption.

It follows from (*) that the tangent space to D at any one of its points, say x, is a translate of its tangent space at e (take $D' = x^{-1}D$). This ensures that D is a local group.

Since D is a local group, the product mapping $D \times D \to D$ is regular in (e, e). It then follows from Lemma 3 that there exist a neighborhood U of (e, e) in $D \times D$ and a function v defined in a neighborhood of e in D so that

$$v(x_1x_2) = u(x_1)u(x_2)$$

for $(x_1, x_2) \in U$. But then, for points x_1, x_2 close enough to e, we have

$$u(x_1)u(x_2) = v(x_1x_2 \cdot e) = u(x_1x_2) \cdot u(e) = u(x_1x_2) ,$$

and u is a local representation.

Finally, if D' is integrable (right or left), it follows from (8) that the tangent space to D' at any one of its points is contained in a translate of the tangent space to D at e. If $e \in D'$, this implies that D' is locally (at e) contained in D.

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